Search and Intersection

O'Rourke, Chapter 7

de Berg *et al.*, Chapter 11

Outline

- Review
  - Duality
  - Linear Programming
- Half-Spaces and Convex Hulls (2D)
- Convex Polygon Intersection

Duality

Definition:

Given a point $p = (\alpha, \beta)$ in the plane, define the *dual line* to be the (non-vertical) line with equation:

$$p^* = \{(x, y) | y = 2\alpha x - \beta\}$$

Given a (non-vertical) line $L = \{(x, y) | y = mx + b\}$, define the *dual point* to be the point with coordinates:

$$L^* = \left(\frac{m}{2}, -b\right)$$

Duality

Properties:

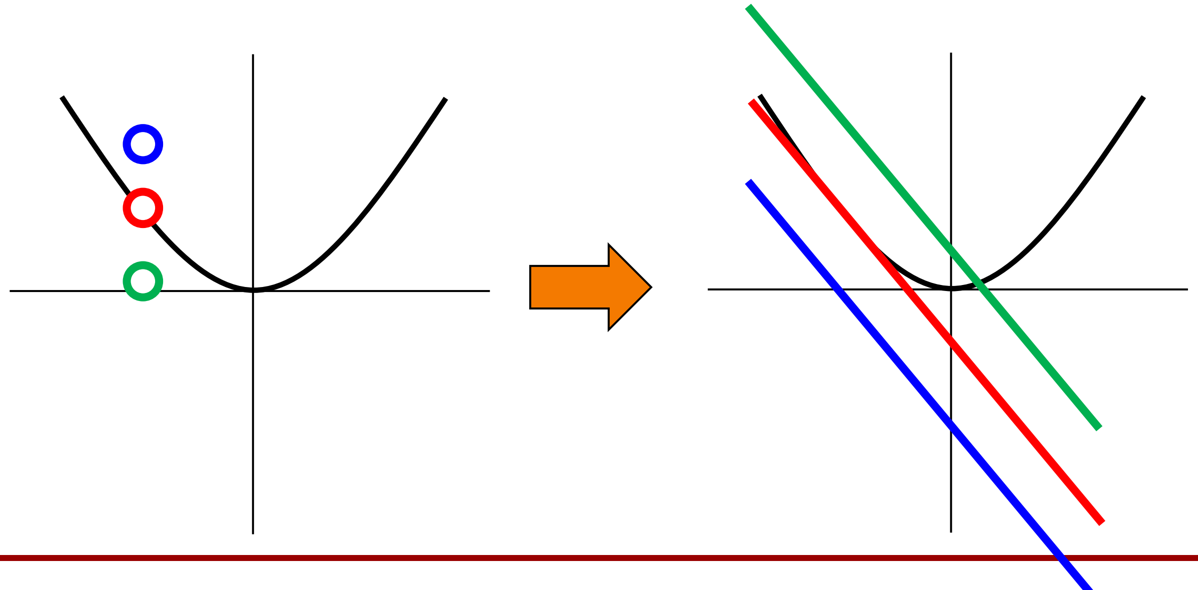
Given a point p and lines L, L_1, and L_2:

- $(p^*)^* = p$ and $(L^*)^* = L$.
- $p \in L$ iff. $L^* \in p^*$.
- $p \in L_1 \cap L_2$ iff. $L_1^*, L_2^* \in p^*$.
- L is below/above p iff. L^* is above/below p^*.
- p is on the parabola $y = x^2$ iff. p^* is tangent to the parabola at p.

Duality

Properties:

- Given a point $p = (\alpha, \beta)$:
  - The slope of p^* is the slope of the tangent to the parabola at (α, α^2).
  - p^* passes through the point $\left(\alpha, \alpha^2 + \left(\alpha^2 - \beta\right)\right)$.



Linear Programming

Goal:

Given a set of linear constraints:

$$C_i = \{p | \langle p, n_i \rangle \geq d_i\}$$

and a linear energy function:

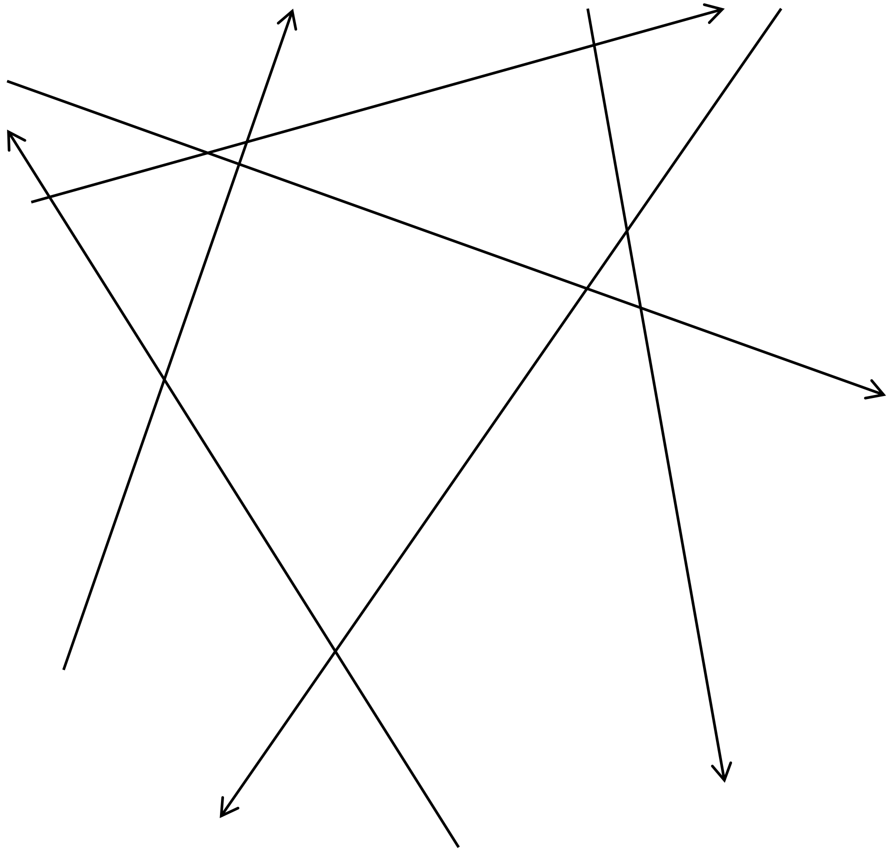
$$E(p) = \langle p, n \rangle + d$$

we would like to find the point p that satisfies the constraints and minimizes the energy.

Linear Programming

Approach:

- Since the constraints are linear, each one defines a half-space of valid solutions.



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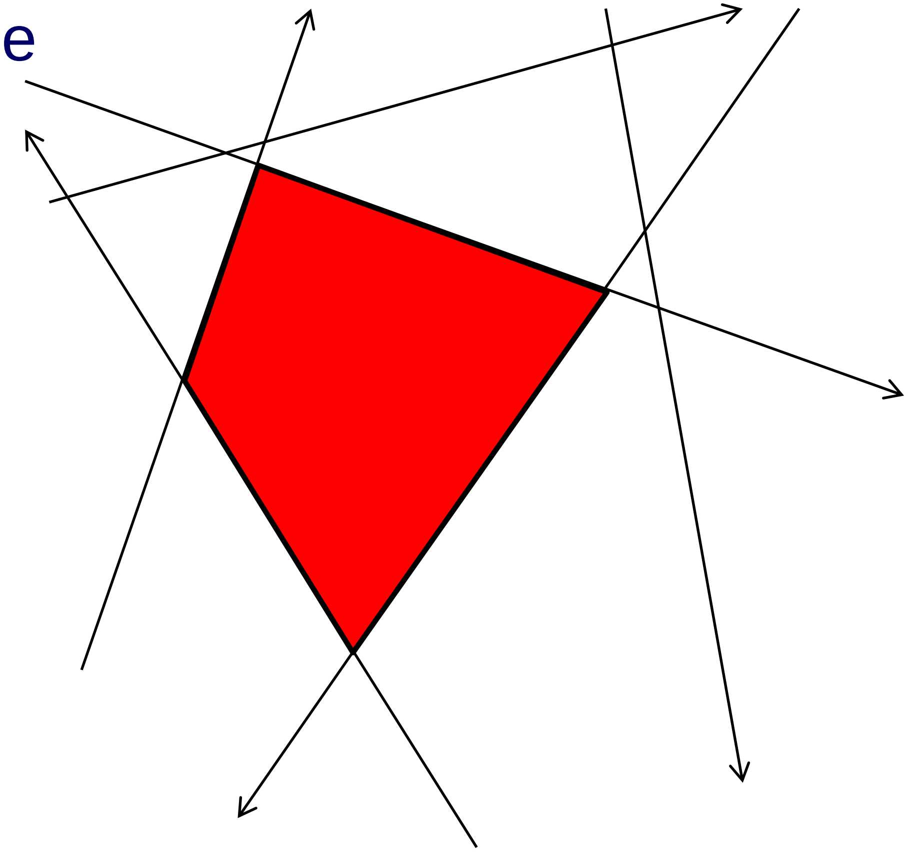
- The intersection of these half-spaces is convex.



