1 Adaptive Weighting

1.1 Problem Statement

Given gradient constraints represented by the function $\vec{g}(x)$, desired pixel values, given by the function $b(x)$, and a non-negative weighting function $\alpha(x)$, the goal is to solve for the function $u(x)$ minimizing:

$$\text{Error}(u) = \|\nabla u - \vec{g}\|^2 + \|\alpha \cdot (u - b)\|^2.$$

Setting f to the divergence of $\vec{g}$, the solution is the function u satisfying:

$$\Delta u + \alpha u = f + \alpha b.$$

1.2 Combining Answers

Suppose that the solution to the gradient constraints is known so that we have a function ϕ minimizing $\|\nabla\phi - \vec{g}\|$. Then solving the adaptive weighting problem is about mixing the two solutions; on the one hand we would like to set $u = \phi$ to satisfy the gradient constraints, on the other we would like to set $u = b$ to satisfy the interpolation constraints.

We can think of this as the problem of simultaneous minimization with respect to two different norms. Specifically, if for a symmetric positive[1] semi-definite matrix M we set the pseudo-inner-product[2] $\langle \cdot\,,\,\cdot \rangle_M$ to be $\langle x, y\rangle_M = x^t M y$, then solving for u amounts to minimizing:

$$\text{Error}(u) = \|u - \phi\|^2_\Delta + \|u - b\|^2_{\alpha^2}.$$

1.3 The Loss of Diagonality

In implementing the multigrid solver, we assume that we have vector spaces V and W with an injective prolongation operator $P : V \hookrightarrow W$, and inner-products $\langle \cdot\,,\,\cdot \rangle_\Delta, \langle \cdot\,,\,\cdot \rangle_{\alpha^2} : W \times W \to \mathbb{R}$ defined over W.

To solve the low-resolution problem over V we define the inner-products by using the prolongation operator to take vectors up to the higher-resolution space and then compute the known inner-products. For $b^0, \phi^0, u^0 \in V$, the error is:

$$\begin{aligned}\text{Error}(u^0) &= \|P(u^0 - \phi^0)\|^2_\Delta + \|P(u^0 - b^0)\|^2_{\alpha^2} \\ &= \|u^0 - \phi^0\|^2_{P^t\Delta P} + \|u^0 - b^0\|^2_{P^t\alpha^2 P}.\end{aligned}$$

The first operator, $P^t\Delta P$ is just the low-resolution Laplacian defined by the Galerkin conditions. The second, $P^t\alpha^2 P$ is the source of trouble in adaptive weighting since even though the weights α^2 act as a diagonal operator in the high-resolution space W, it is not true that the weighting matrix defined at lower resolutions will continue to be diagonal.

[1] Positivity is required to ensure that a minimum is attained where the gradient is zero.
[2] *Pseudo* because the matrix M is not assumed to be definite.

1.4 Thoughts

If we want to maintain efficiency in our streaming solver by ensuring that the adaptive weights do not swamp the computation, we may want to explore how using just the diagonal entries of the restricted weighting function at coarser resolution affects the convergence.

Alternatively, since we know that the operator is diagonal at the highest resolution, and since this is where most of the computational effort happens, maybe the loss of efficiency at lower resolutions won't make too much of a difference. (Note that if we do implement the correct solution, the weighting stencil will vary from point to point, but the neighborhood over which it is defined will never be larger than the neighborhood over which the Laplacian is defined, so for second-order elements, the stencil will remaing 5×5.)