Motion Planning

O'Rourke, Chapter 8

Outline

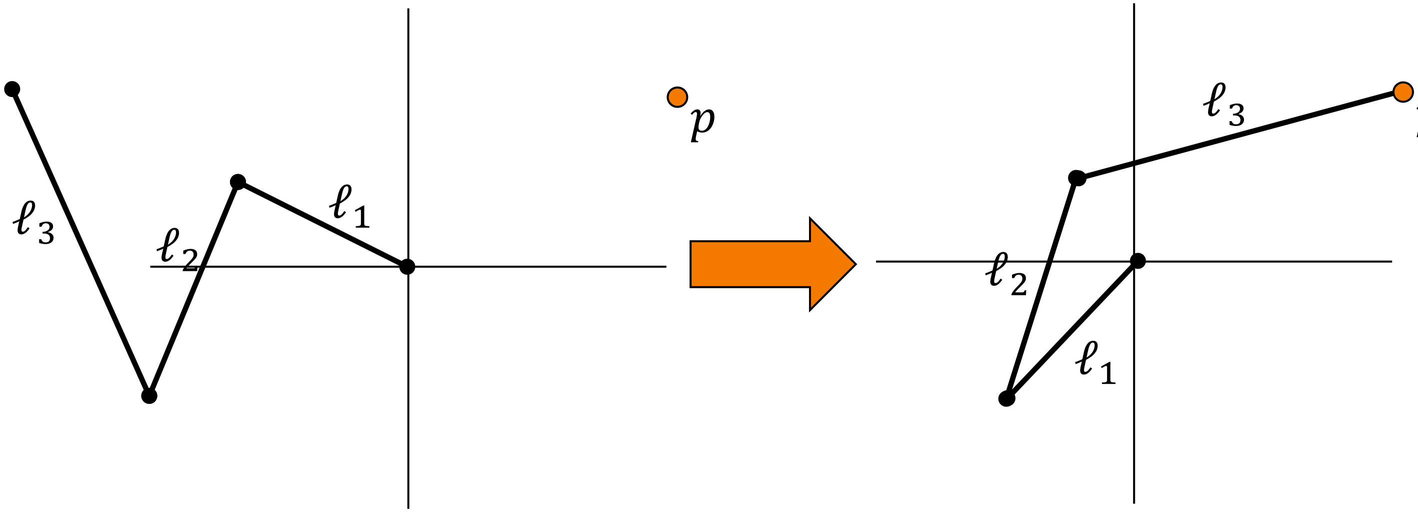
- Robot Arm
- Separability

Robot Arm

Goal:

Given a jointed arm, rooted at the origin, with link lengths $L = \{\ell_1, \ldots, \ell_n\}$ and given $p \in \mathbb{R}^2$:

1. Is there a configuration of joint angles for which the arm reaches p?
2. If there is a configuration, what is it?

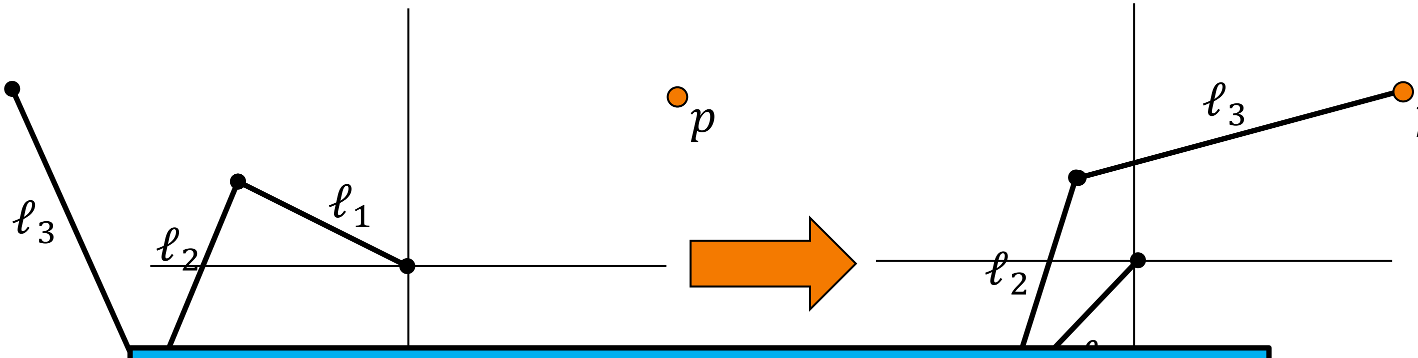


Robot Arm

Goal:

Given a jointed arm, rooted at the origin, with link lengths $L = \{\ell_1, \dots, \ell_n\}$ and given $p \in \mathbb{R}^2$:

1. Is there a configuration of joint angles for which the arm reaches p?
2. If there is a configuration, what is it?



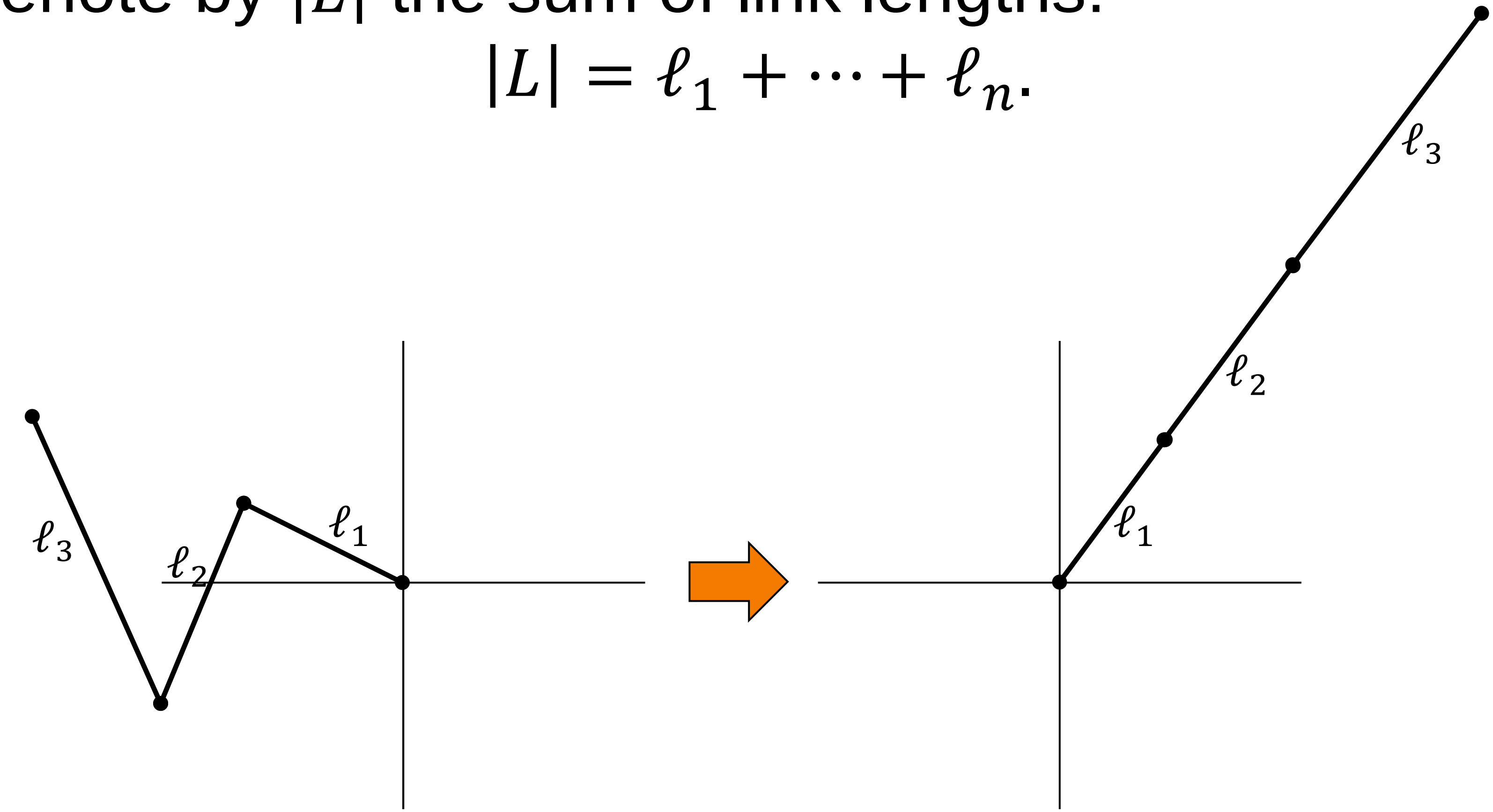
There may be more than one set of angles that has the arm reach p.

Robot Arm

Notation:

Given an arm with link lengths $L = \{\ell_1, \ldots, \ell_n\}$, denote by $|L|$ the sum of link lengths:

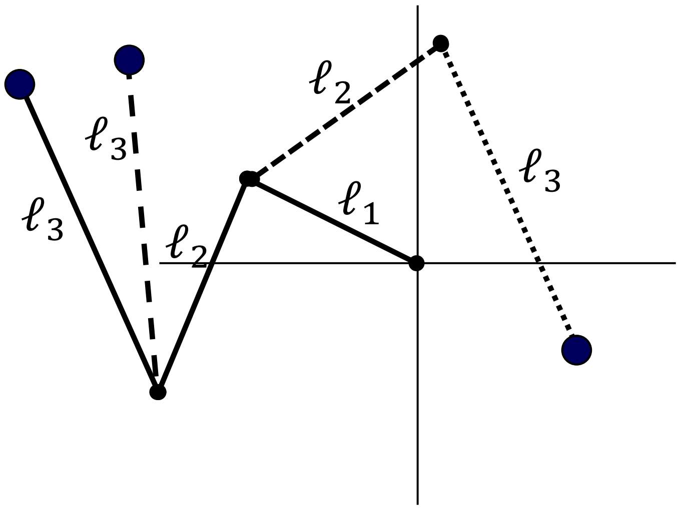
$$|L| = \ell_1 + \cdots + \ell_n.$$



Robot Arm

<u>Definition</u>:

Given an arm with link lengths $L = \{\ell_1, \ldots, \ell_n\}$, the *reach* of the arm is the set of points $p \in \mathbb{R}^2$ that can be reached by some configuration of joint angles.

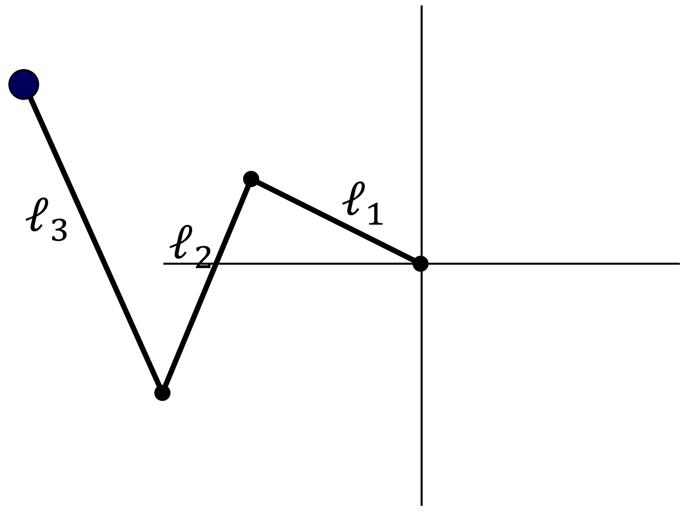


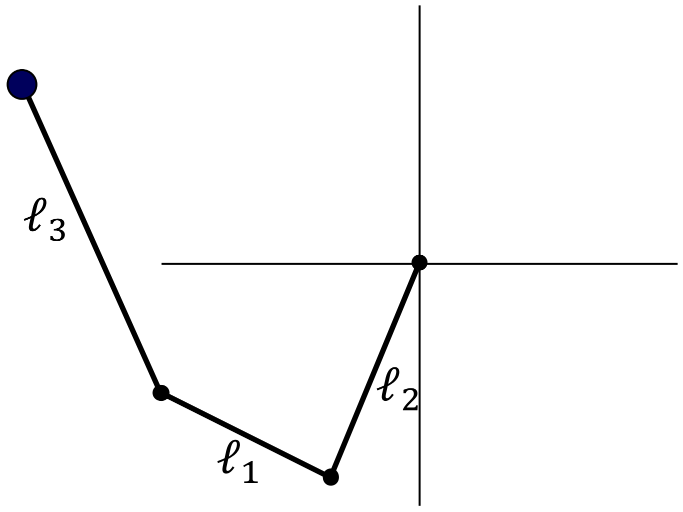
Robot Arm

Definition:

Given an arm with link lengths $L = \{\ell_1, \ldots, \ell_n\}$, the *reach* of the arm is the set of points $p \in \mathbb{R}^2$ that can be reached by some configuration of joint angles.

Because vector addition is commutative, the reach is independent of the order of the links.

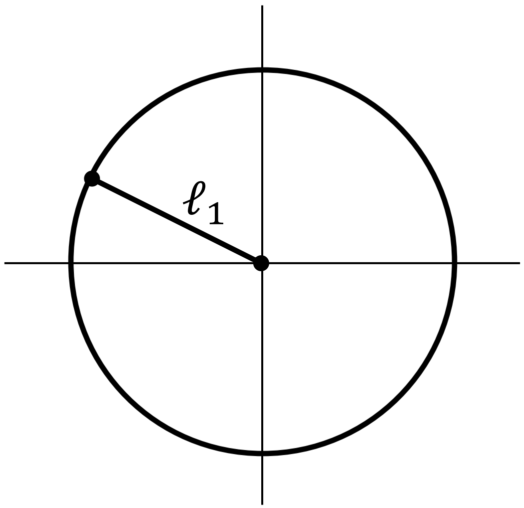




Robot Arm

What is the reach of an n-link arm?

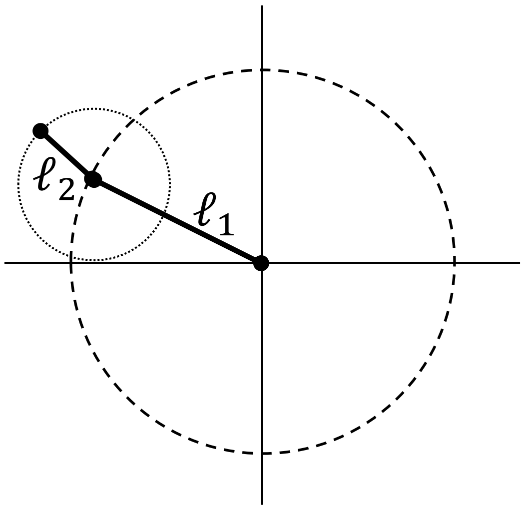
- $n = 1$: A circle with radius ℓ_1.



Robot Arm

What is the reach of an n-link arm?

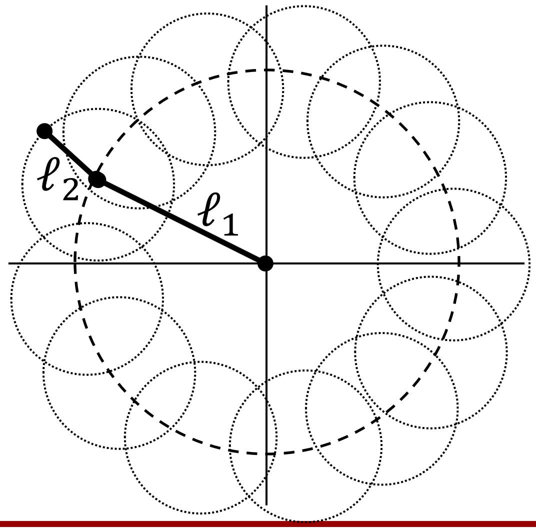
- $n = 1$: A circle with radius ℓ_1.
- $n = 2$: An annulus with outer radius $r_o = \ell_1 + \ell_2$ and inner radius $r_i = |\ell_1 - \ell_2|$.