Linear Programming

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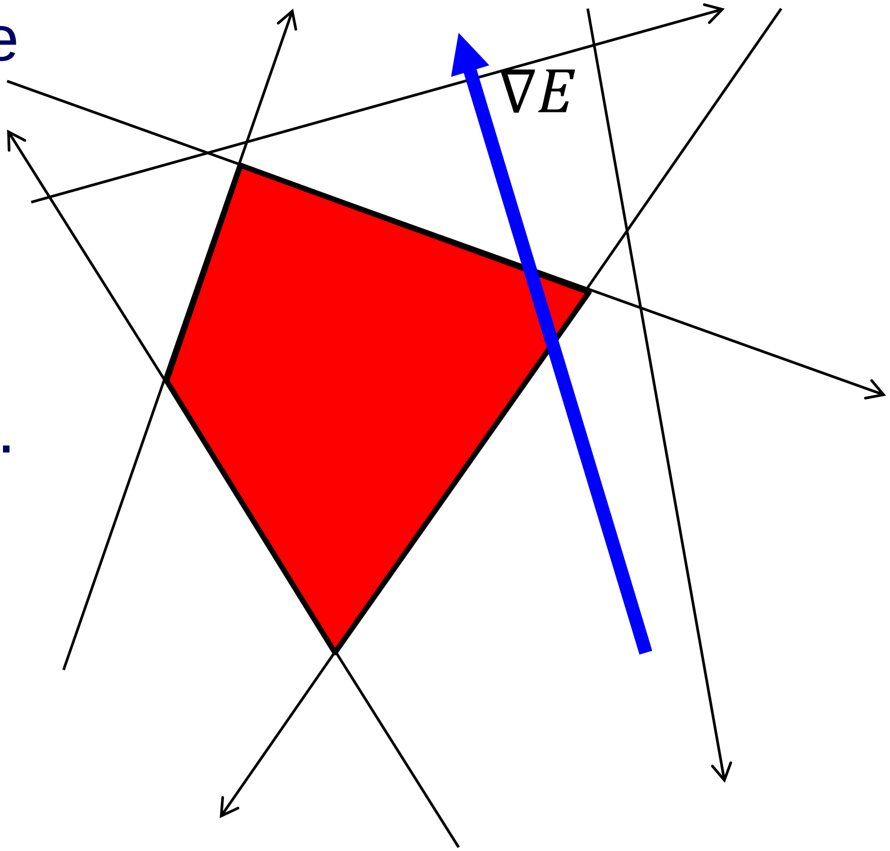
- The intersection of these half-spaces is convex.
- Since the energy is linear, it has a constant gradient ∇E pointing away from the minimum.



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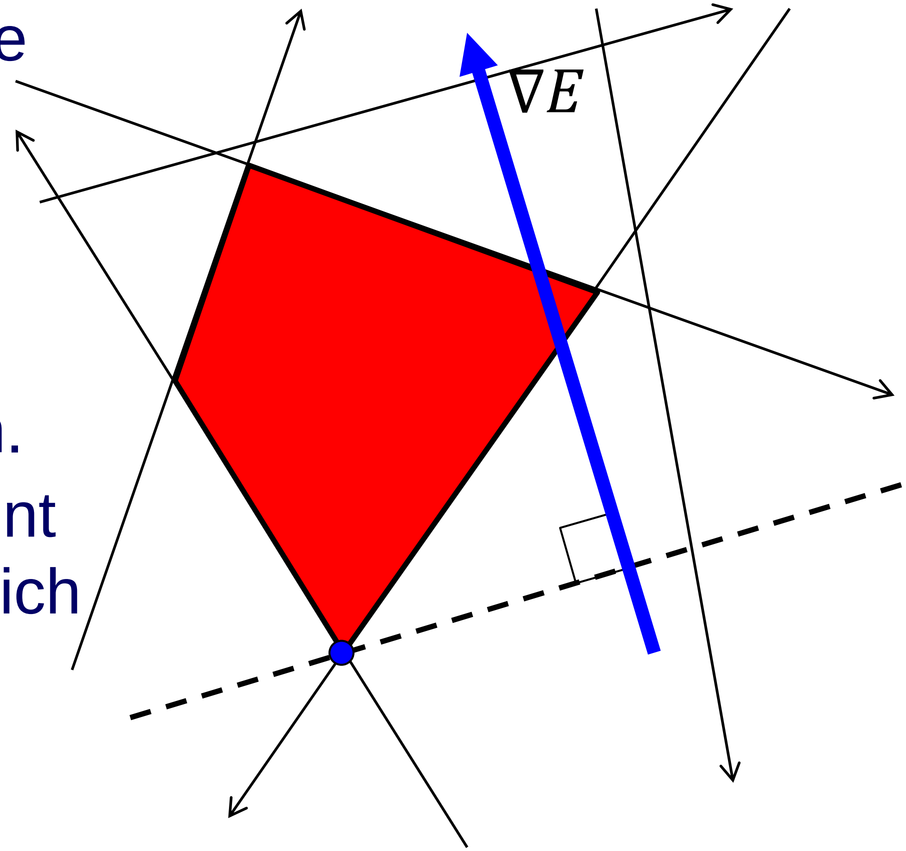
- The minimizer is the point in the convex region which is extreme along direction ∇E.



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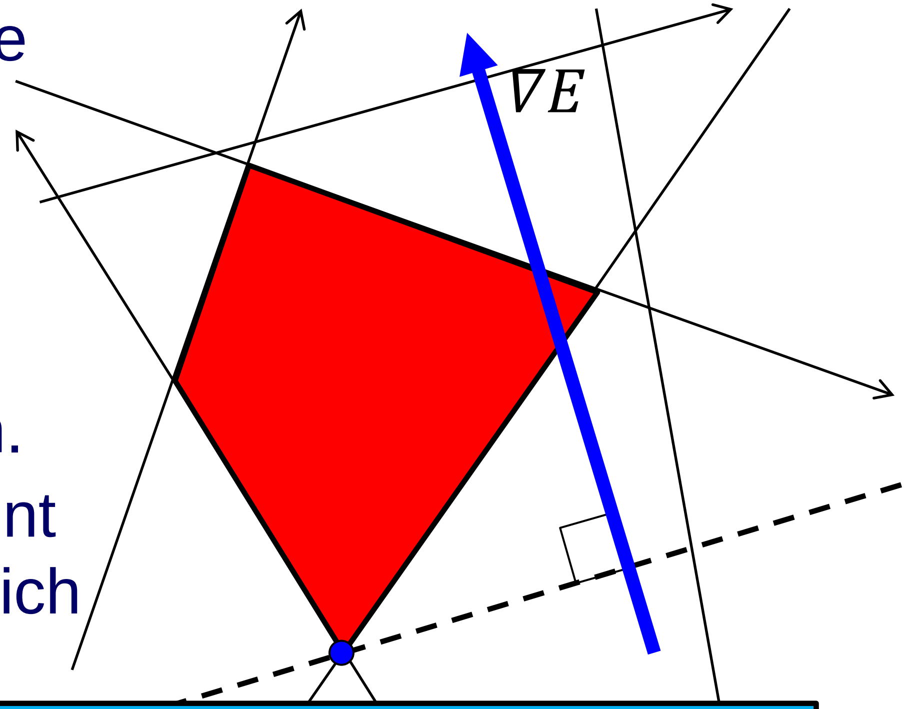


How do we compute the convex hull corresponding to the intersection of half spaces?

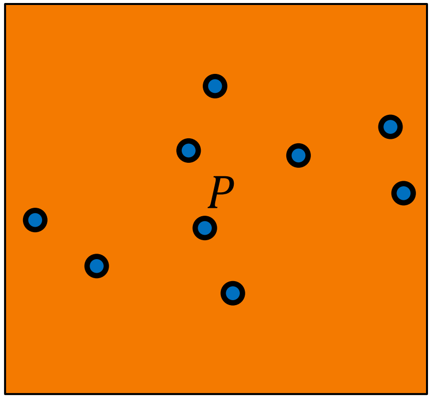
Outline

- Review
- Half-Spaces and Convex Hulls (2D)
- Convex Polygon Intersection

Half-Spaces and Convex Hulls (2D)

Notation:

Given a set of points, $P = \{p_1, \ldots, p_n\}$:

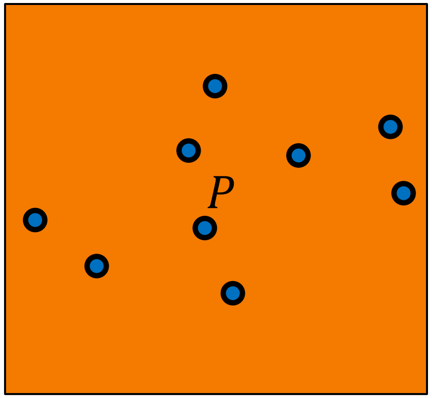


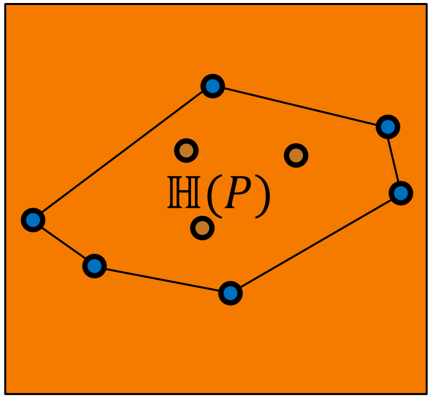
Half-Spaces and Convex Hulls (2D)

Notation:

Given a set of points, $P = \{p_1, \ldots, p_n\}$:

- Denote the convex hull of the points as: $\mathbb{H}(P)$



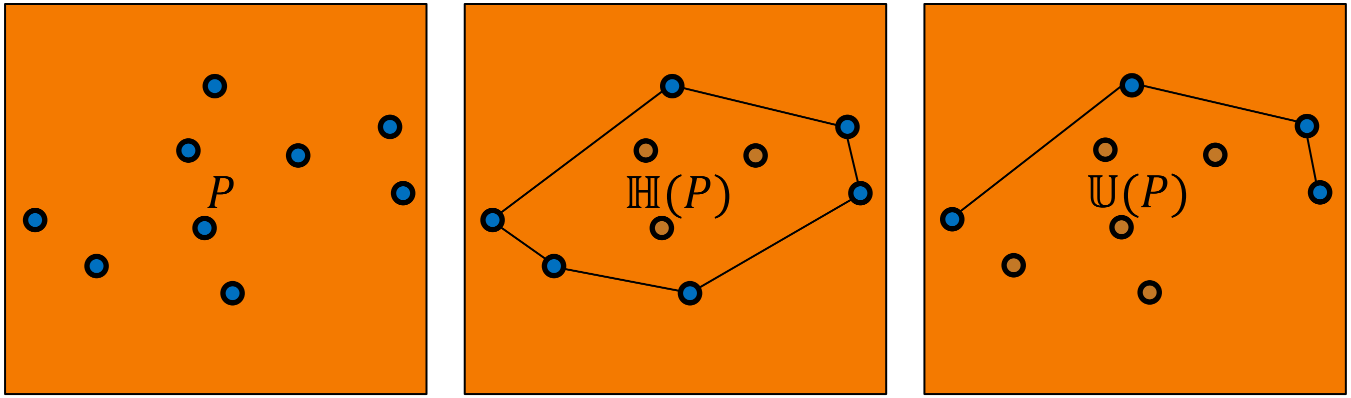


Half-Spaces and Convex Hulls (2D)

Notation:

Given a set of points, $P = \{p_1, \dots, p_n\}$:

- Denote the convex hull of the points as: $\mathbb{H}(P)$
- Denote the upper hull of the points as: $\mathbb{U}(P)$



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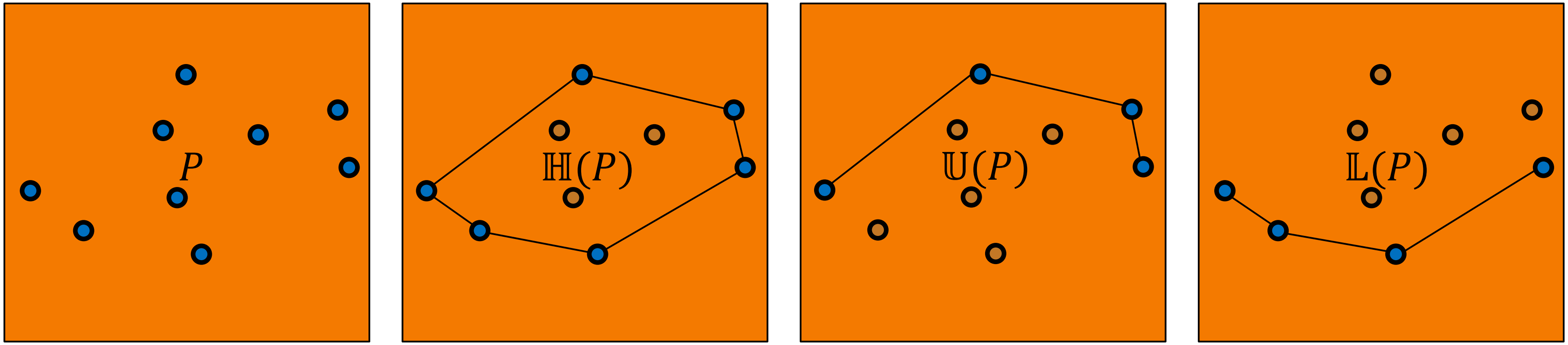
Given a set of points, $P = \{p_1, \ldots, p_n\}$:

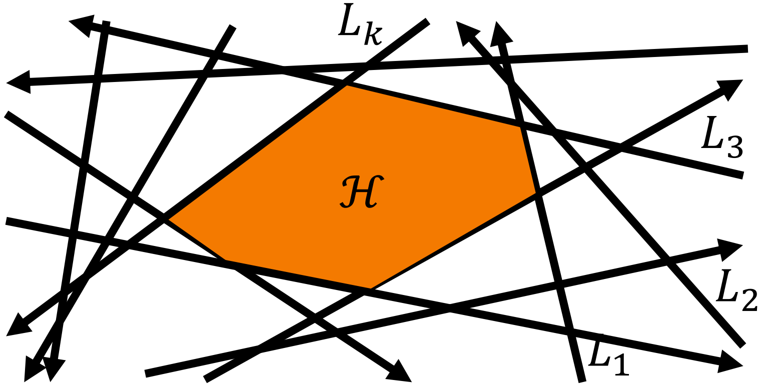
- Denote the convex hull of the points as: $\mathbb{H}(P)$
- Denote the upper hull of the points as: $\mathbb{U}(P)$
- Denote the lower hull of the points as: $\mathbb{L}(P)$



Half-Spaces and Convex Hulls (2D)

Goal:

Given a set of half-spaces, represented by directed lines $\{L_1, \ldots, L_n\}$ compute the convex hull, $\mathcal{H}$, corresponding to the boundary of their intersection.

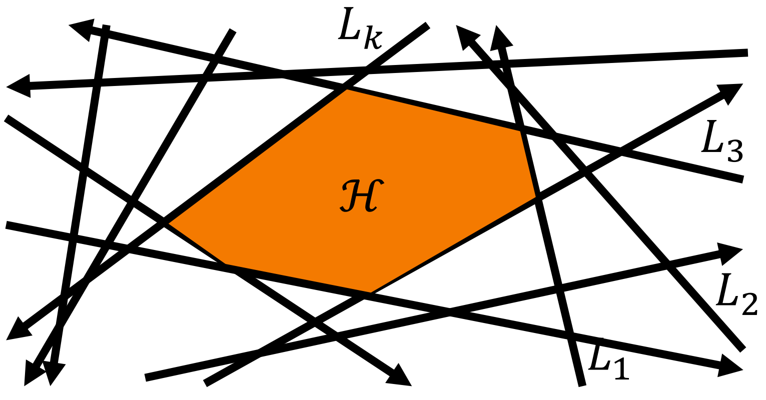


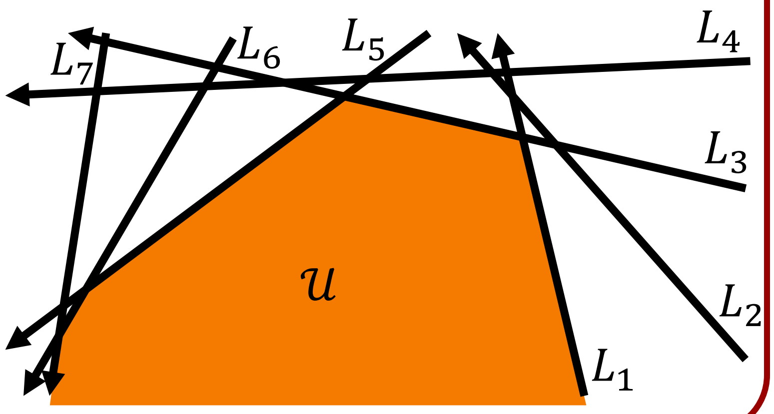
Half-Spaces and Convex Hulls (2D)

Approach:

Consider the upper/lower hulls, $\mathcal{U}/\mathcal{L}$, independently.

- For the upper (resp. lower) part, assume all line segments have $(0,-\infty)$ (resp. $(0,\infty)$) to their left.





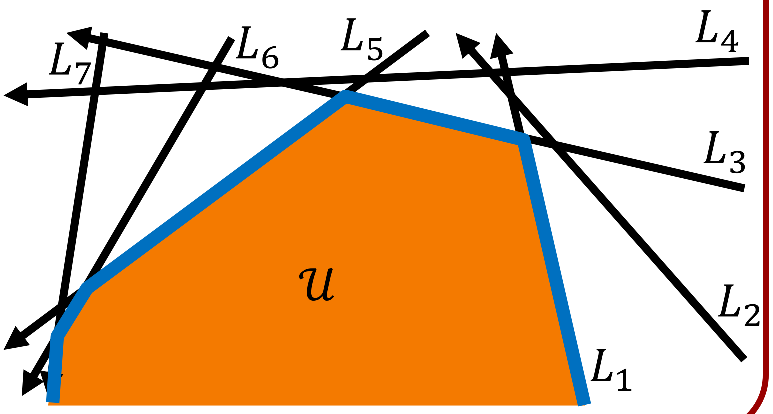
Half-Spaces and Convex Hulls (2D)

Approach:

Consider the upper/lower hulls, $\mathcal{U}/\mathcal{L}$, independently.

An edge $e \subset \mathcal{U}$ is the set of points on a line that are below all the other lines:

$$e \subset L_i \quad \text{and} \quad \text{Below}(v, L_k)\ \forall k \neq i, v \in e$$



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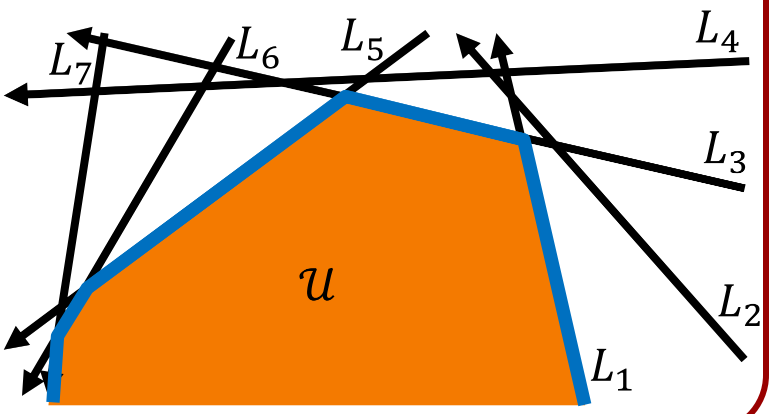
Dually:

$$L_i^* \in v^* \quad \text{and} \quad \text{Below}(L_k^*, v^*)\ \forall k \neq i\ , v \in e$$



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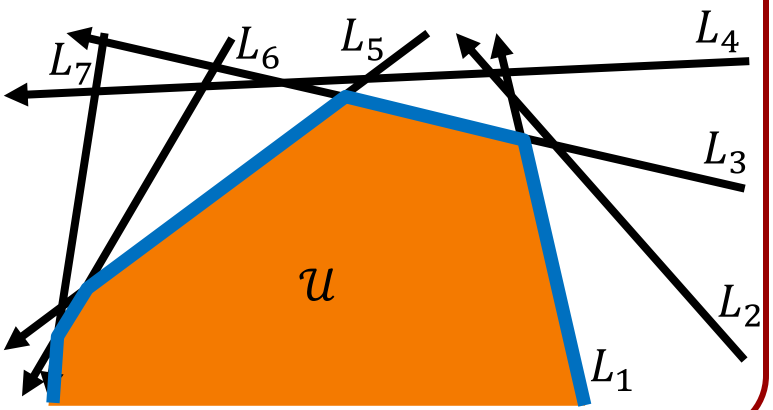

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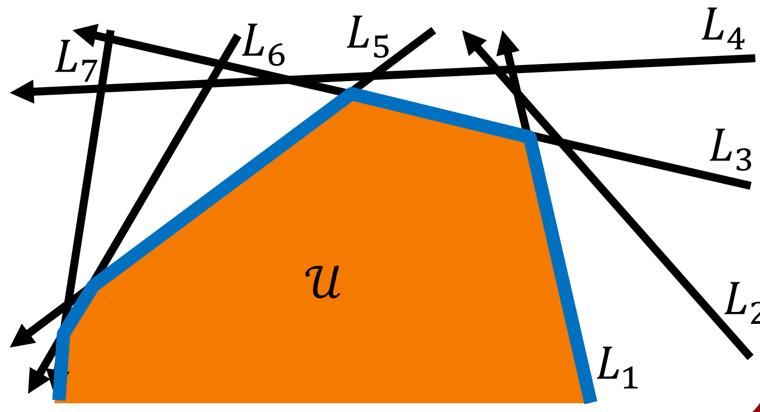
$\Leftrightarrow$ Lines v^*, with $v \in e$, pass through L_i^* and have all other $\{L_1^*, \ldots, L_n^*\}$ below.