Robot Arm

What is the reach of an n-link arm?

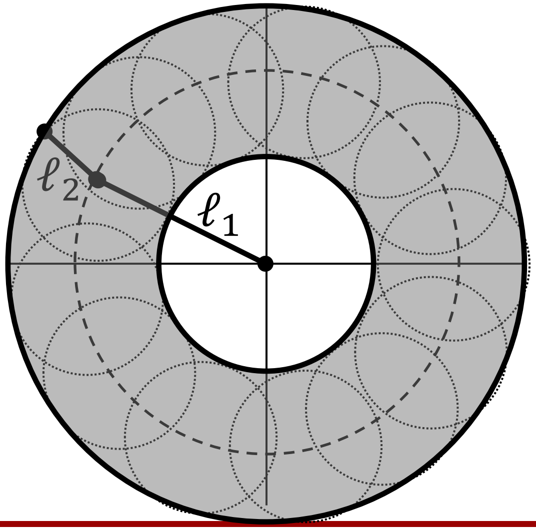
- $n = 1$: A circle with radius ℓ_1.
- $n = 2$: An annulus with outer radius $r_o = \ell_1 + \ell_2$ and inner radius $r_i = |\ell_1 - \ell_2|$.



Robot Arm

What is the reach of an n-link arm?

- $n = 1$: A circle with radius ℓ_1.
- $n = 2$: An annulus with outer radius $r_o = \ell_1 + \ell_2$ and inner radius $r_i = |\ell_1 - \ell_2|$.



Robot Arm

What is the reach of an n-link arm?

- $n = 1$: A circle with radius ℓ_1.
- $n = 2$: An annulus with outer radius $r_o = \ell_1 + \ell_2$ and inner radius $r_i = |\ell_1 - \ell_2|$.
- $n = k$: The Minkowski Sum of the reach of the arm with lengths $\{\ell_1, \ldots, \ell_{k-1}\}$ and the circle with radius ℓ_k.

Robot Arm

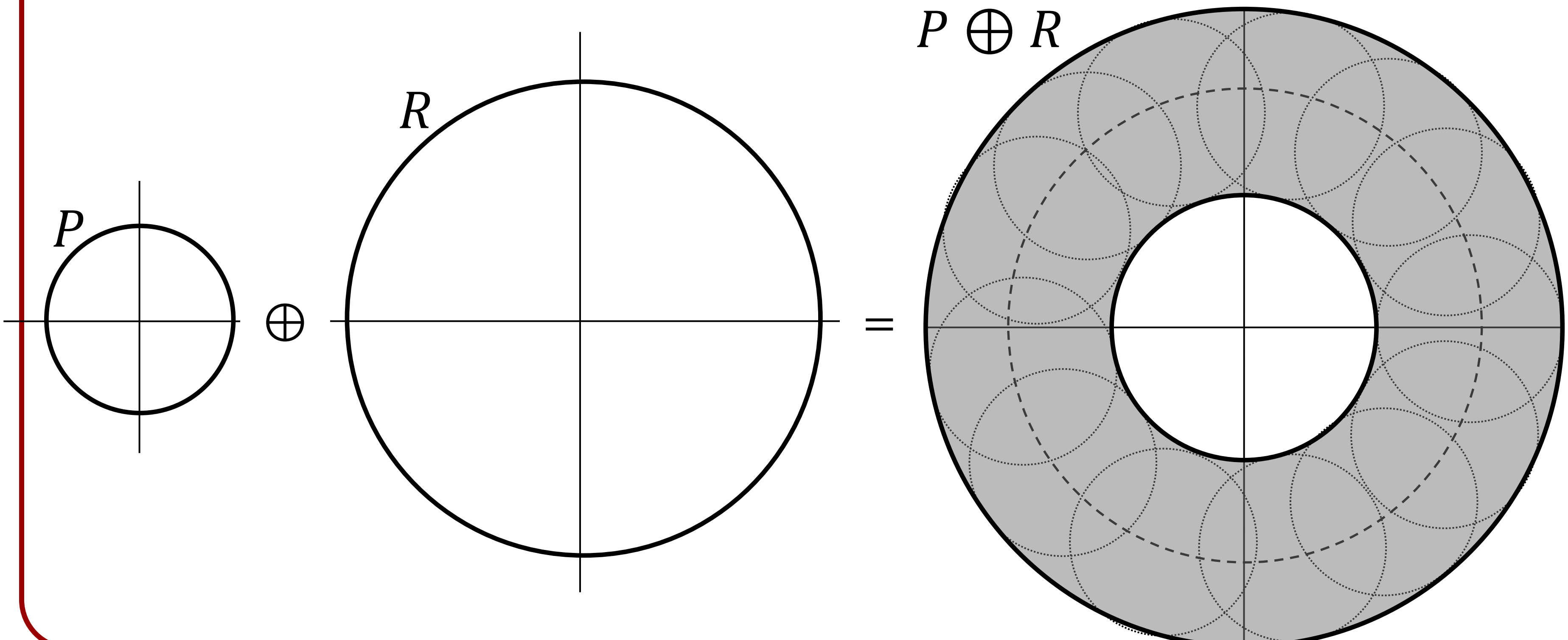
Claim:

The reach of an n-link arm is a (possibly degenerate) annulus.

Robot Arm

Lemma:

If $P, R \subset \mathbb{R}^2$ are path connected, then their Minkowski Sum $P \oplus R$ is path connected.



Robot Arm

Proof (Lemma):

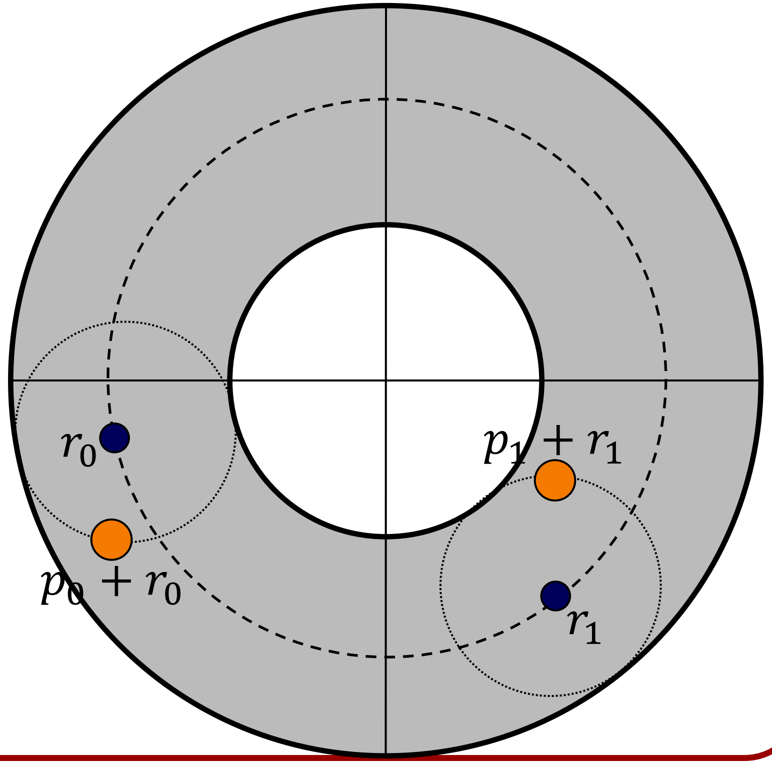
Given $p_0 + r_0, p_1 + r_1 \in P \oplus R$, set:

$$\pi: [0,1] \to P \quad \text{and} \quad \rho: [0,1] \to R$$

to be the paths with:

$\pi(0) = p_0, \pi(1) = p_1$

$\rho(0) = r_0, \ \rho(1) = r_1$



Robot Arm

Proof (Lemma):

Given $p_0 + r_0, p_1 + r_1 \in P \oplus R$, set:

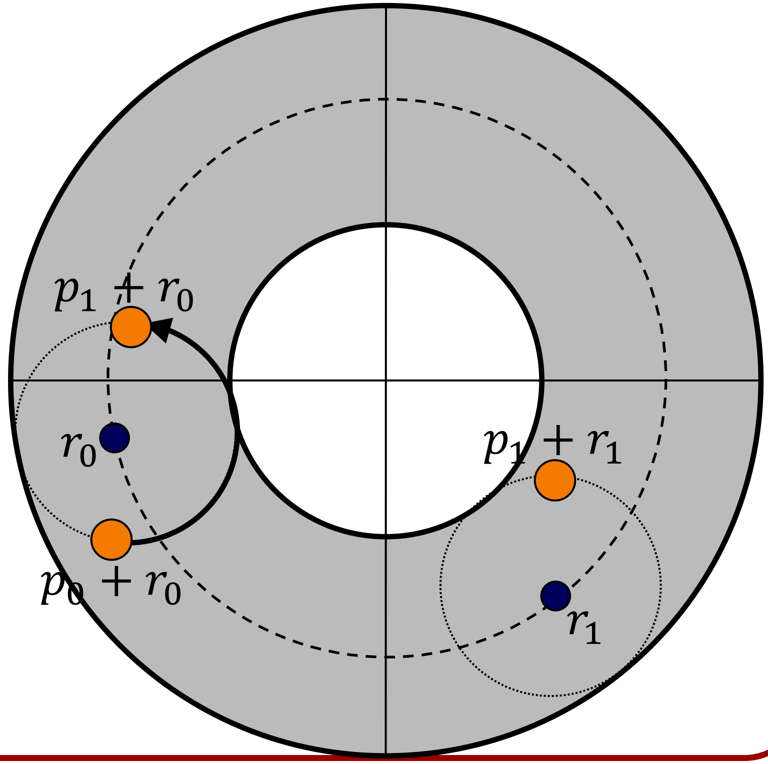
$$\pi: [0,1] \rightarrow P \quad \text{and} \quad \rho: [0,1] \rightarrow R$$

to be the paths with:

$\pi(0) = p_0, \pi(1) = p_1$

$\rho(0) = r_0, \ \rho(1) = r_1$

First use π to travel from $p_0 + r_0$ to $p_1 + r_0$.



Robot Arm

Proof (Lemma):

Given $p_0 + r_0, p_1 + r_1 \in P \oplus R$, set:

$$\pi: [0,1] \to P \quad \text{and} \quad \rho: [0,1] \to R$$

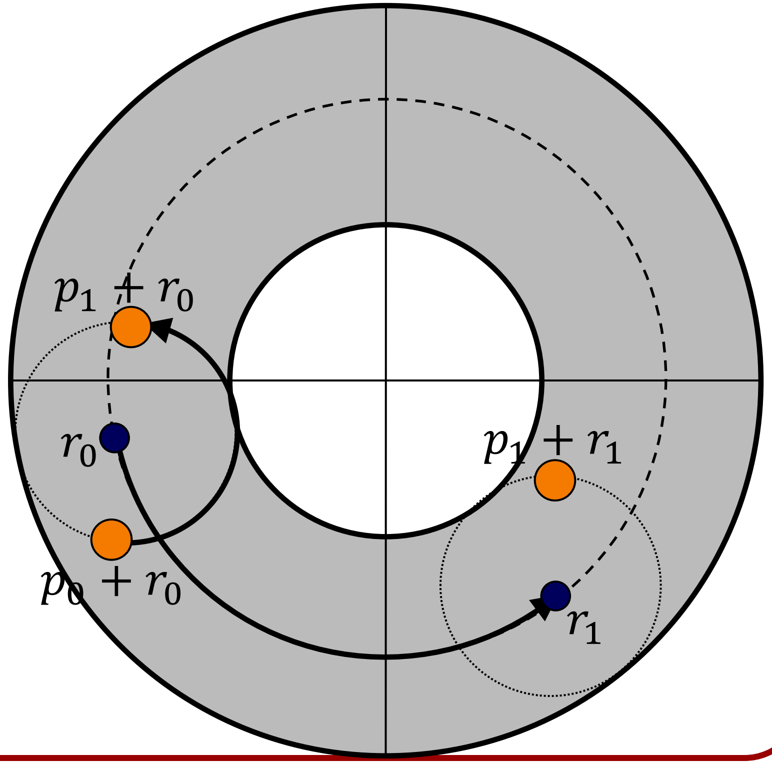
to be the paths with:

$\pi(0) = p_0, \pi(1) = p_1$

$\rho(0) = r_0,\ \rho(1) = r_1$

First use π to travel from $p_0 + r_0$ to $p_1 + r_0$.

Then use ρ to travel from $p_1 + r_0$ to $p_1 + r_1$.



Robot Arm

<u>Proof (Lemma)</u>:

Given $p_0 + r_0, p_1 + r_1 \in P \oplus R$, set:

$$\pi: [0,1] \rightarrow P \quad \text{and} \quad \rho: [0,1] \rightarrow R$$

to be the paths with:

$\pi(0) = p_0, \pi(1) = p_1$

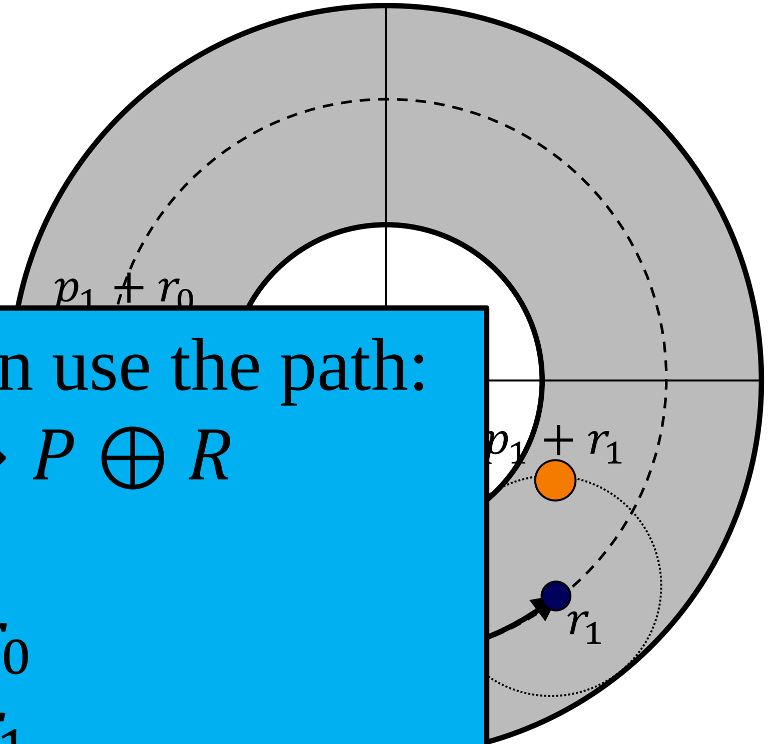
$\rho(0) = r_0, \ \rho(1) = r_1$

First use π to travel from

$p_0 + r_0$

Then u

$p_1 + r_0$



Still more simply, we can use the path:

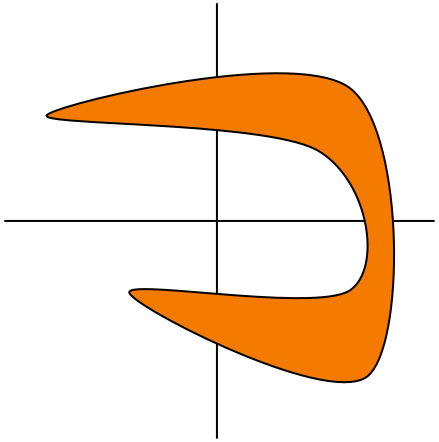
$$(\pi + \rho): [0,1] \rightarrow P \oplus R$$

which satisfies:

- $(\pi + \rho)(0) = p_0 + r_0$
- $(\pi + \rho)(1) = p_1 + r_1$

Robot Arm

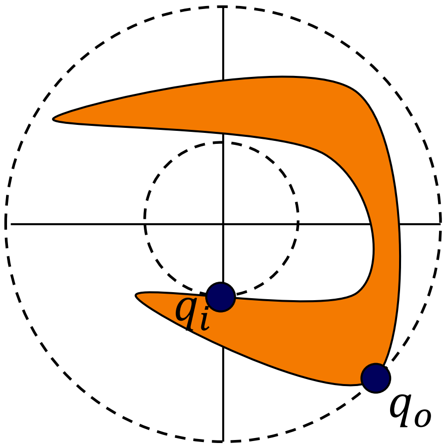
Proof (Claim):



Robot Arm

Proof (Claim):

Let q_i and q_o be the points in the reach which are closest/furthest from the origin.