$\Leftrightarrow$ L_i^* is a vertex of:

$$\mathcal{U}^* = \mathbb{U}(L_1^*, \ldots, L_n^*)$$



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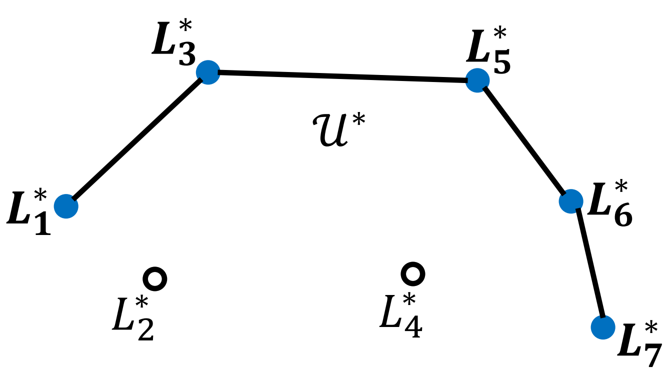
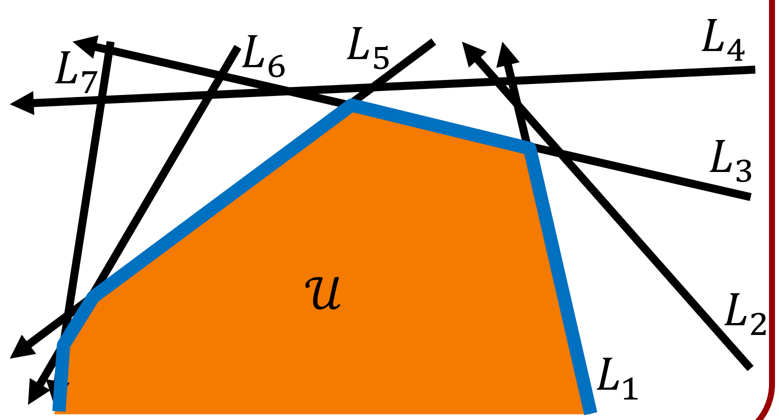




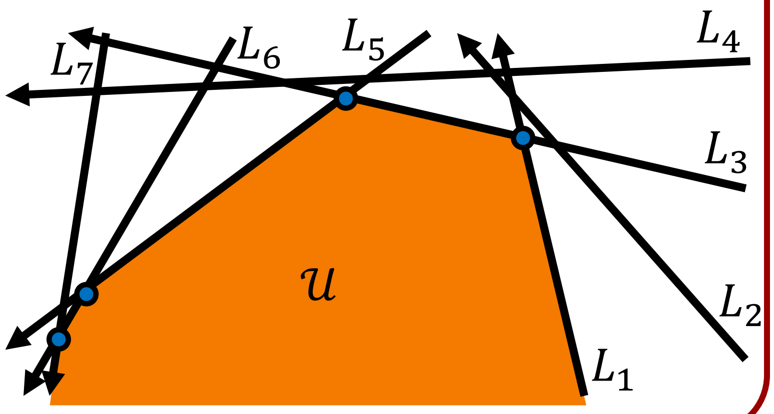
Half-Spaces and Convex Hulls (2D)

<u>Approach</u>:

Consider the upper/lower hulls, $\mathcal{U}/\mathcal{L}$, independently.

A vertex $v \in \mathcal{U}$ lies on the intersection of two lines and is below all the other lines:

$$v \in L_i \cap L_j \quad \text{and} \quad \text{Below}(v, L_k)\ \forall k \neq i, j$$



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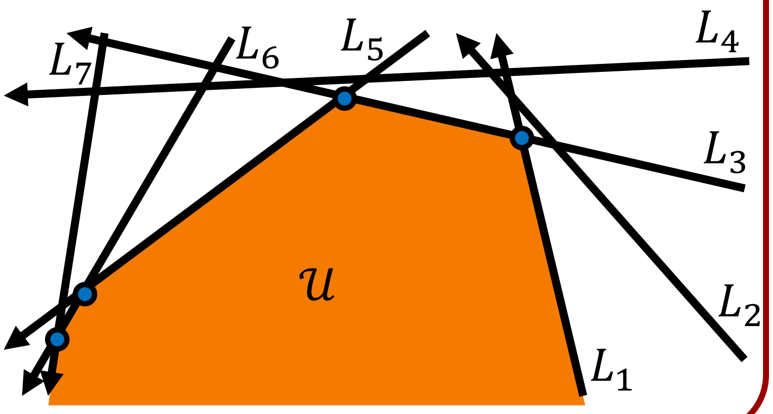

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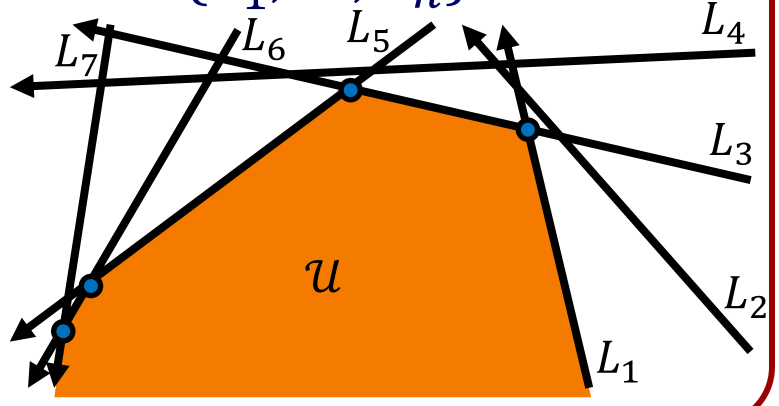

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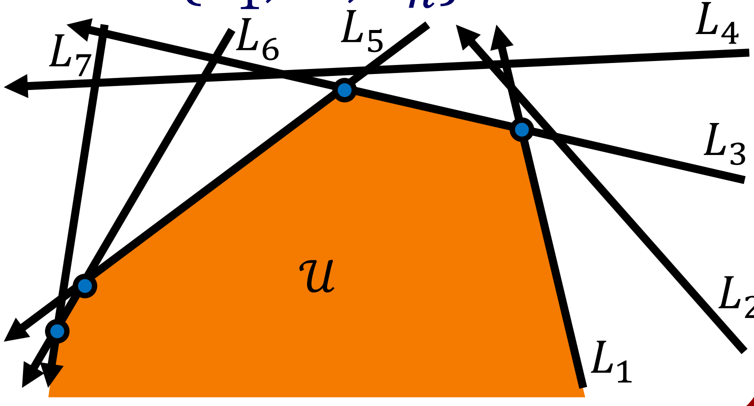
⇔ The line segment $\overline{L_i^* L_j^*}$ has all other $\{L_1^*, \ldots, L_n^*\}$ below.

⇔ $\overline{L_i^* L_j^*}$ is an edge of

$$\mathcal{U}^* = \mathbb{U}(L_1^*, \ldots, L_n^*)$$



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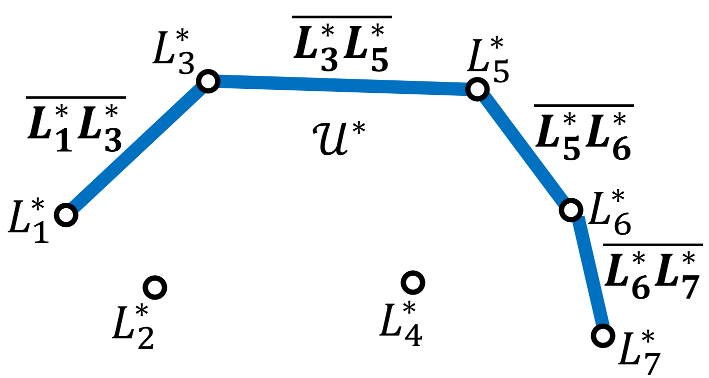
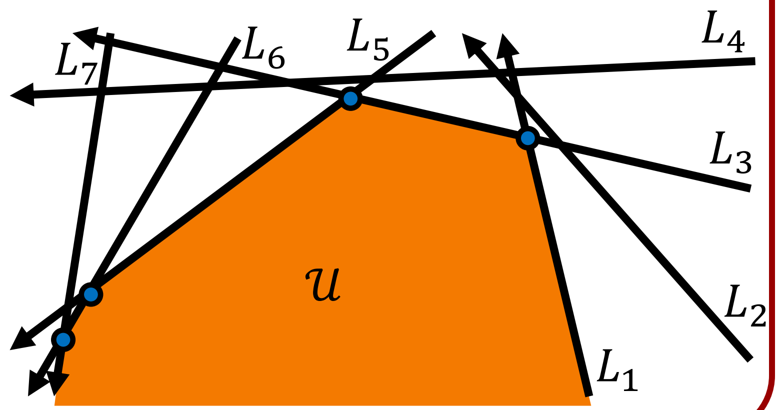




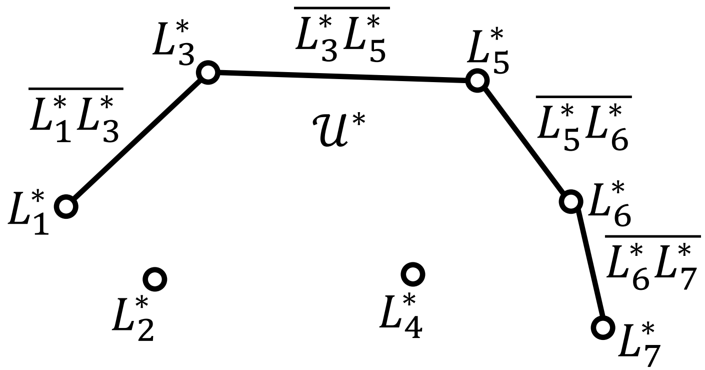
Half-Spaces and Convex Hulls (2D)

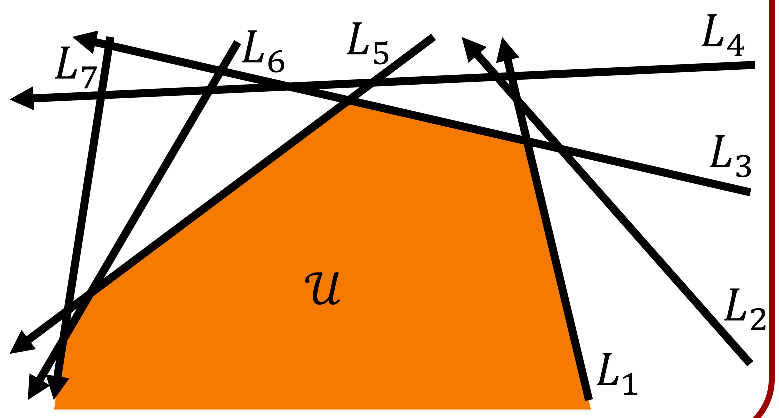
<u>Implementation</u>:

UpperHull($\{L_1, \dots, L_n\}$)

- $\{v_1, \dots, v_m\} \leftarrow$ UpperHull($\{L_1^*, \dots, L_n^*\}$)
- return $\{v_m^*, \dots, v_1^*\}$

This gives the lines in the (CCW) order in which they appear on the intersection of half-spaces.





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LowerHull($\{L_1, \ldots, L_n\}$)
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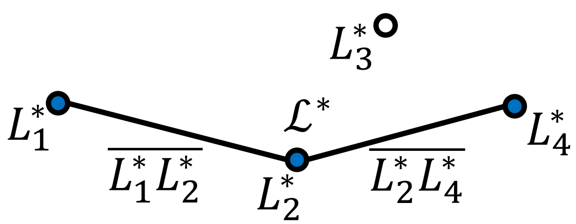
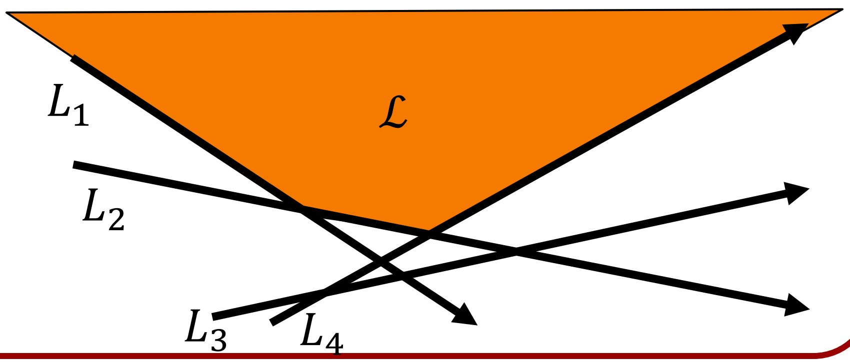
- return $\{v_1^*, \ldots, v_m^*\}$





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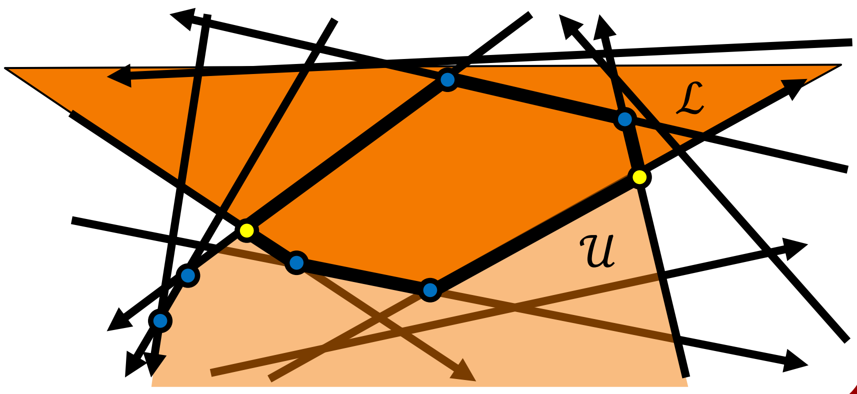
Taking the intersection (in linear time), we get the convex hull.



Half-Spaces and Convex Hulls (2D)

Implementation:

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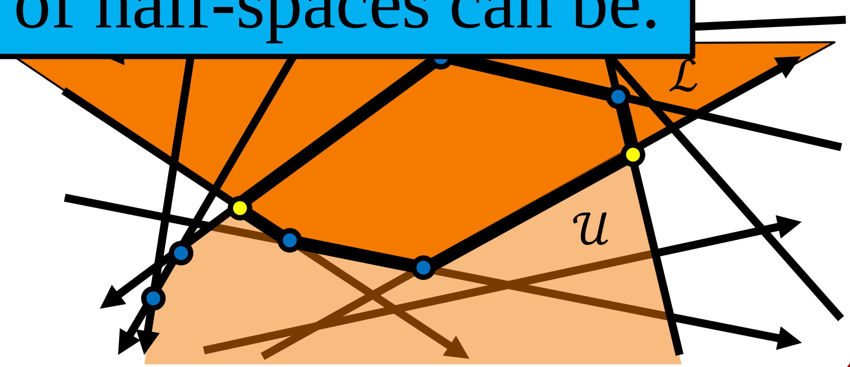
We have to separately compute the upper and lower hulls because the dual map is undefined (discontinuous) as lines approach vertical.

Is there an alternate definition of duality that would allow us to compute the hulls simultaneously?

No! The convex hull of the dual cannot be empty, but the intersection of half-spaces can be.

Taking the intersection (in linear time), we get the convex hull.



Half-Spaces and Convex Hulls

Writing $p \in \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ as:

$$p = (\alpha_1, \ldots, \alpha_{n-1}, \beta) = (\vec{\alpha}, \beta)$$

we can define duality in n-dimensional space.

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we can define duality in n-dimensional space.

Given a point $p = (\vec{\alpha}, \beta)$, define the *dual hyper-plane* to be the (non-vertical) plane:

$$p^* = \{(\vec{x}, y) \in \mathbb{R}^{n-1} \times \mathbb{R} \,|\, y = 2\langle \vec{x}, \vec{\alpha} \rangle - \beta\}$$

Half-Spaces and Convex Hulls

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$$p^* = \{(\vec{x}, y) \in \mathbb{R}^{n-1} \times \mathbb{R} \,\big|\, y = 2\langle \vec{x}, \vec{\alpha} \rangle - \beta\}$$

Given a plane $H = \{(\vec{x}, y) | y = \langle \overrightarrow{m}, \vec{x} \rangle + b\}$, define the *dual point* to be the point with coordinates:

$$H^* = \left(\frac{\overrightarrow{m}}{2}, -b\right)$$

Half-Spaces and Convex Hulls

Writing $p \in \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ as:

$$p \equiv (\alpha_1, \ldots, \alpha_{n-1}, \beta) \equiv (\vec{\alpha}, \beta)$$

The same properties hold in n-dimensions:

- $(p^*)^* = p$ and $(H^*)^* = H$.
- $p \in H$ iff. $H^* \in p^*$.
- $p \in H_1 \cap H_2$ iff. $H_1^*, H_2^* \in p^*$.
- H is below/above p iff. H^* is above/below p^*.
- $p = (\vec{x}, y)$ is on the parabola $y = ||\vec{x}||^2$ iff. p^* is tangent to the parabola at p.

Given a plane $H = \{(\vec{x}, y) | y = \langle \vec{m}, \vec{x} \rangle + b\}$, define the *dual point* to be the point with coordinates:

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Half-Spaces and Convex Hulls

In 3D:

Let $\{H_1, \ldots, H_m\}$ be oriented hyper-planes in $\mathbb{R}^3$ with $(0,0,-\infty)$ to the left and let $\mathcal{U}$ be the intersection of the associate half-spaces.

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$$\{H_i^*, H_j^*, H_k^*\} \text{ is a triangle of } \mathbb{U}(\{H_1^*, \ldots, H_m^*\})$$

$$\Updownarrow$$

$$H_i \cap H_j \cap H_k \text{ is a vertex of } \mathcal{U}.$$

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$$\{H_i^*, H_j^*\} \text{ is an edge of } \mathbb{U}(\{H_1^*, \ldots, H_m^*\})$$

$$\Updownarrow$$

$$H_i \cap H_j \text{ contains an edge of } \mathcal{U}.$$

Half-Spaces and Convex Hulls

<u>In 3D</u>:

Let $\{H_1, \dots, H_m\}$ be oriented hyper-planes in $\mathbb{R}^3$ with $(0,0,-\infty)$ to the left and let $\mathcal{U}$ be the intersection of the associate half-spaces.

$$H_i^* \text{ is a vertex of } \mathbb{U}(\{H_1^*, \dots, H_m^*\})$$

$$\Updownarrow$$

$$H_i \text{ contains a face of } \mathcal{U}.$$

Half-Spaces and Convex Hulls

<u>More Generally</u>:

Let $\{H_1, \ldots, H_m\}$ be oriented hyper-planes in $\mathbb{R}^n$ with $(0, \ldots, 0, -\infty)$ to the left and let $\mathcal{U}$ be the intersection of the associate half-spaces.