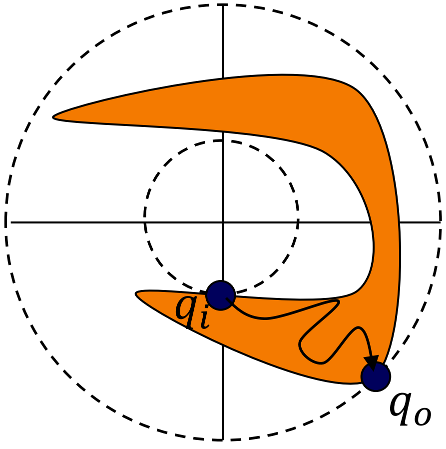


Robot Arm

Proof (Claim):

Let q_i and q_o be the points in the reach which are closest/furthest from the origin.

By induction, there is a path from q_i to q_o.



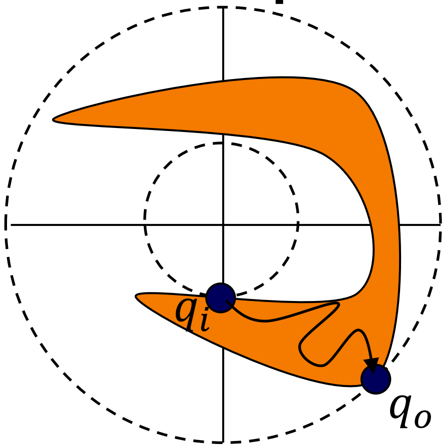
Robot Arm

Proof (Claim):

Let q_i and q_o be the points in the reach which are closest/furthest from the origin.

By induction, there is a path from q_i to q_o.

By the mean-value theorem, for all d with $|q_i| \leq d \leq |q_o|$, there is a point on the path with distance d from the origin.



Robot Arm

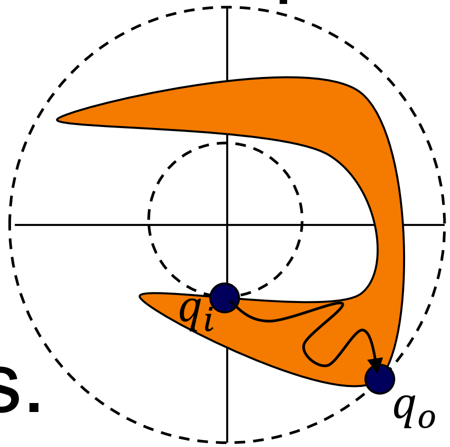
Proof (Claim):

Let q_i and q_o be the points in the reach which are closest/furthest from the origin.

By induction, there is a path from q_i to q_o.

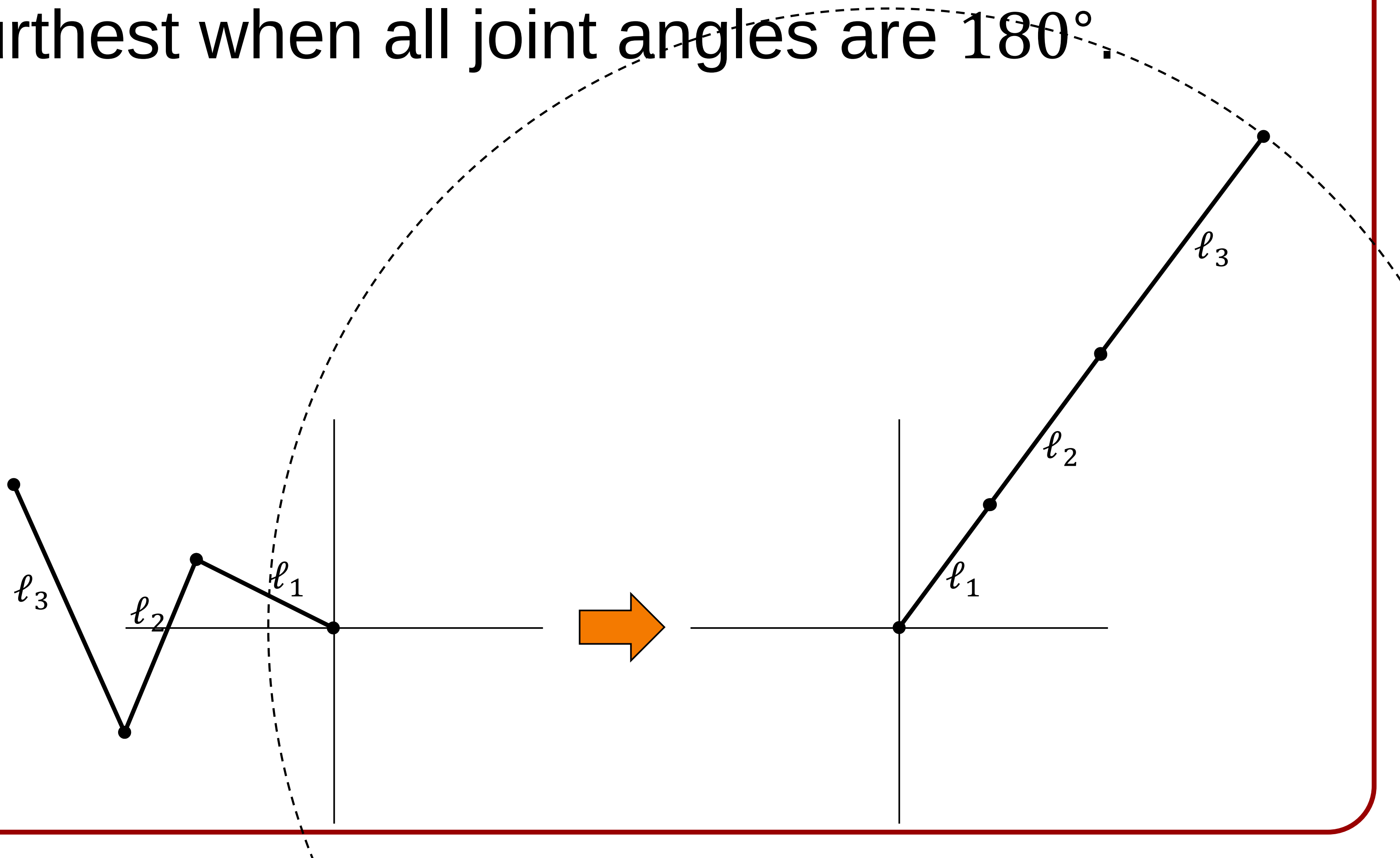
By the mean-value theorem, for all d with $|q_i| \leq d \leq |q_o|$, there is a point on the path with distance d from the origin.

Since the set of reachable points is invariant to rotation about the origin, it's an annulus.



Robot Arm (Outer Radius)

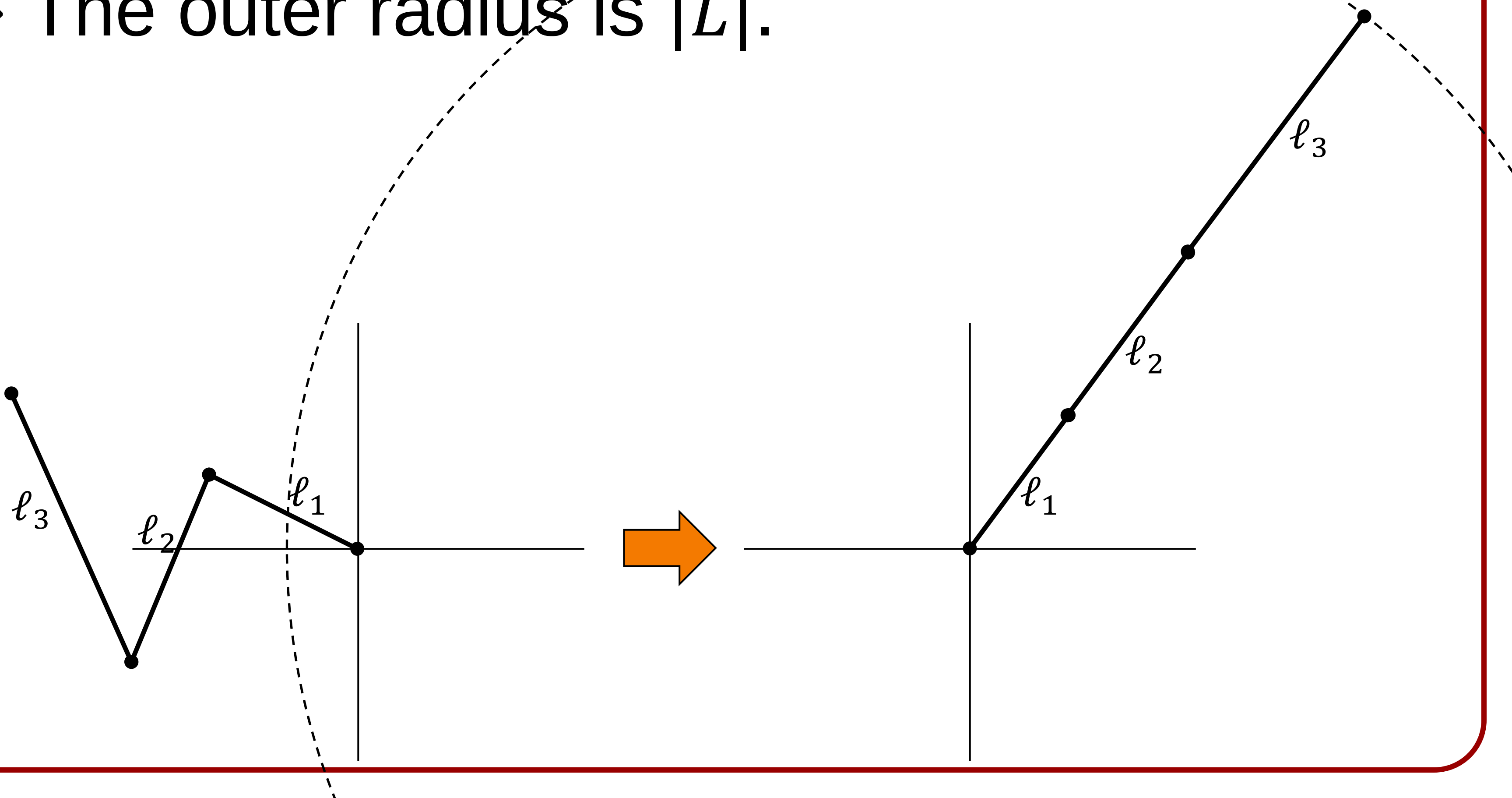
Given $L = \{\ell_1, \ldots, \ell_n\}$ the arm extends furthest when all joint angles are $180°$.



Robot Arm (Outer Radius)

Given $L = \{\ell_1, \ldots, \ell_n\}$ the arm extends furthest when all joint angles are $180°$.

⇒ The outer radius is $|L|$.

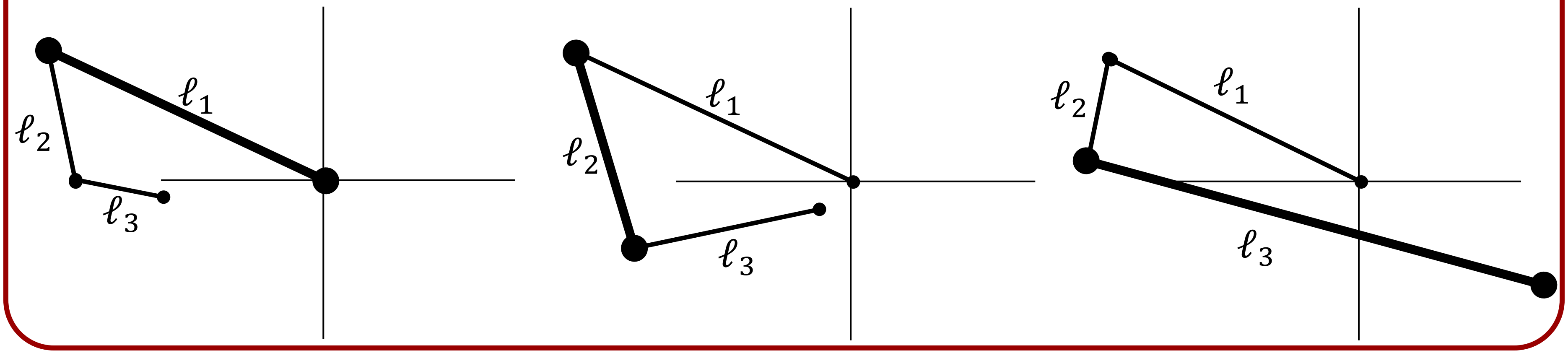


Robot Arm (Inner Radius)

<u>Definition</u>:

Given link lengths $\{\ell_1, \ldots, \ell_n\}$, the *median link* is the link ℓ_M containing the mid-point:

$$\sum_{i=1}^{M-1} \ell_i \leq \frac{|L|}{2} < \sum_{i=1}^{M} \ell_i$$



Robot Arm (Inner Radius)

Definition:

Given link lengths $\{\ell_1, \dots, \ell_n\}$, the *median link* is the link ℓ_M containing the mid-point:

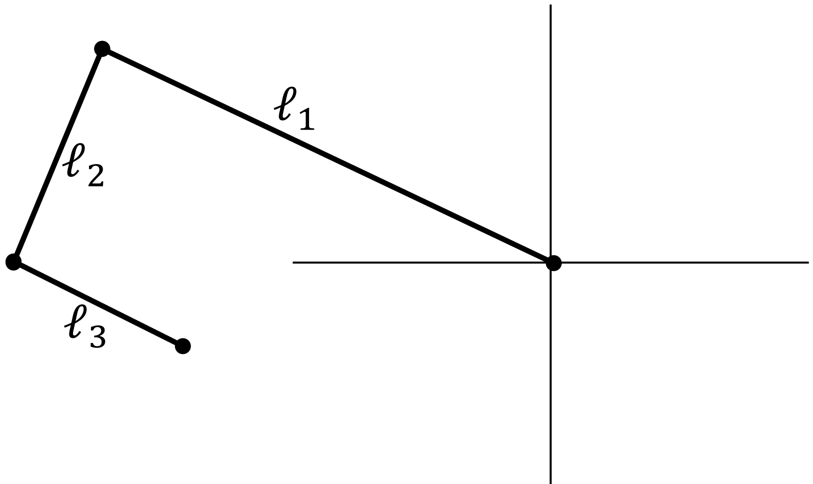
$$\sum_{i=1}^{M-1} \ell_i \leq \frac{|L|}{2} < \sum_{i=1}^{M} \ell_i$$

- Either $\ell_M > |L|/2$.
- Or there is no link with $\ell_k > |L|/2$ (and $M \neq 1, n$).