Half-Spaces and Convex Hulls

<u>More Generally</u>:

Let $\{H_1, \ldots, H_m\}$ be oriented hyper-planes in $\mathbb{R}^n$ with $(0, \ldots, 0, -\infty)$ to the left and let $\mathcal{U}$ be the intersection of the associate half-spaces.

$$\left\{H_{i_1}^*, \ldots, H_{i_k}^*\right\} \text{ is a } k\text{-simplex of } \mathbb{U}(\{H_1^*, \ldots, H_m^*\})$$

$$\Updownarrow$$

$$\cap_{j=1}^{k} H_{i_j} \text{ contains one } (n-d)\text{-dimensional face of } \mathcal{U}.$$

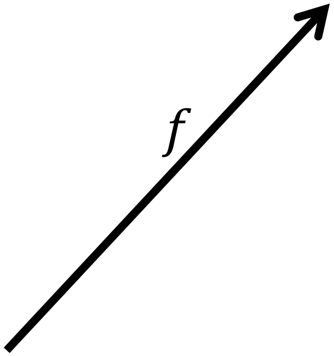
Outline

- Review
- Half-Spaces and Convex Hulls (2D)
- Convex Polygon Intersection

Convex Polygon Intersection

Notation:

Given a (directed) edge $f = (a, b)$ we refer to b as the head of f.

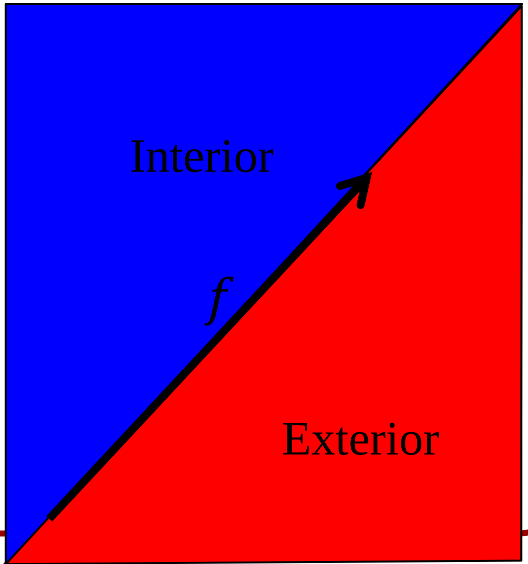


Convex Polygon Intersection

Notation:

Given a (directed) edge $f=(a,b)$ we refer to b as the head of f.

Given edges e and f we say that e is interior / exterior to f if the head of e is left / right of f.



Convex Polygon Intersection

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Given a (directed) edge $f = (a, b)$ we refer to b as the head of f.

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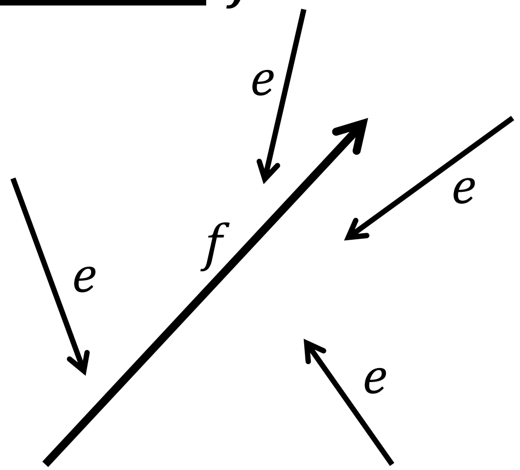
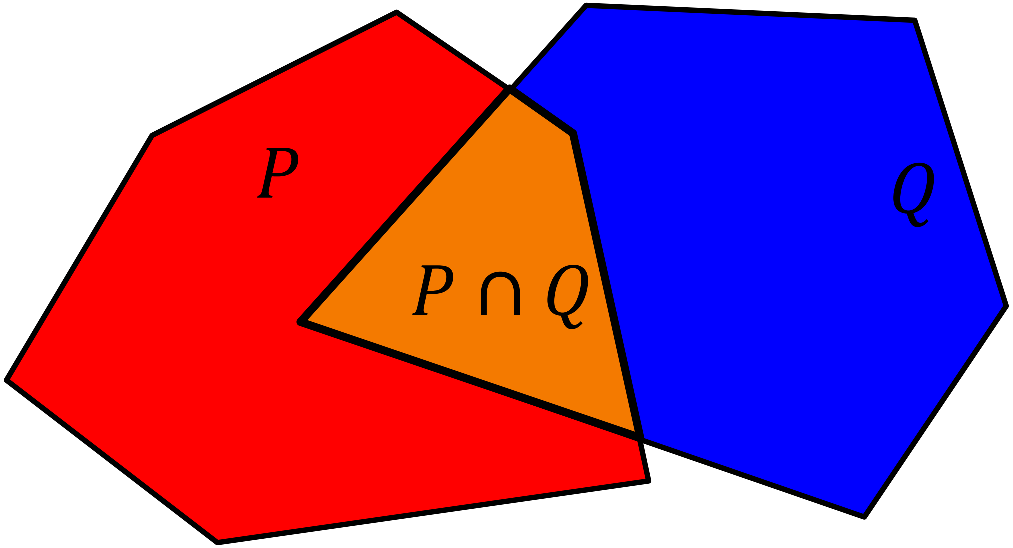
Given edges e and f we say that e aims at f if moving forward along e we intersect the line containing f.



Convex Polygon Intersection

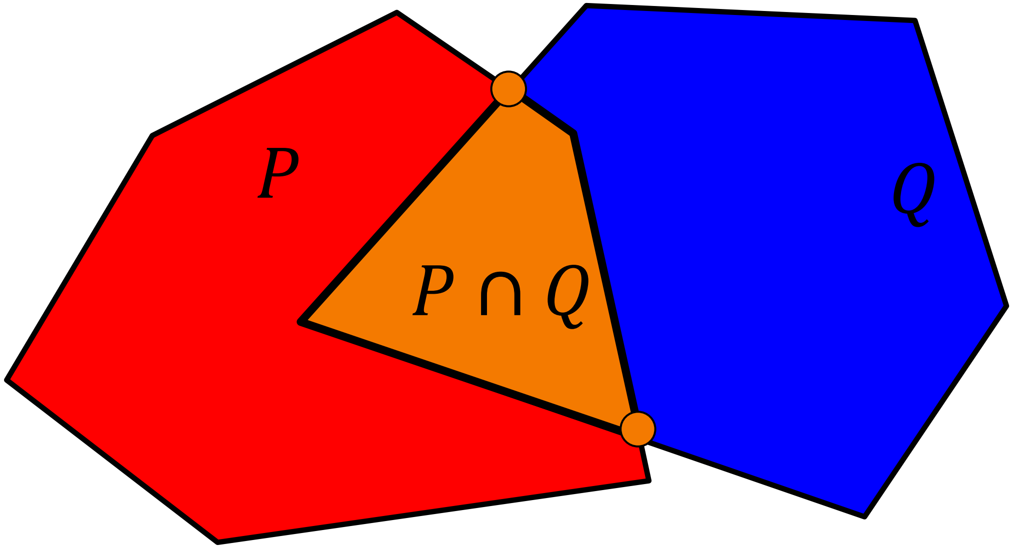
Given convex polygons P and Q, find the (convex) intersection $P \cap Q$.



Convex Polygon Intersection

Given convex polygons P and Q, find the (convex) intersection $P \cap Q$.

Note that if ∂P and ∂Q intersect (non-degenerately) there will be either two or four points of intersection.

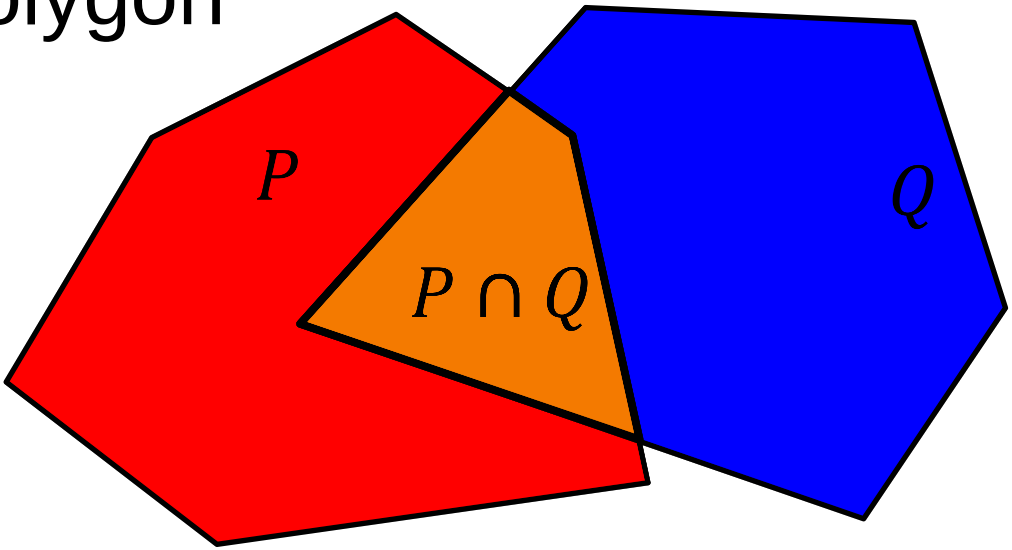


Convex Polygon Intersection

Given convex polygons P and Q, find the (convex) intersection $P \cap Q$.

Approach:

Find the intersections between ∂P and ∂Q and track which polygon is interior between successive crossings.

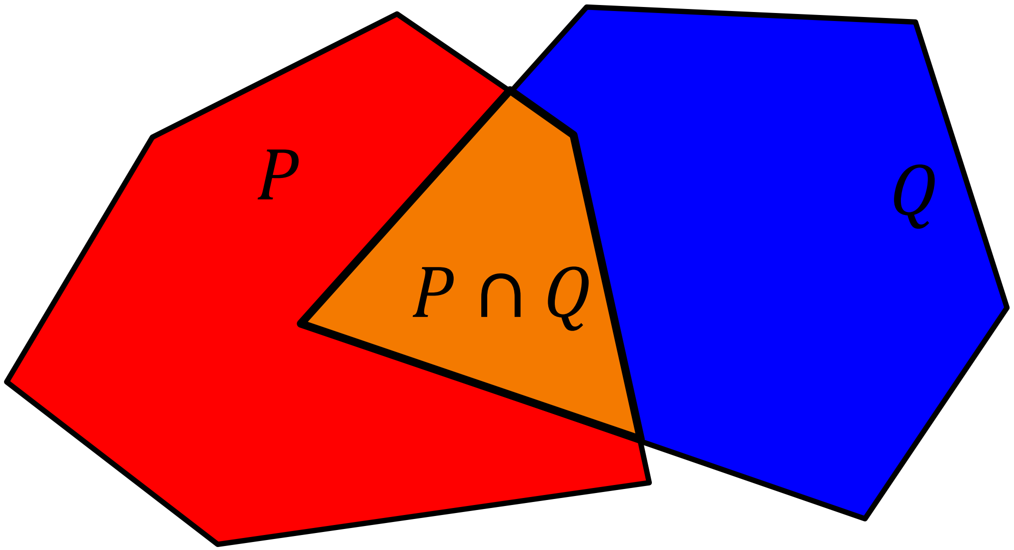


Convex Polygon Intersection

"Simple" Algorithm:

Start with some edge $e \in \partial P$ and $f \in \partial Q$.
Successively try:

1. Advance on e (resp. f) while it is exterior
2. Advance on e (resp. f) if it aims at f (resp. e)
3. Can't happen

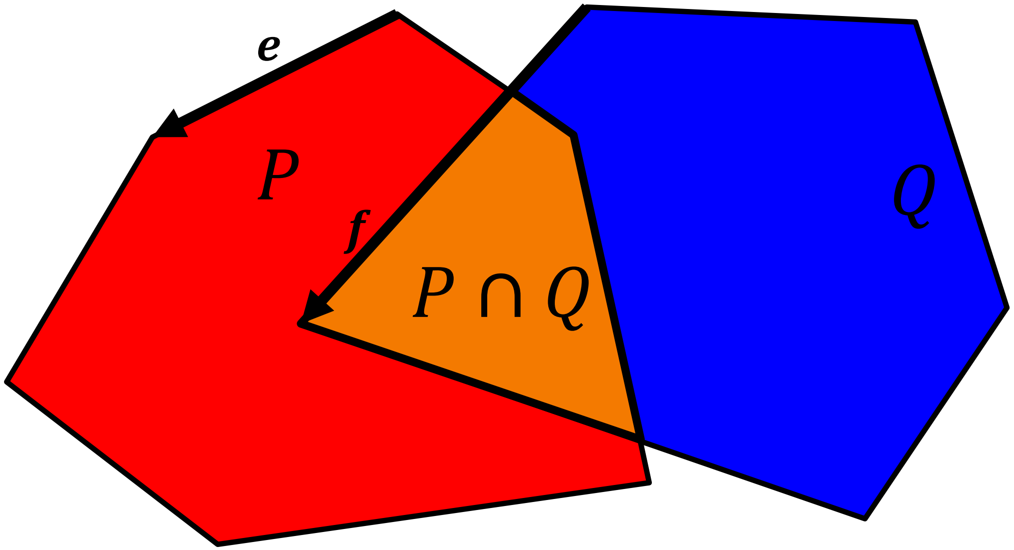


Convex Polygon Intersection

1. Advance e (resp. f) while it is exterior

If e is exterior to f:

⇒ The head of e is to the right of f



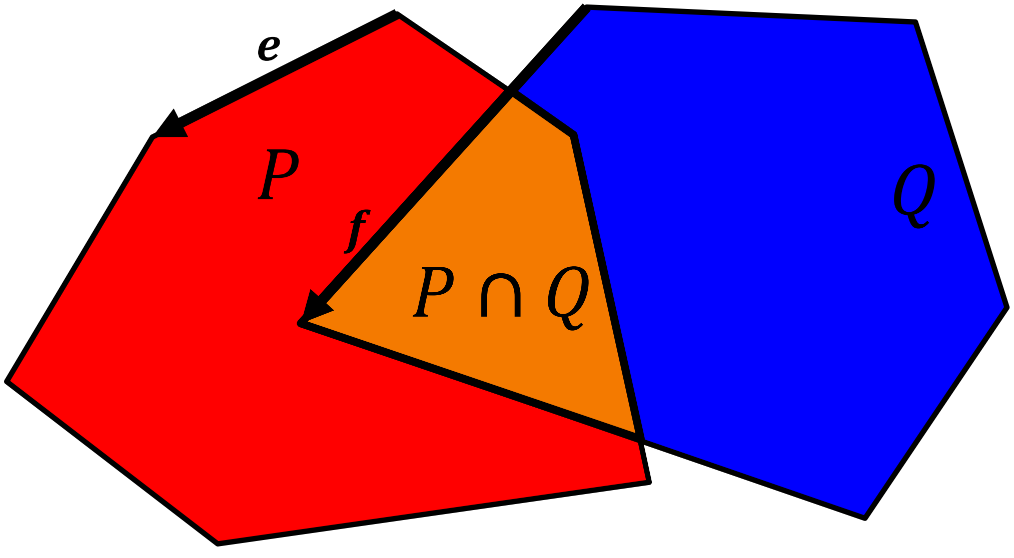
Convex Polygon Intersection

1. Advance e (resp. f) while it is exterior

If e is exterior to f:

⇒ The head of e is to the right of f

⇒ The head of e is outside of Q

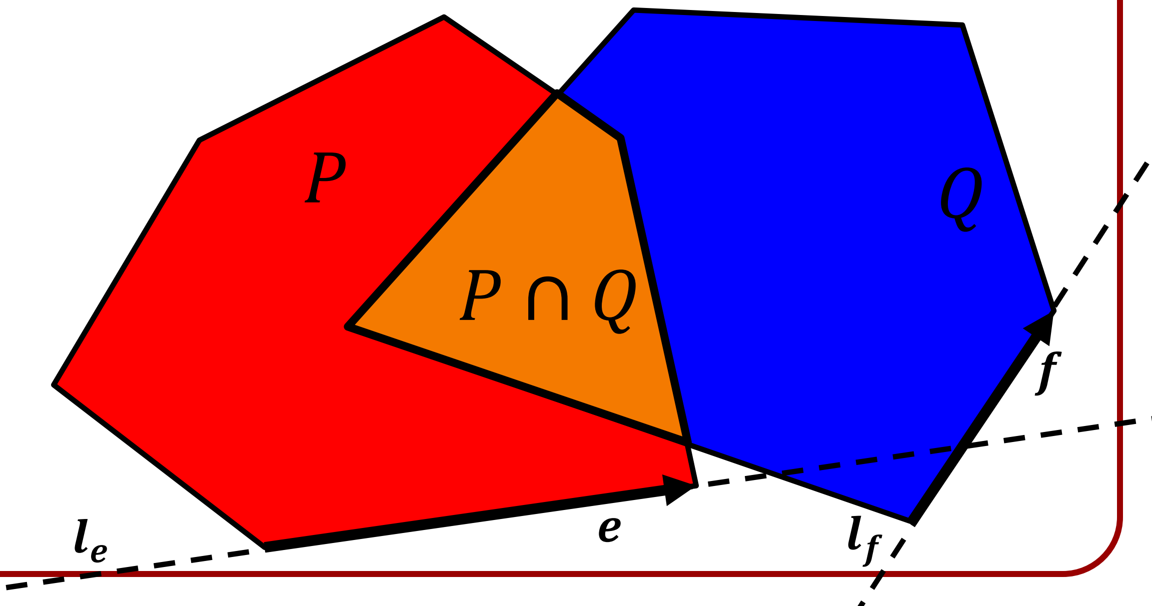


Convex Polygon Intersection

2. Advance on e (resp. f) if it aims at f (resp. e)

If e aims at f:

- Approximate P by the line l_e through e and approximate the Q by the line l_f through f.



Convex Polygon Intersection

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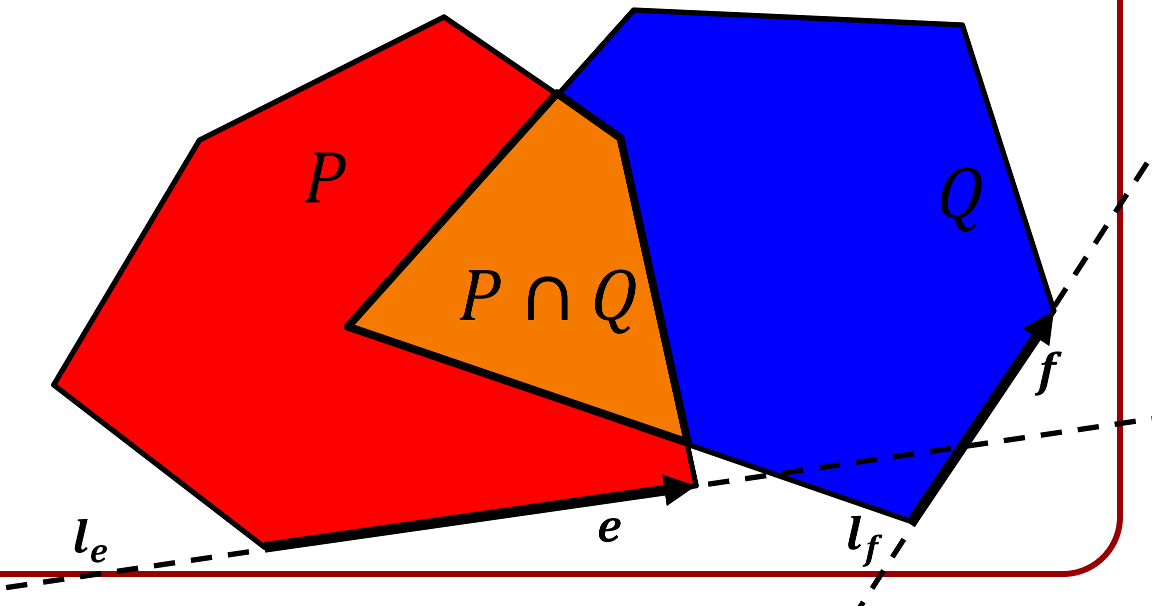
<u>If e aims at f:</u>

- Approximate P by the line l_e through e and approximate the Q by the line l_f through f.
- Advancing e is approximated by moving forward on l_e.