Robot Arm (Inner Radius)

Case $\ell_M > |L|/2$:

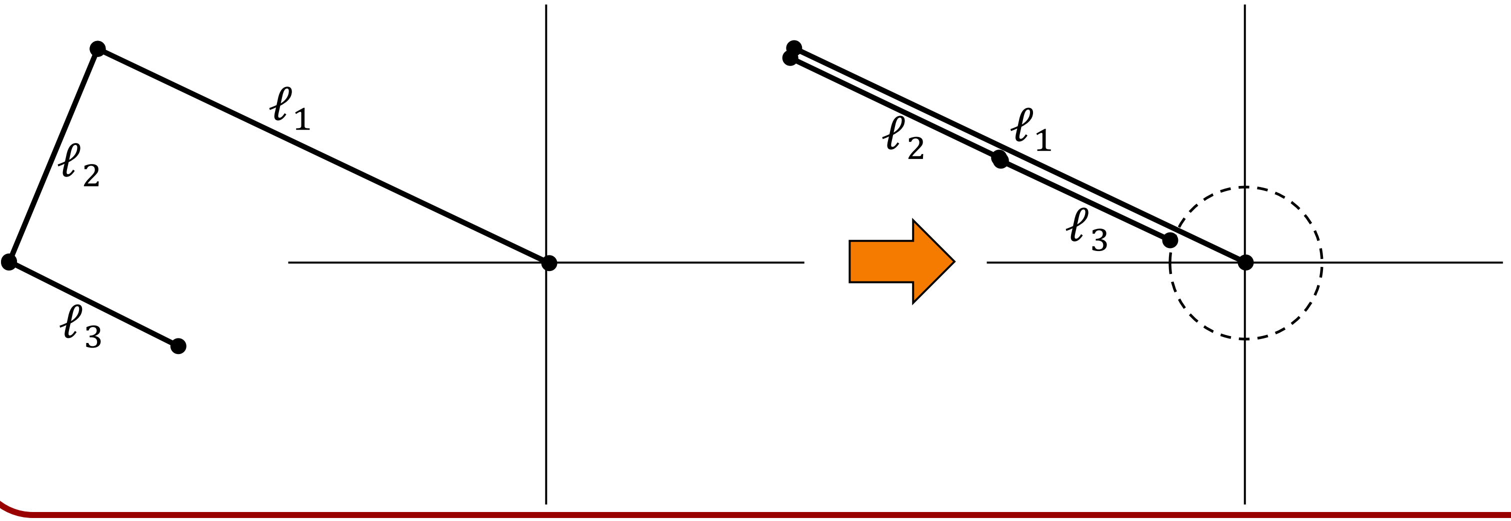
- Since the reach is independent of order, we can assume $M = 1$.



Robot Arm (Inner Radius)

<u>Case $\ell_M > |L|/2$</u>:

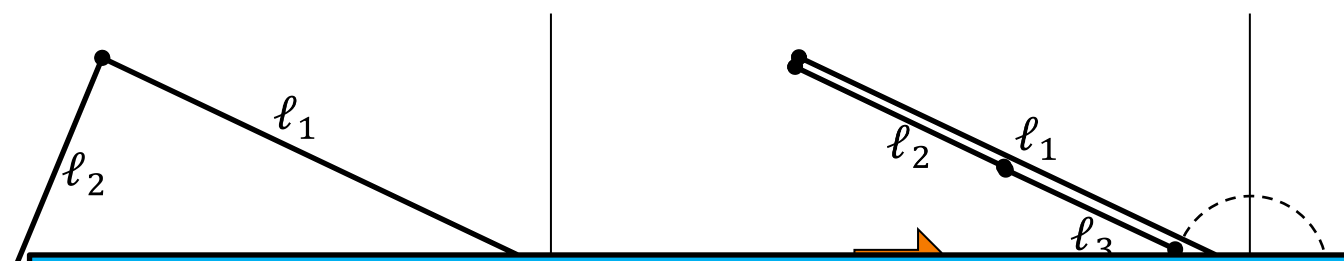
- Since the reach is independent of order, we can assume $M = 1$.
- The inner radius is $\ell_1 - \ell_2 - \cdots - \ell_n$ since we can't get closer than that to the origin.



Robot Arm (Inner Radius)

<u>Case $\ell_M > |L|/2$</u>:

- Since the reach is independent of order, we can assume $M = 1$.
- The inner radius is $\ell_1 - \ell_2 - \cdots - \ell_n$ since we can't get closer than that to the origin.



If we don't reorder, the configuration can be obtained by:

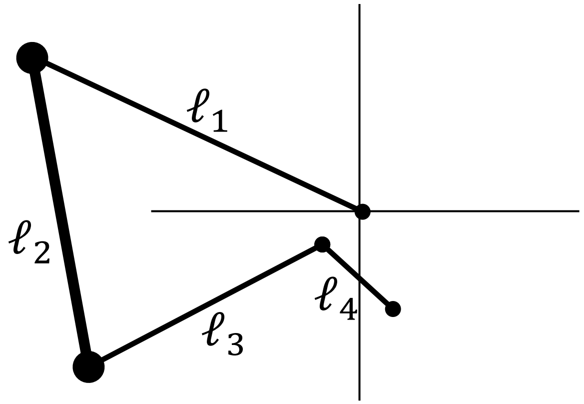
- Setting the joint angles around the median to 0°
- Setting all other joint angles to 180°.

Robot Arm (Inner Radius)

Case $\ell_k \leq |L|/2 \ \forall k \in [1, n]$:

Keeping all but the joints on the median link straight, we obtain a 3-link arm with lengths:

$$\sum_{i=1}^{M-1} \ell_i = \tilde{\ell}_1, \ \ell_M = \tilde{\ell}_2, \ \sum_{i=M+1}^{n} \ell_i = \tilde{\ell}_3$$

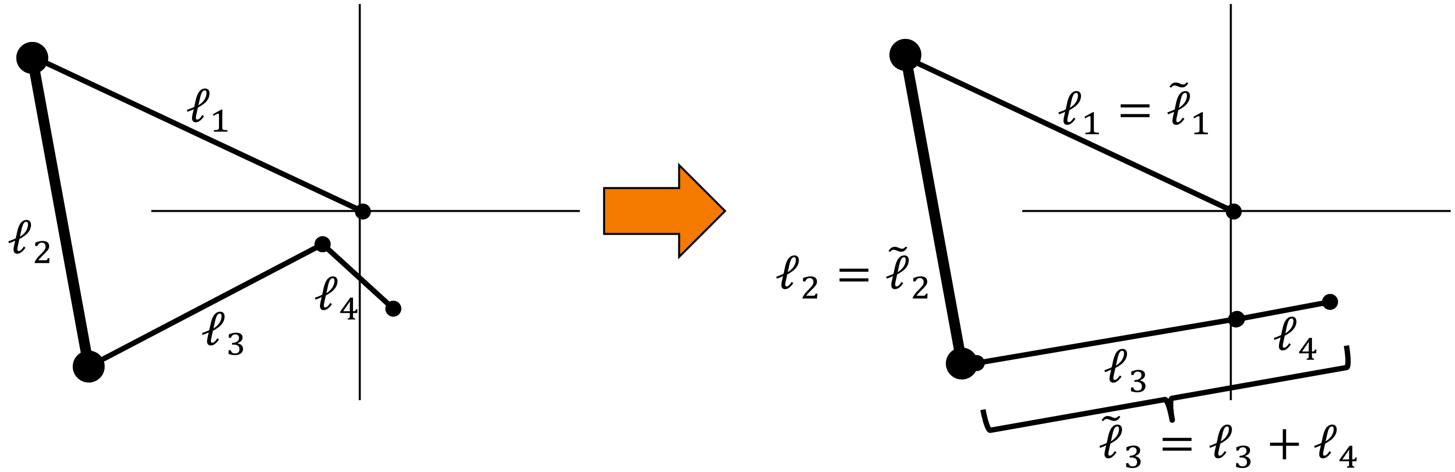


Robot Arm (Inner Radius)

<u>Case $\ell_k \leq |L|/2 \;\forall k \in [1, n]$</u>:

Keeping all but the joints on the median link straight, we obtain a 3-link arm with lengths:

$$\sum_{i=1}^{M-1} \ell_i = \tilde{\ell}_1,\ \ell_M = \tilde{\ell}_2,\ \sum_{i=M+1}^{n} \ell_i = \tilde{\ell}_3$$

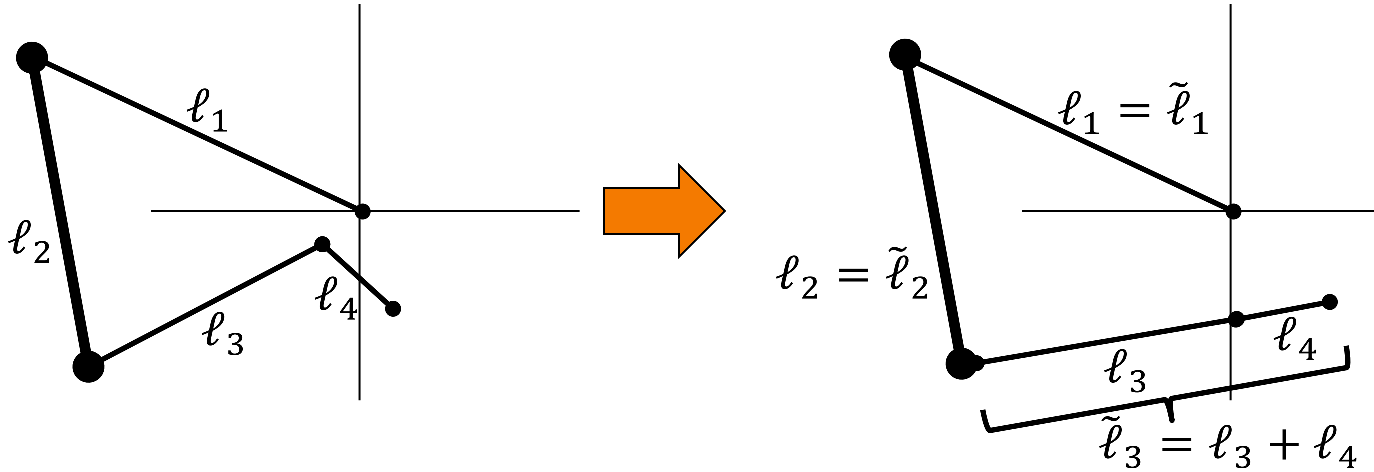


Robot Arm (Inner Radius)

<u>Case $\ell_k \le |L|/2 \ \forall k \in [1,n]$</u>:

Keeping all but the joints on the median link straig… gths:

> <u>Note</u>:
>
> For all distinct $i,j,k \in \{1,2,3\}$, we have:
>
> $$\tilde{\ell}_i + \tilde{\ell}_j \ge \frac{|L|}{2} \ge \tilde{\ell}_k$$



Robot Arm (Inner Radius)

<u>Case $\ell_k \leq |L|/2 \ \forall k \in [1, n]$</u>:

Keeping all but the joints on the median link

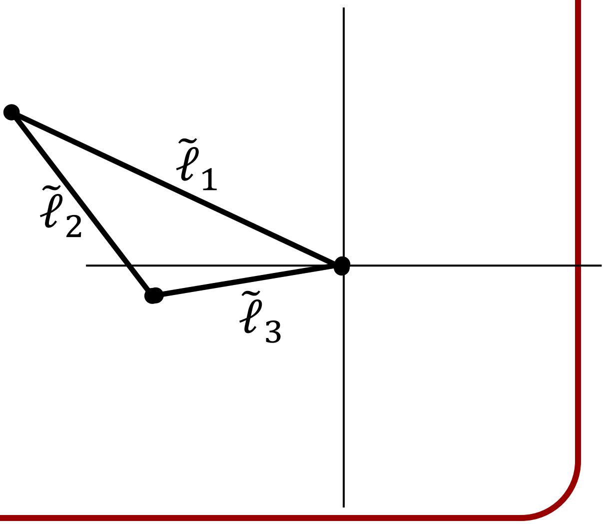
straig … gths:

<u>Note</u>:

For all distinct $i, j, k \in \{1,2,3\}$, we have:

$$\tilde{\ell}_i + \tilde{\ell}_j \geq \frac{|L|}{2} \geq \tilde{\ell}_k$$

$\Rightarrow$ The lengths $\tilde{\ell}_1$, $\tilde{\ell}_2$, and $\tilde{\ell}_3$ can be realized by a triangle.



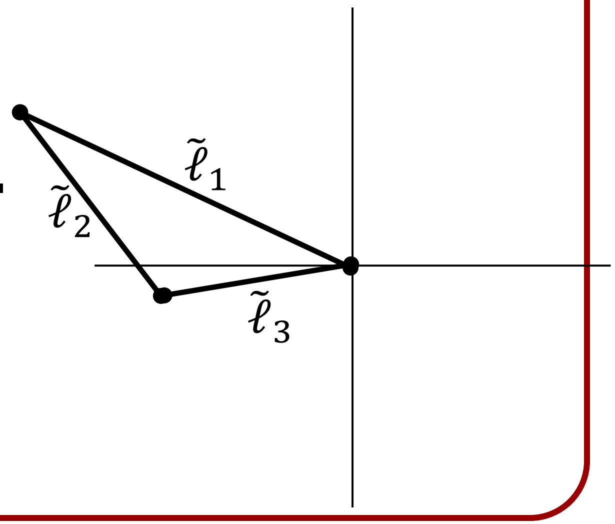
Robot Arm (Inner Radius)

Case $\ell_k \leq |L|/2 \ \forall k \in [1, n]$:

Keeping all but the joints on the median link straight, we obtain a 3-link arm with lengths:

$$\sum_{i=1}^{M-1} \ell_i = \tilde{\ell}_1, \ \ell_M = \tilde{\ell}_2, \ \sum_{i=M+1}^{n} \ell_i = \tilde{\ell}_3$$

⇒ The lengths $\tilde{\ell}_1$, $\tilde{\ell}_2$, and $\tilde{\ell}_3$ can be realized by a triangle.



⇒ The inner radius is 0.

Robot Arm (Inner Radius)

<u>Case $\ell_k \leq |L|/2\ \forall k \in [1, n]$</u>:

Keeping all but the joints on the median link straight, we obtain a 3-link arm with lengths:

$$\sum_{i=1}^{M-1} \ell_i = \tilde{\ell}_1,\ \ell_M = \tilde{\ell}_2,\ \sum_{i=M+1}^{n} \ell_i = \tilde{\ell}_3$$

⇒ The lengths $\tilde{\ell}$, $\tilde{\ell}$, and $\tilde{\ell}$

The inner and outer radii can be reached by keeping all but the joints around the median straight.

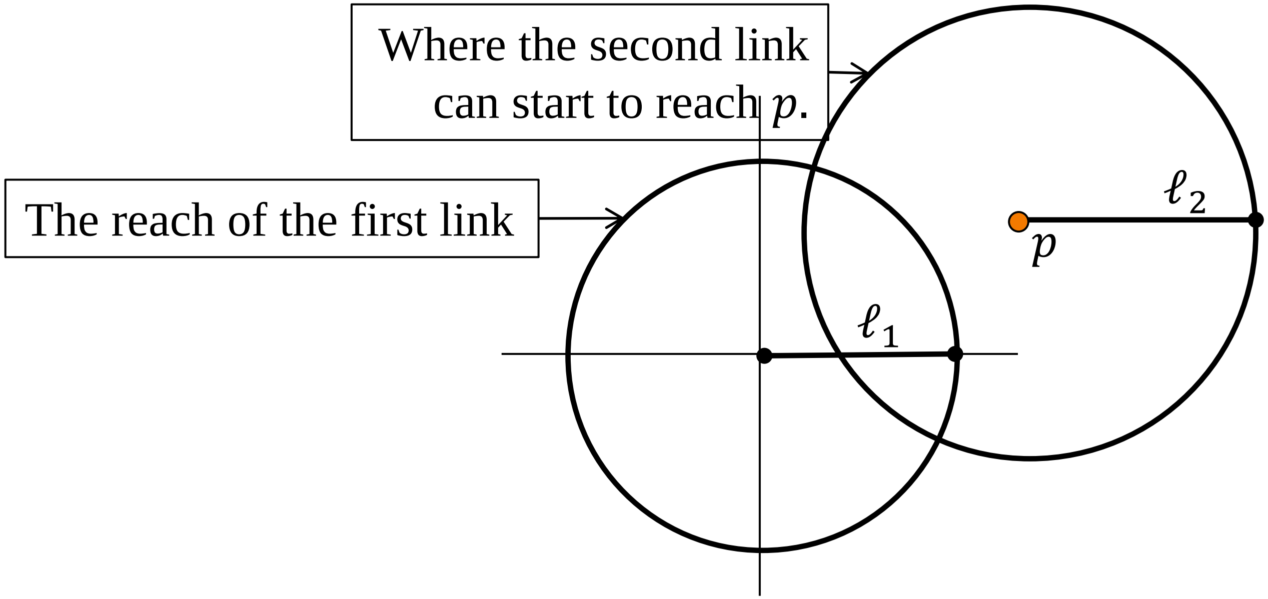
⇒

The problem of finding a reaching configuration of an n-link arm reduces to the problem of finding a reaching configuration of a 3-link arm.

Finding the Configuration

2-link $\{\ell_1, \ell_2\}$:

The arm reaches $p \in \mathbb{R}^2$ if the circle about the origin with radius ℓ_1 intersects the circle about p with radius ℓ_2.

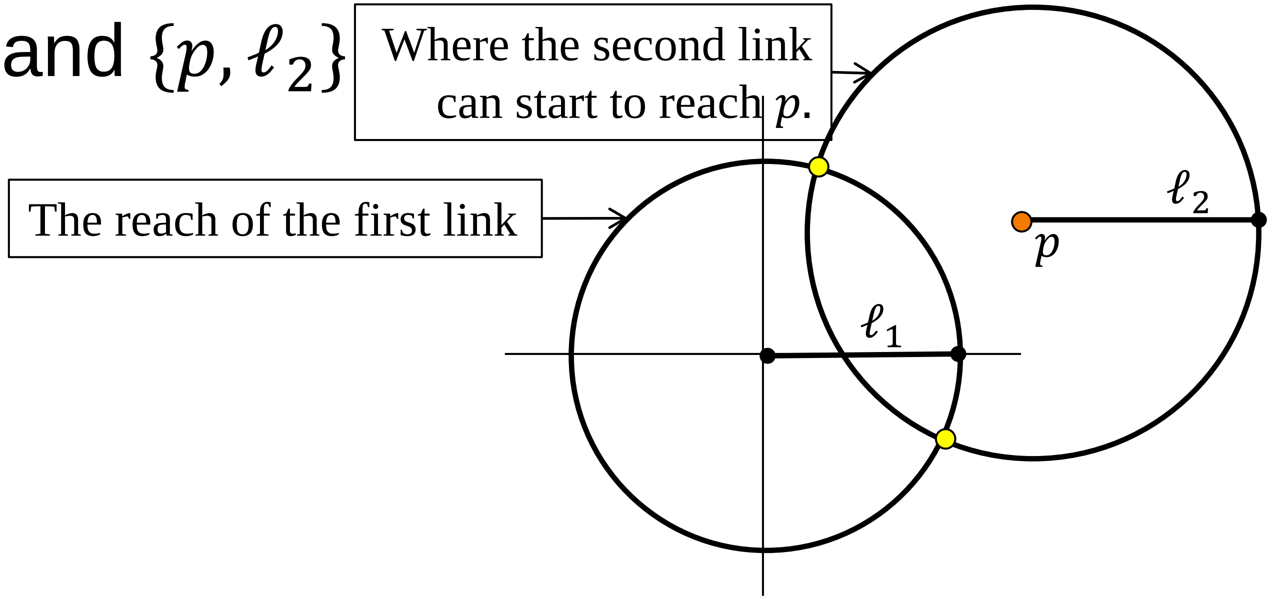


Finding the Configuration

2-link $\{\ell_1, \ell_2\}$:

The arm reaches $p \in \mathbb{R}^2$ if the circle about the origin with radius ℓ_1 intersects the circle about p with radius ℓ_2.

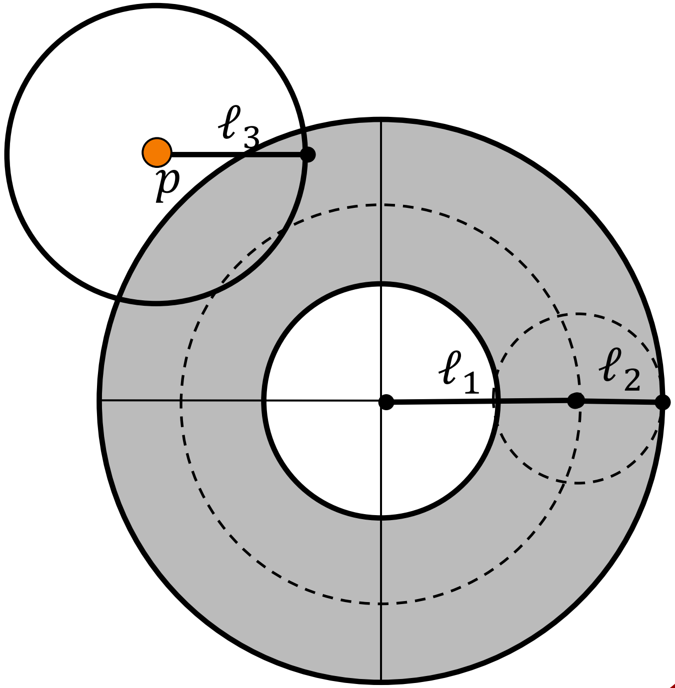
⇒ Two-circle intersection:
$\{(0,0), \ell_1\}$ and $\{p, \ell_2\}$



Finding the Configuration

3-link $\{\ell_1, \ell_2, \ell_3\}$:

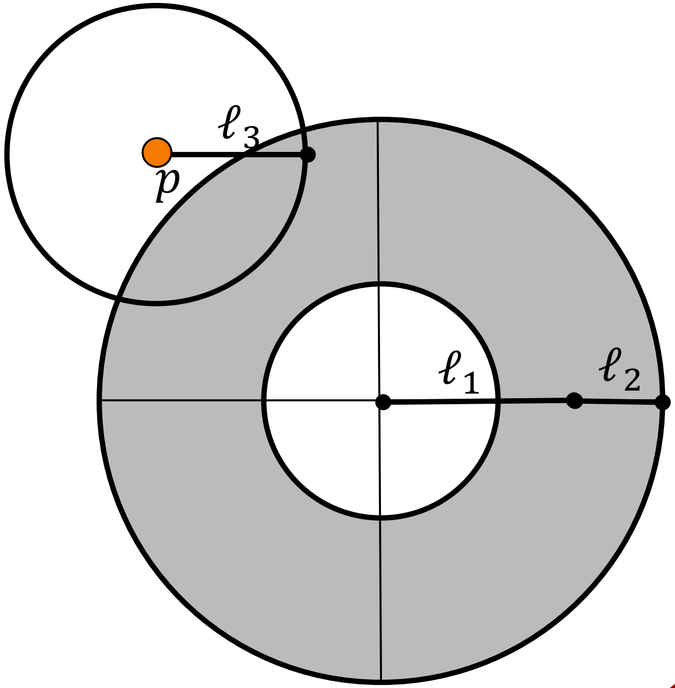
The arm reaches $p \in \mathbb{R}^2$ if the annuls about the origin with radii $|\ell_1 - \ell_2|$ and $\ell_1 + \ell_2$ intersects the circle about p with radius ℓ_3.



Finding the Configuration

3-link $\{\ell_1, \ell_2, \ell_3\}$: [Case 1]

The circle about p intersects the outer radius.

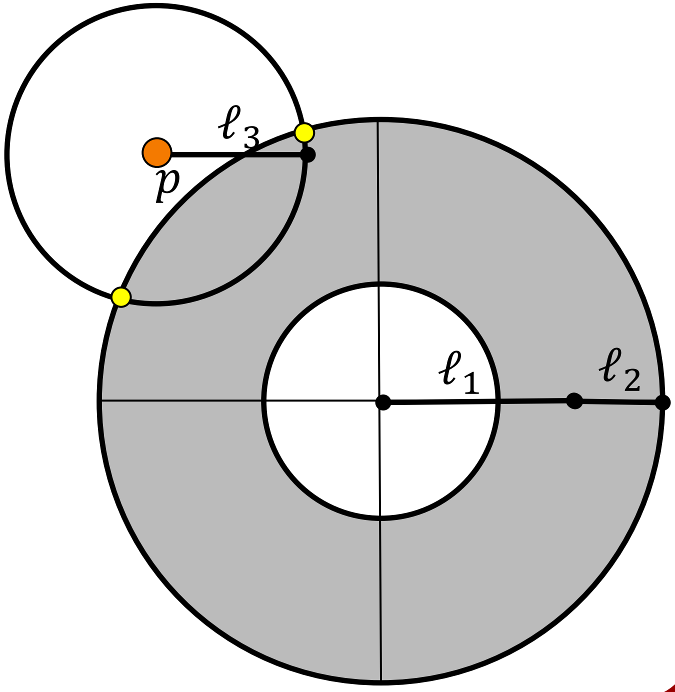


Finding the Configuration

3-link $\{\ell_1, \ell_2, \ell_3\}$: [Case 1]

The circle about p intersects the outer radius.

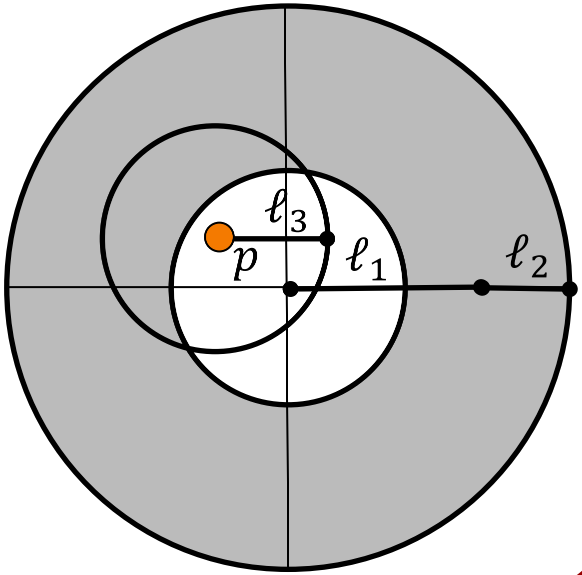
$\Rightarrow$ Two-circle intersection:
$\{(0,0), \ell_1 + \ell_2\}$ and $\{p, \ell_3\}$



Finding the Configuration

3-link $\{\ell_1, \ell_2, \ell_3\}$: [Case 2]

The circle about p intersects the inner radius.

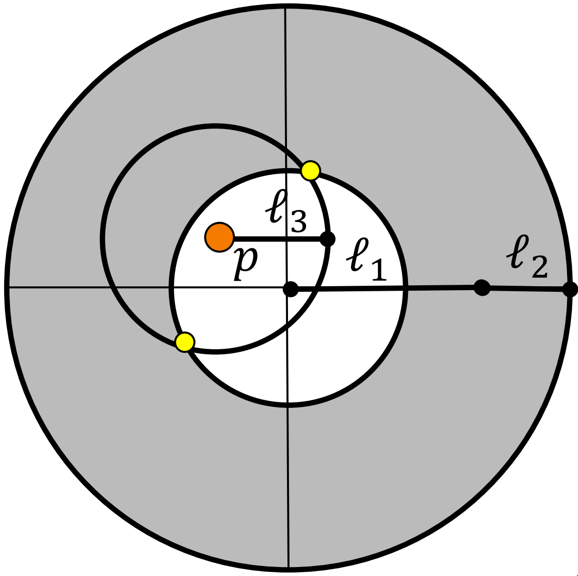


Finding the Configuration

3-link $\{\ell_1, \ell_2, \ell_3\}$: [Case 2]

The circle about p intersects the inner radius.

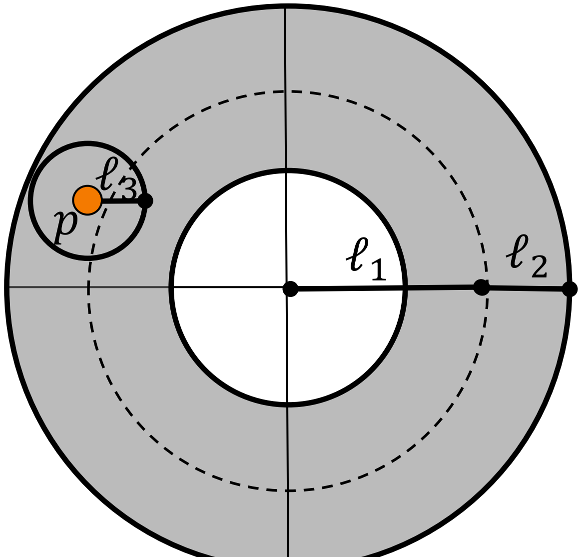
$\Rightarrow$ Two-circle intersection:
$\{(0,0), |\ell_1 - \ell_2|\}$ and $\{p, \ell_3\}$



Finding the Configuration

3-link $\{\ell_1, \ell_2, \ell_3\}$: [Case 3a]

The circle about p does not intersect either boundary and doesn't contain the origin.

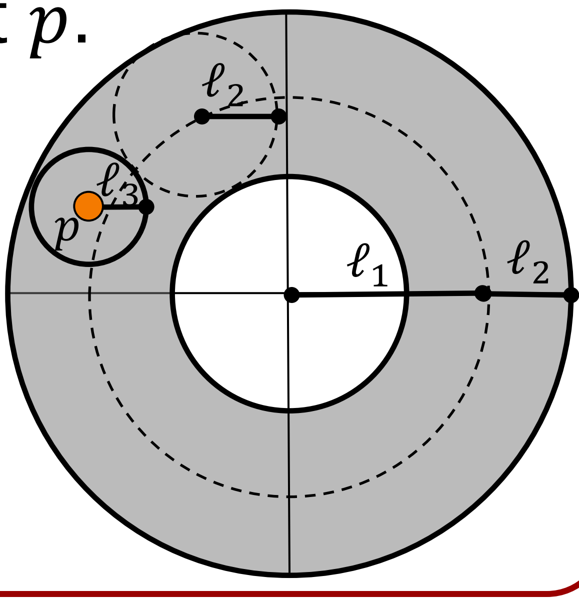


Finding the Configuration

3-link $\{\ell_1, \ell_2, \ell_3\}$: [Case 3a]

The circle about p does not intersect either boundary and doesn't contain the origin.