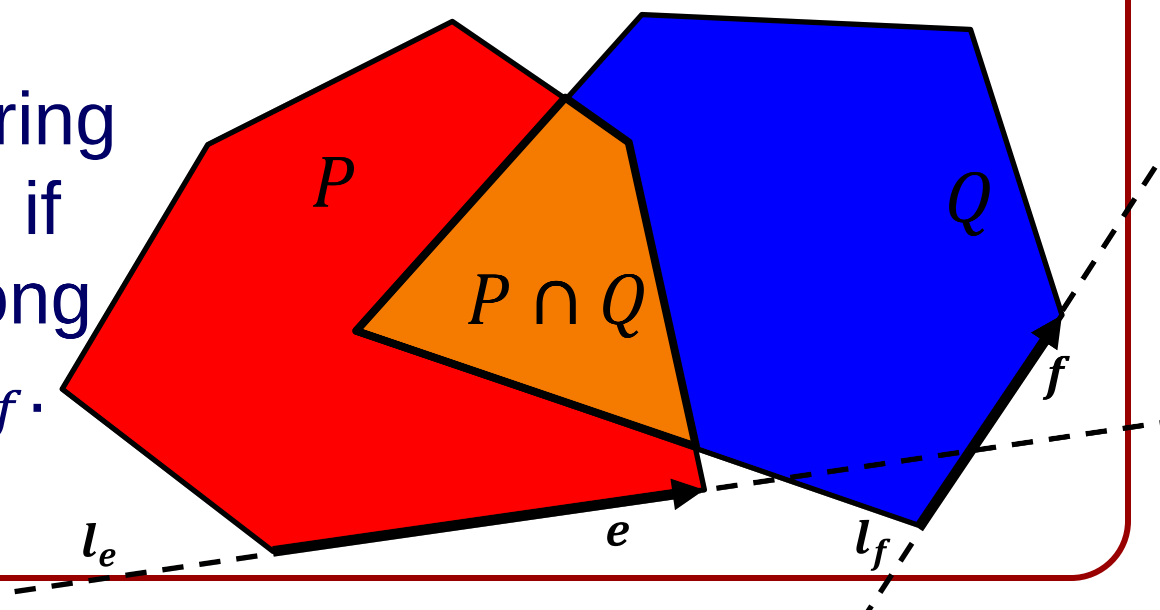


Convex Polygon Intersection

2. Advance on e (resp. f) if it aims at f (resp. e)

If e aims at f:

- Approximate P by the line l_e through e and approximate the Q by the line l_f through f.
- Advancing e is approximated by moving forward on l_e.
- This should bring us closer to Q if advancing along l_e gets us to l_f.



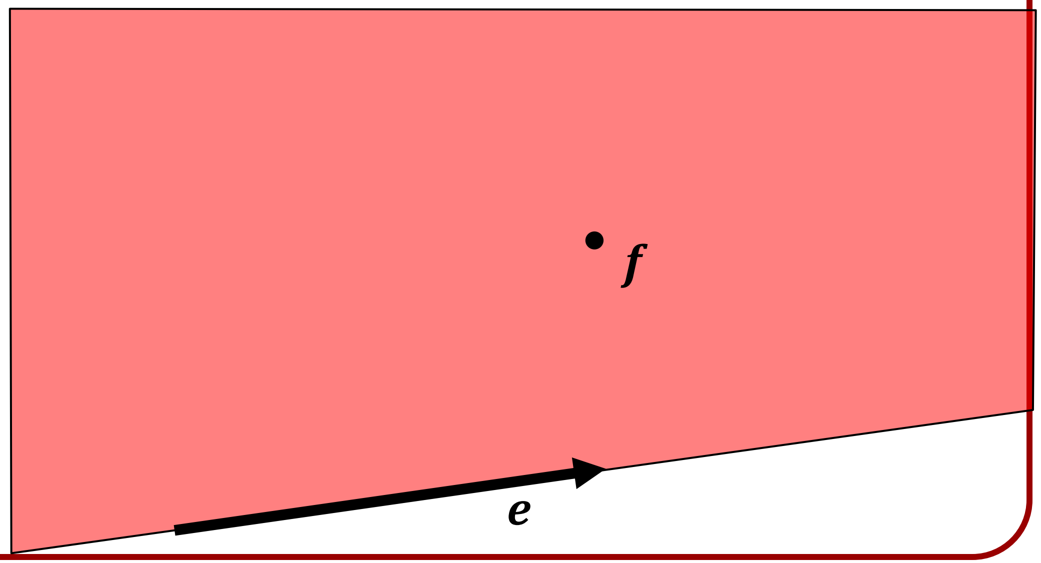
Convex Polygon Intersection

3. Can't happen

e and f interior and neither aims at the other:

Given e:

- The head of f must be to the left of e



Convex Polygon Intersection

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e and f interior and neither aims at the other:

Given e:

- The head of f must be to the left of e
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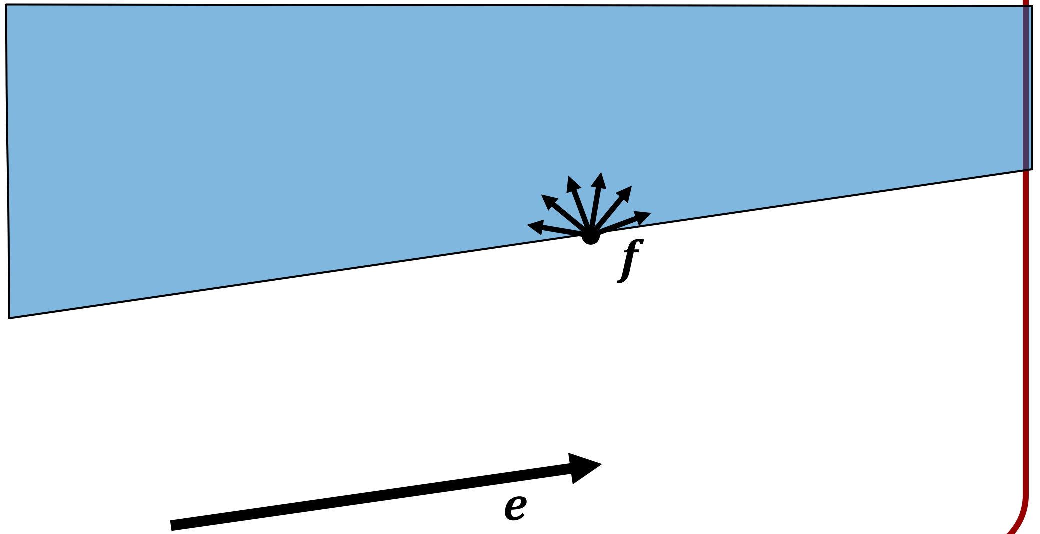

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Given e:

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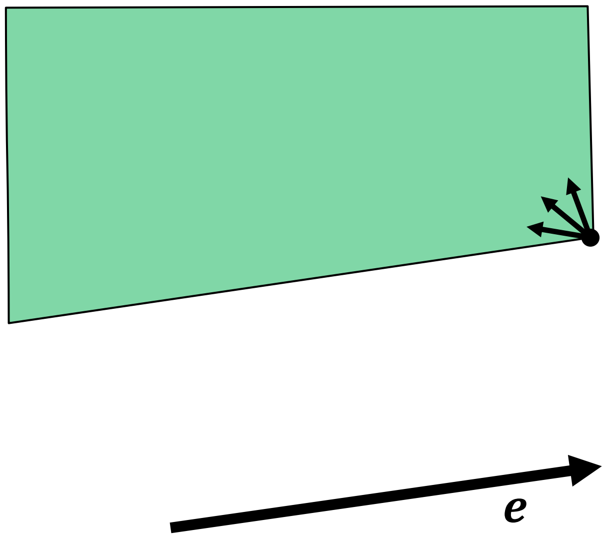

Convex Polygon Intersection

3. Can't happen

<u>e and f interior and neither aims at the other:</u>

Given e:

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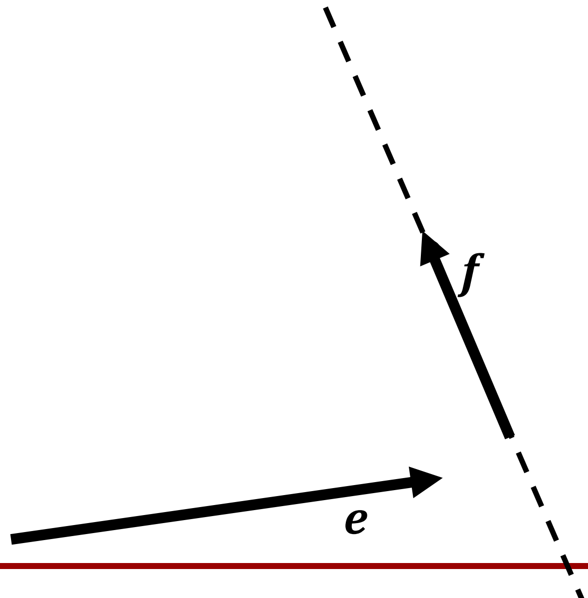
- f aims away from e
- The head of e must be to the left of f

$\Rightarrow$ e aims at f



Convex Polygon Intersection

"Simple" Algorithm:

One can show that:

1. Once we iterate enough, one of the two edges advances to an intersection.

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All in linear time.

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Animation