$\Rightarrow$ There is a circle with radius ℓ_2 centered on a point on the circle about the origin with radius ℓ_1 that is tangent to the circle about p.



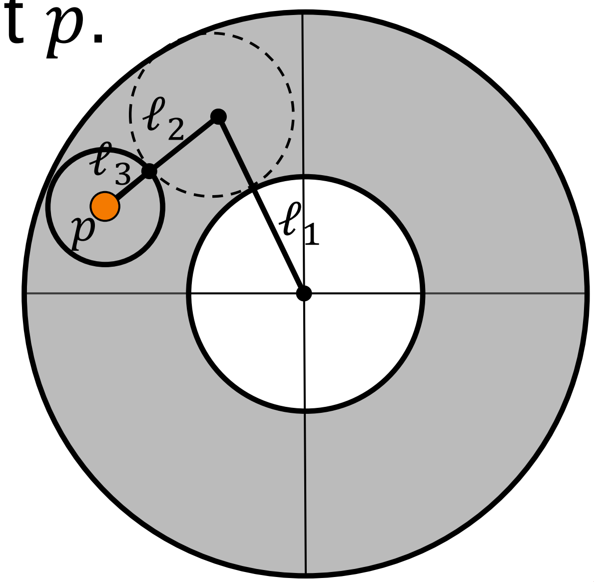
Finding the Configuration

<u>3-link $\{\ell_1, \ell_2, \ell_3\}$</u>: [Case 3a]

The circle about p does not intersect either boundary and doesn't contain the origin.

⇒ There is a circle with radius ℓ_2 centered on a point on the circle about the origin with radius ℓ_1 that is tangent to the circle about p.

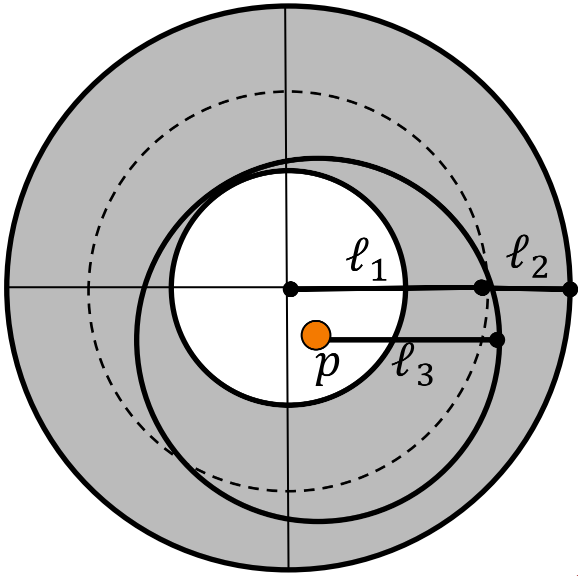
⇒ Two-circle intersection: $\{(0,0), \ell_1\}$ and $\{p, \ell_2 + \ell_3\}$



Finding the Configuration

3-link $\{\ell_1, \ell_2, \ell_3\}$: [Case 3b]

The circle about p does not intersect either boundary and contains the origin.

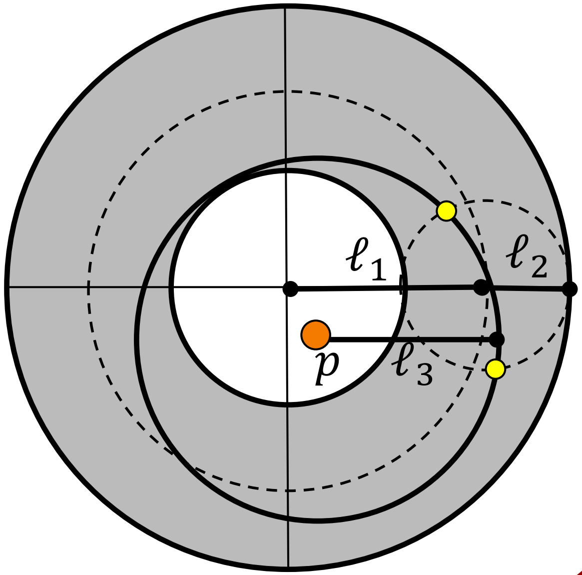


Finding the Configuration

3-link $\{\ell_1, \ell_2, \ell_3\}$: [Case 3b]

The circle about p does not intersect either boundary and contains the origin.

$\Rightarrow$ The circle with radius ℓ_2 centered on any point on the circle about the origin with radius ℓ_1 intersects the circle about p.



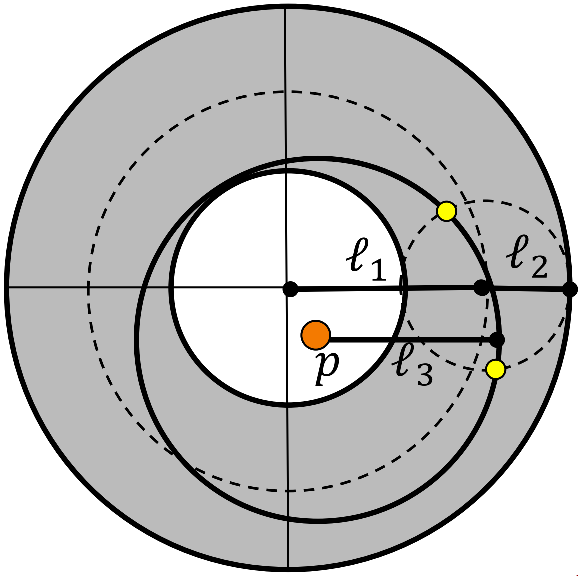
Finding the Configuration

<u>3-link $\{\ell_1, \ell_2, \ell_3\}$</u>: [Case 3b]

The circle about p does not intersect either boundary and contains the origin.

⇒ The circle with radius ℓ_2 centered on any point on the circle about the origin with radius ℓ_1 intersects the circle about p.

⇒ Two-circle intersection: $\{(|\ell_1|, 0), \ell_2\}$ and $\{p, \ell_3\}$

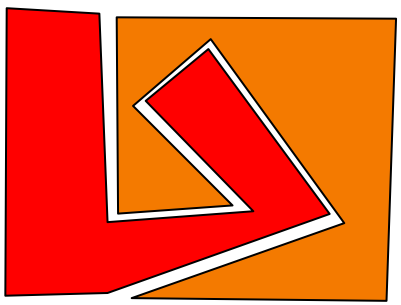


Outline

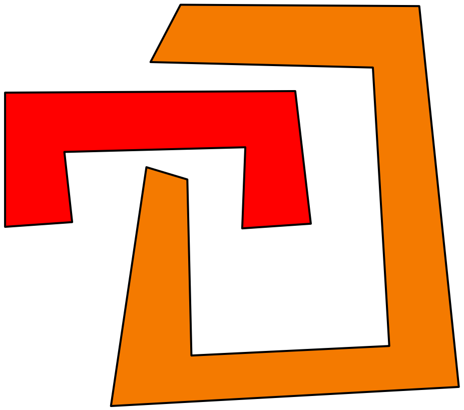
- Robot Arm
- Separability

Separability

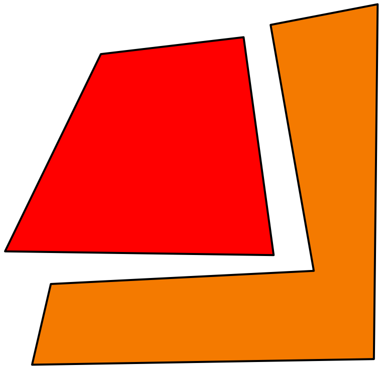
Are these polygons separable?



No



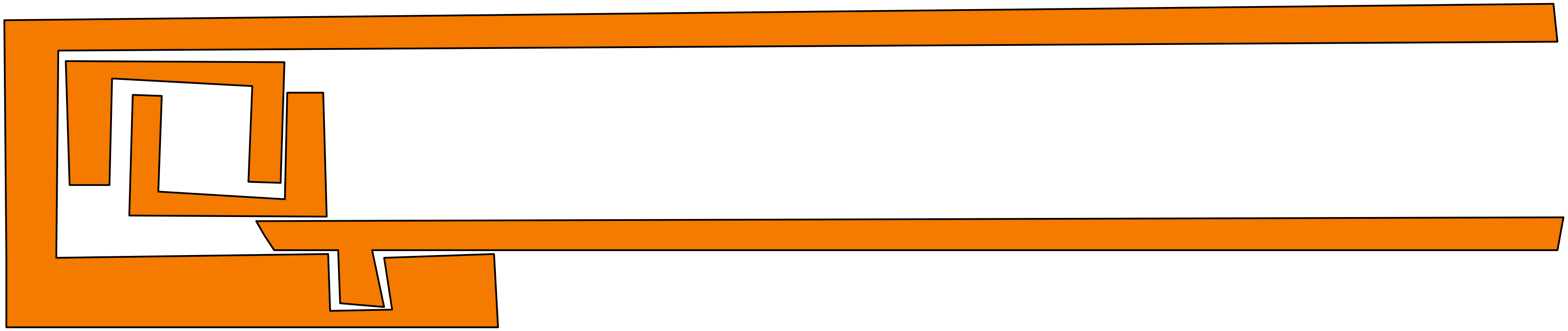
No if:
- Translations only



No if:
- Translations only
- Along a fixed direction

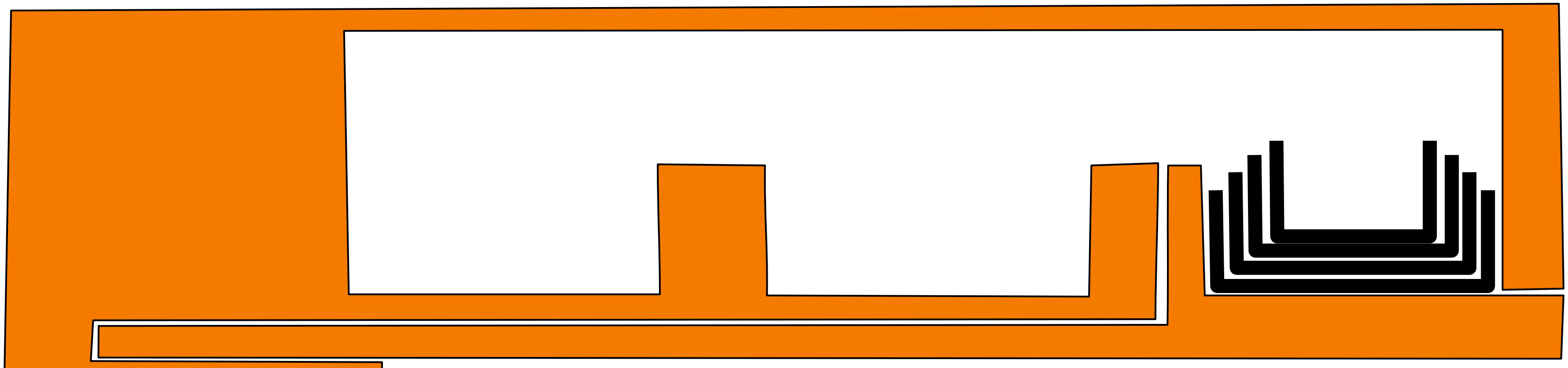
Separability

Are these polygons easily separable?



No if:

- Polygons are constrained to move one at a time

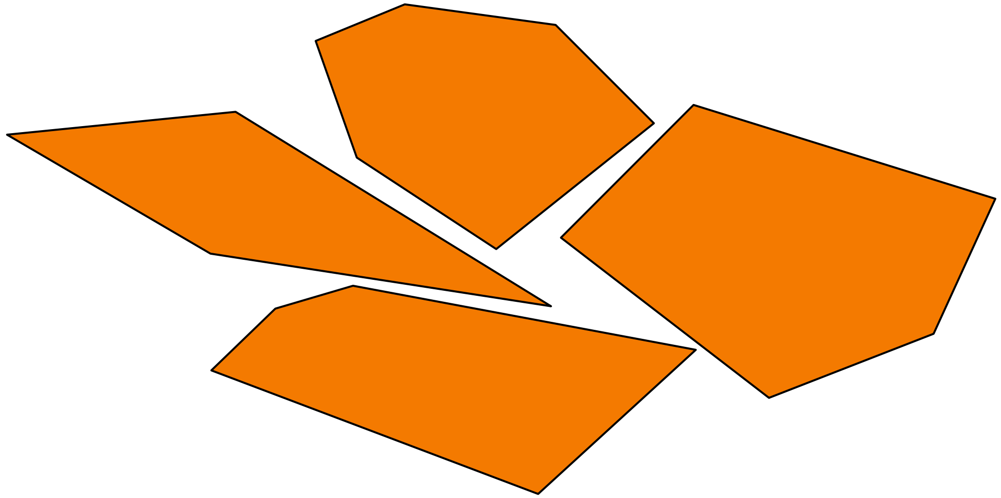


Separability

Convex Polygons [Guibas and Yao, 1983]

Given a set of convex polygons, the polygons can be translated arbitrarily far (w/o loss of generality) to the right, without crossing, by:

- applying a single translation to each polygon
- applying the translations one at a time

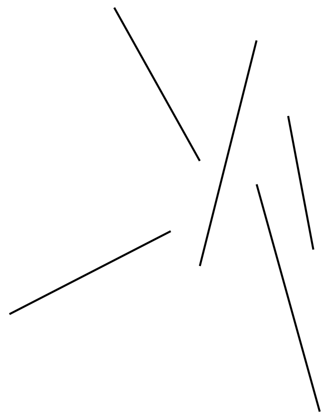


Separability

Lemma:

Given a set of (non-intersecting) line segments, the segments can be translated arbitrarily far (w/o loss of generality) to the right, without crossing, by:

- applying a single translation to each line segment
- applying the translations one at a time

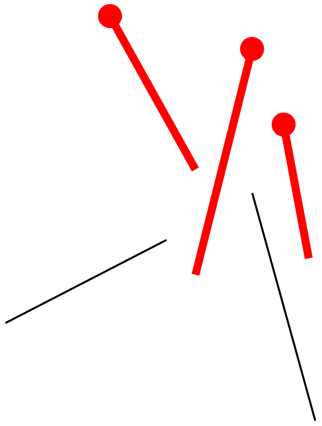


Separability

Proof (Lemma):

Identify the set of segments L whose top vertex is unobstructed from the right.*

- Note that the line segment with highest (right-most) vertex must be in this set.

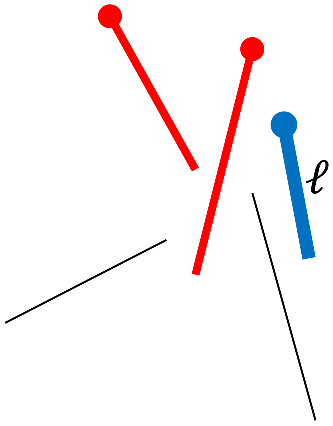


*e.g. In $O(n \log n)$ time, for example, with the sweep-line algorithm.

Separability

Proof (Lemma):
Identify the set of segments L...

Claim:
Segment $\ell \in L$ with lowest top vertex can be moved.

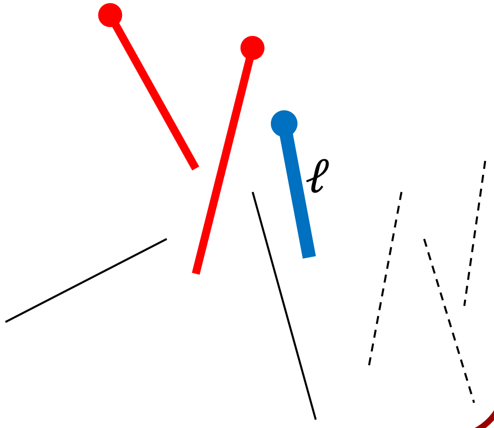


Separability

Proof (Lemma):
Identify the set of segments L...

Claim:
Segment $\ell \in L$ with lowest top vertex can be moved.

Proof:
If part of ℓ is obstructed, the obstructor's top vertex has to be below the top vertex of ℓ.



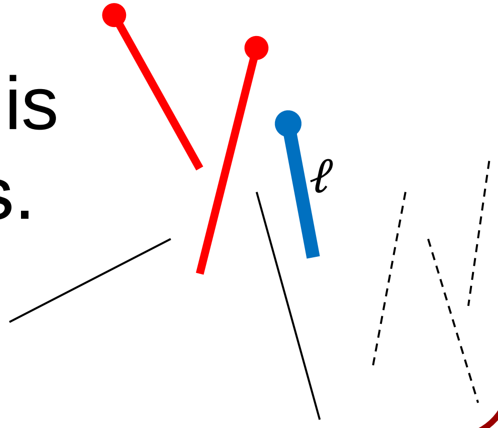
Separability

Proof (Lemma):
Identify the set of segments L...

Claim:
Segment $\ell \in L$ with lowest top vertex can be moved.

Proof:
If part of ℓ is obstructed, the obstructor's top vertex has to be below the top vertex of ℓ.

The obstructor with highest top vertex is in L but has a top vertex lower than ℓ's.



Separability

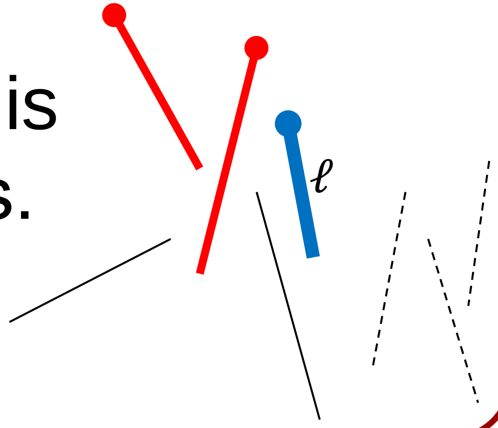
<u>Proof (Lemma)</u>:
Identify the set of segments L...

<u>Claim</u>:
Segment $\ell \in L$ with lowest top vertex can be moved.

<u>Proof</u>:
If part of ℓ is obstructed, the obstructor's top vertex has to be below the top vertex of ℓ.

The obstructor with highest top vertex is in L but has a top vertex lower than ℓ's.

ℓ did not have the lowest top vertex.

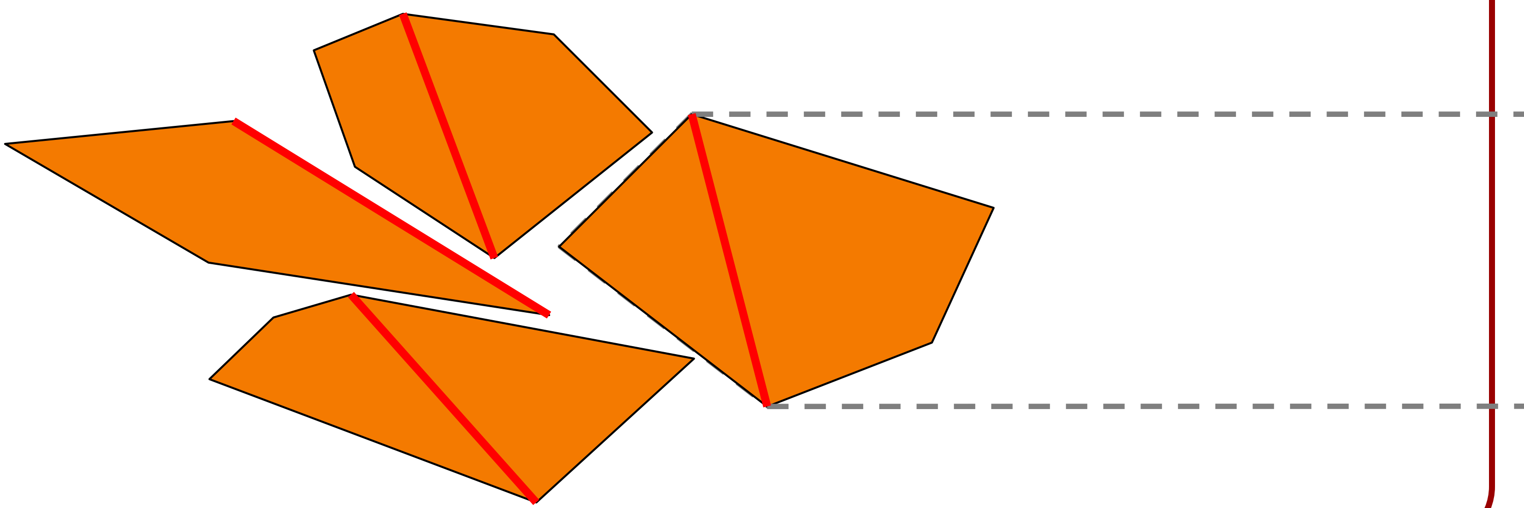


Separability

Proof:

Apply the Lemma to the line segments connecting the (vertically) extremal vertices of the polygons.

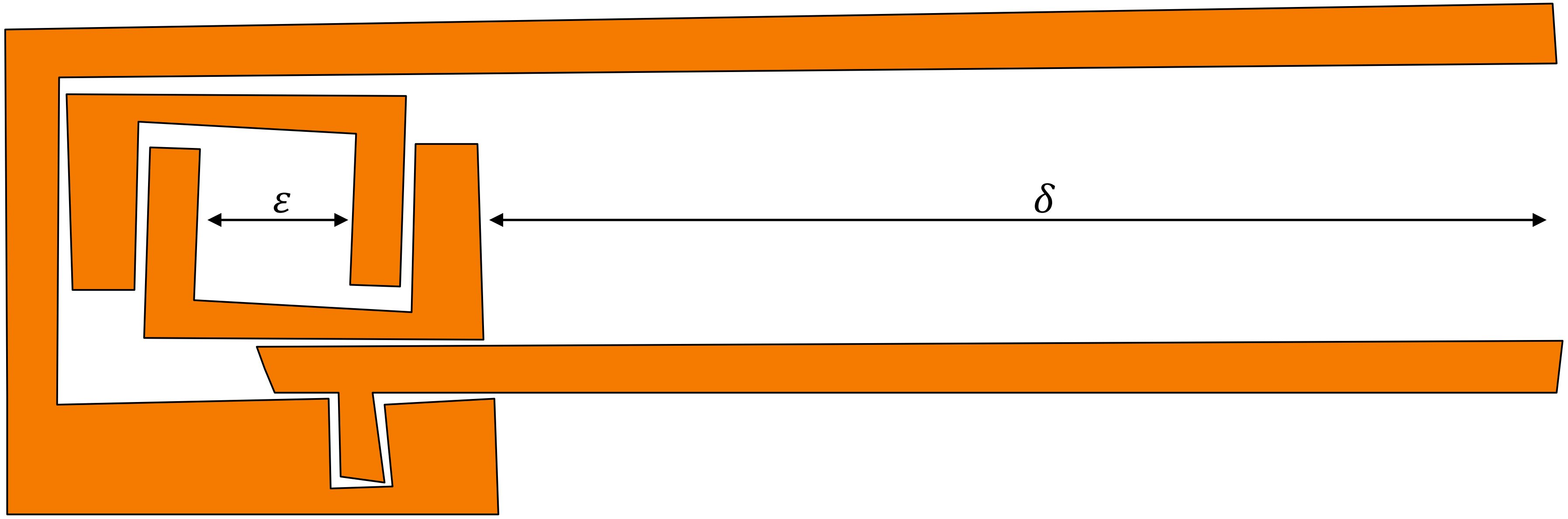
- The sweep of the first line segment, unioned with the associated polygon, contains the right translation of the polygon and is empty of all others.



Separability

Why This is Hard [Take 1]:

There are configurations of polygons of constant size that require arbitrarily many moves:



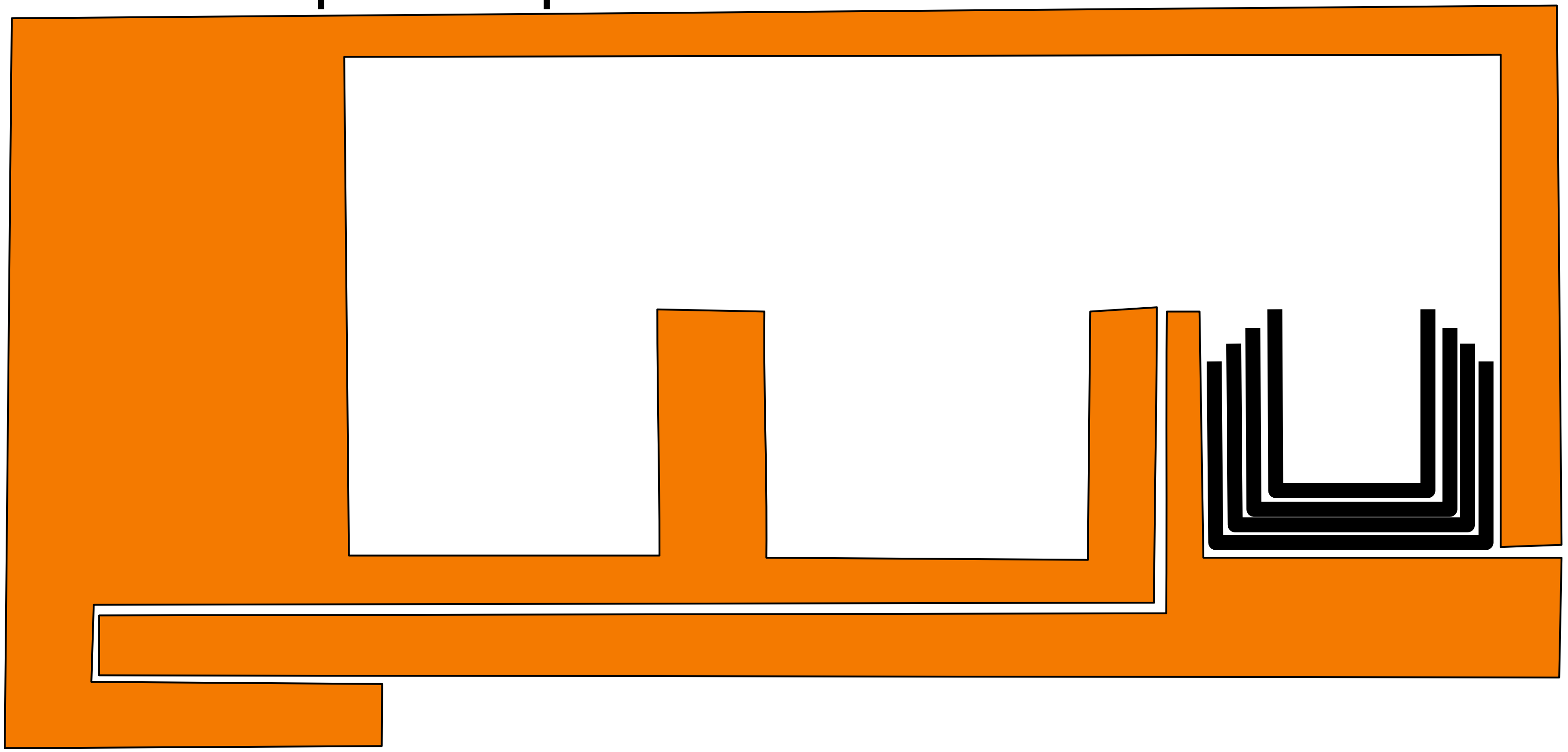
Requires approximately δ/ε moves.

But, this only requires on the order of δ "work".

Separability

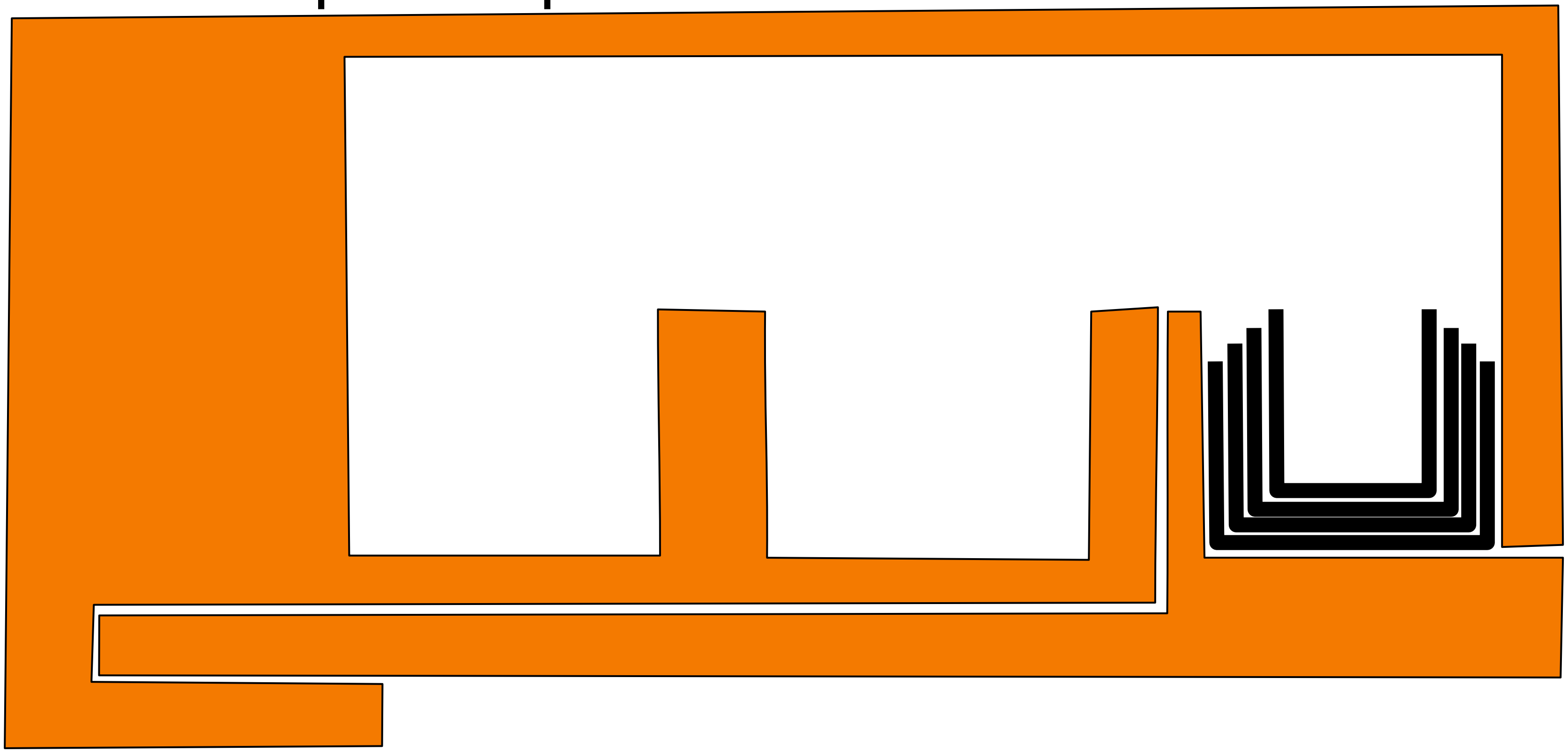
Why This is Hard [Take 2]:

There are configurations of polygons of constant size that require exponential amount of work:



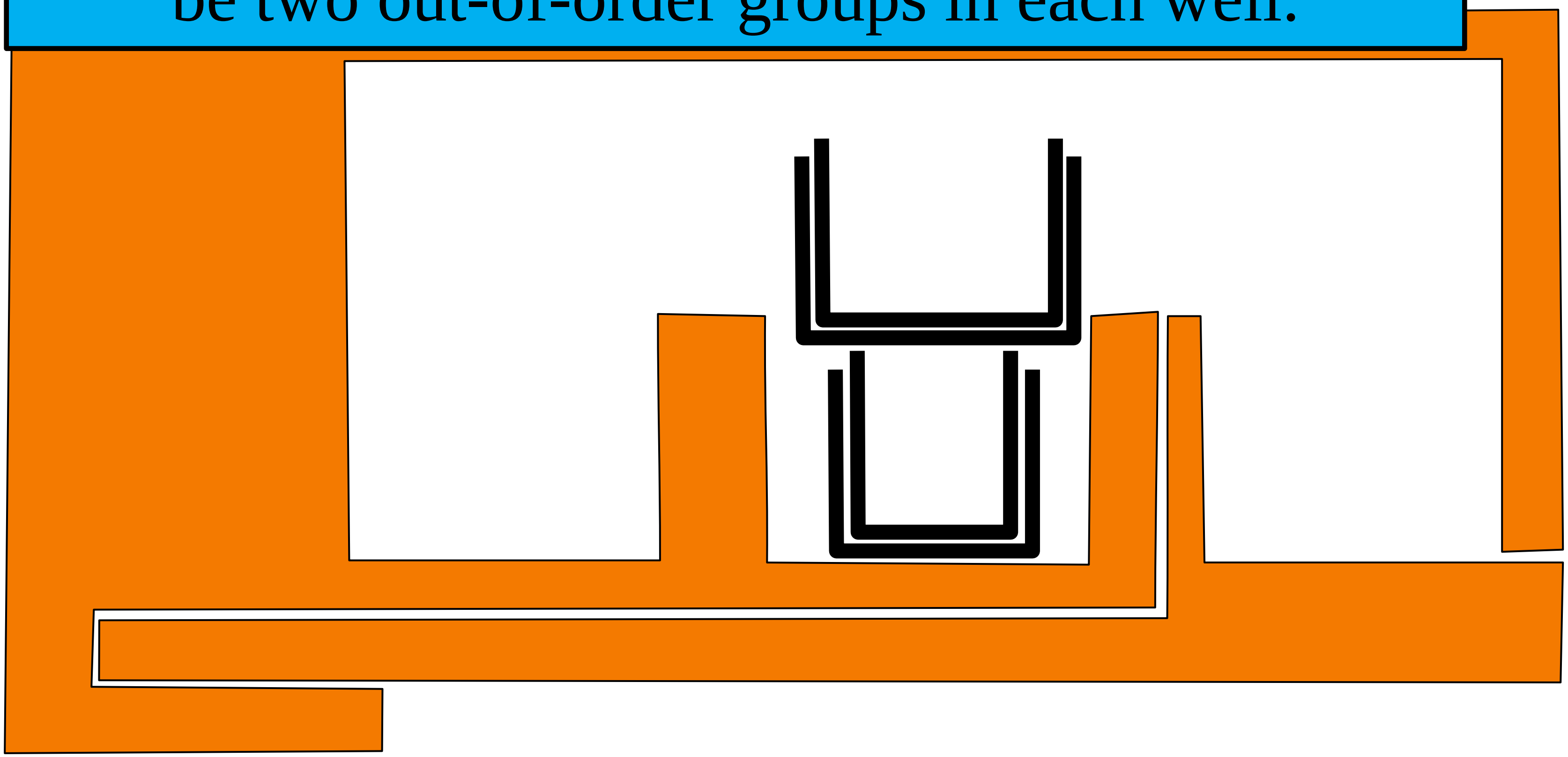
The polygons can be separated if the U-shaped pieces are moved out of the right well into the other two wells

There are configurations of polygons of constant size that require exponential amount of work:



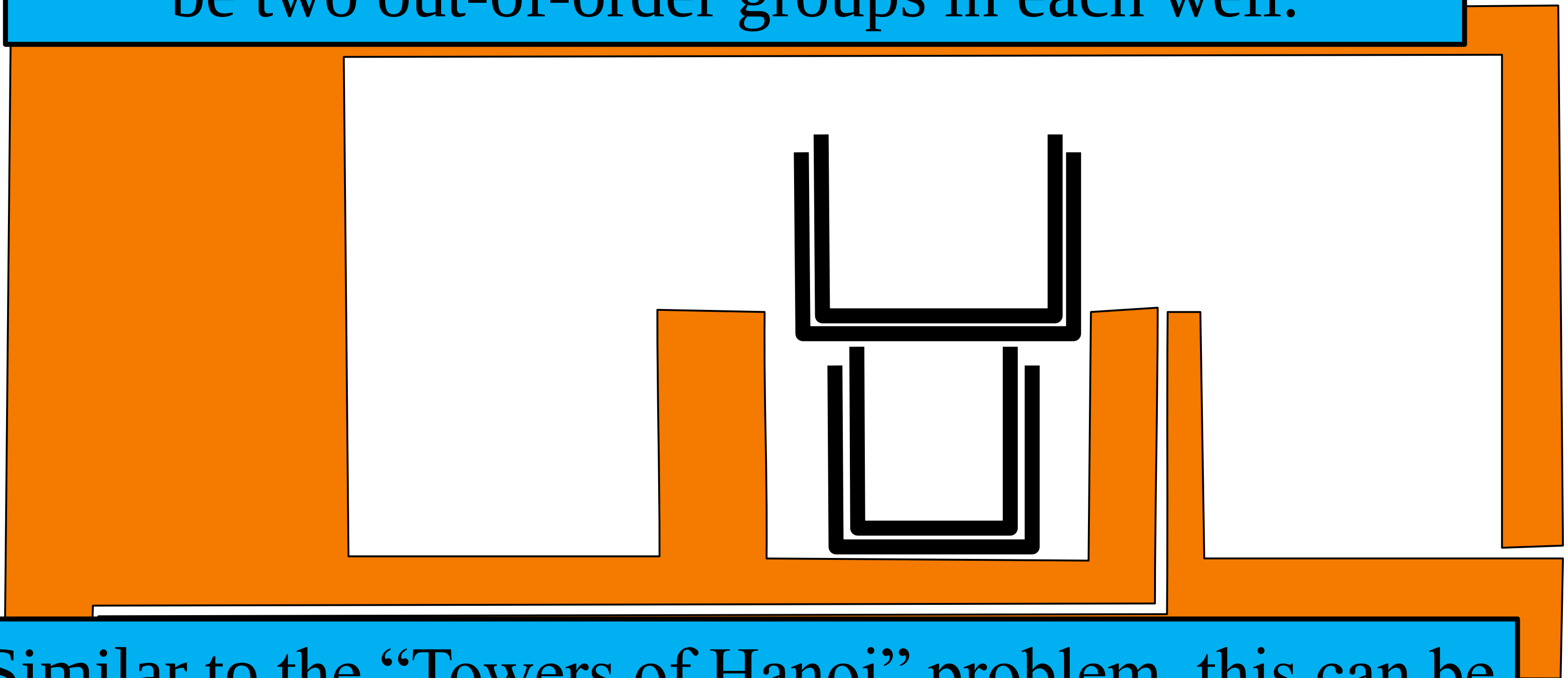
The polygons can be separated if the *U*-shaped pieces are moved out of the right well into the other two wells

Because the depth of the wells is (roughly) twice the height of the *U*-shaped pieces, there can only be two out-of-order groups in each well.



The polygons can be separated if the *U*-shaped pieces are moved out of the right well into the other two wells

Because the depth of the wells is (roughly) twice the height of the *U*-shaped pieces, there can only be two out-of-order groups in each well.



Similar to the "Towers of Hanoi" problem, this can be shown to require an exponential number of moves.

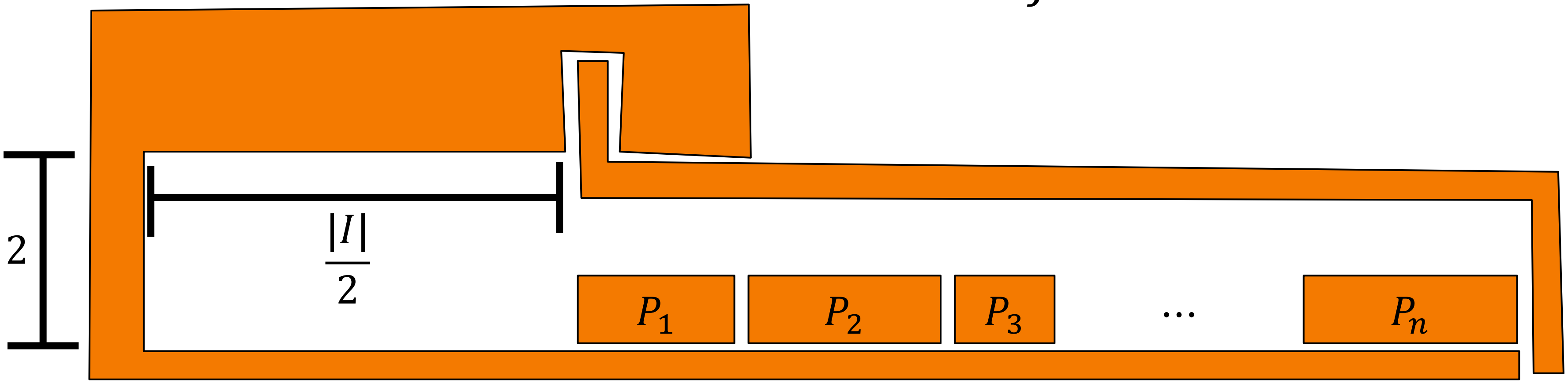
Separability

Why This is Hard [Theoretically]:

Given a set of positive integers $I = \{i_1, \ldots, i_n\}$, set:

$$|I| = i_1 + \cdots + i_n$$

and build the following configuration, where P_j is a rectangle with height 1 and width i_j:



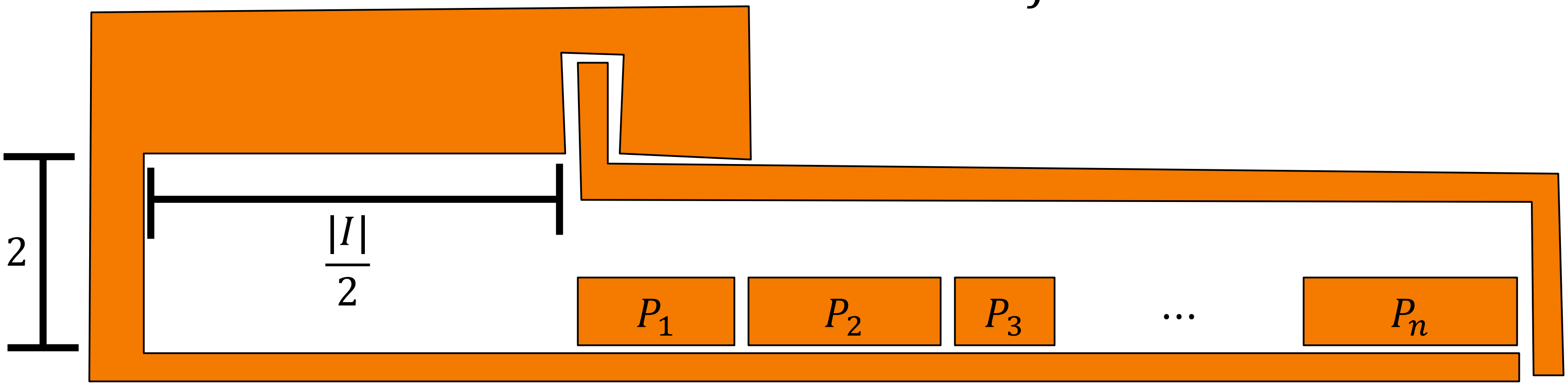
Separability

Why This is Hard [Theoretically]:

Given a set of positive integers $I = \{i_1, \dots, i_n\}$, set:

$$|I| = i_1 + \cdots + i_n$$

and build the following configuration, where P_j is a rectangle with height 1 and width i_j:



The pieces are separable iff. we can partition I into two subsets whose sums are equal (to $|I|/2$).

Separability

Why This is Hard [Theoretically]:

Given a set of positive integers $I = \{i_1, \ldots, i_n\}$, set:

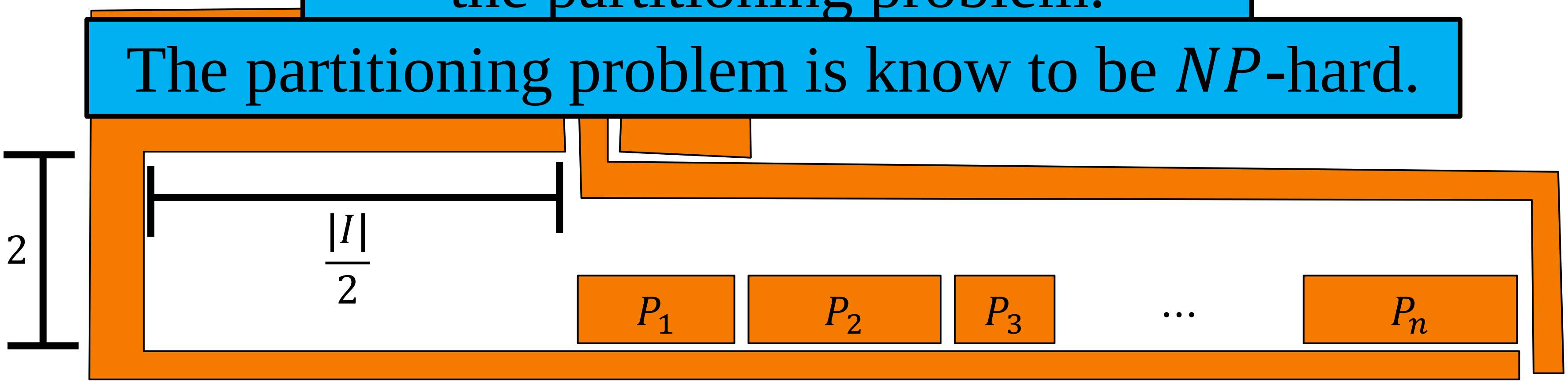
$$|I| = i_1 + \cdots + i_n$$

and build … e P_j is a

rectangle …

Determining separability solves the partitioning problem.

The partitioning problem is know to be NP-hard.



The pieces are separable iff. we can partition I into two subsets whose sums are equal (to $|I|/2$).