Arrangements

O’Rourke, Chapter 6
Outline

• Voronoi Diagrams
• Arrangements
Voronoi Diagrams

Recall:

We can compute the Delaunay Triangulation by raising the points to a paraboloid and computing the projection of the lower hull.
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Given a point $P(p) = (p, \|p\|^2)$ on the paraboloid, the tangent plane is given by:

$$z_p(r) = 2\langle p, r \rangle - \|p\|^2$$
Voronoi Diagrams

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Given a point \( P(p) = (p, \|p\|^2) \) on the paraboloid, the tangent plane is given by:

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For any point \( r \) the (vertical) distance between its position on the parabola and its position on the tangent plane at \( p \) is:

\[
P(r) - z_p(r) = \|r\|^2 - (2\langle r, p \rangle - \|p\|^2)
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Voronoi Diagrams

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P(r) - z_p(r) = \|r\|^2 - (2\langle r, p \rangle - \|p\|^2) = \|p - r\|^2
\]
Voronoi Diagrams

Given points \( p \) and \( q \), wherever the tangent plane at \( q \) is higher than the tangent plane at \( p \), we are closer to \( q \) than to \( p \).

\[
z_p(r) \leq z_q(r)
\]

\[
P(r) - z_p(r) \geq P(r) - z_q(r)
\]

\[
\|p - r\|^2 \geq \|q - r\|^2
\]
Voronoi Diagrams

⇒ Given points $p$ and $q$, wherever the tangent plane at $p$ is higher than the tangent plane at $q$, we are closer to $p$ than to $q$.

⇒ We can visualize the Voronoi diagram by drawing the tangent planes at the sites and looking down the $z$-axis.
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Arrangements

Definition:

An arrangement of lines is a set of lines in the plane, inducing a partition of the domain into (convex) faces, edges, and vertices.
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An *arrangement of lines* is a set of lines in the plane, inducing a partition of the domain into (convex) faces, edges, and vertices.

An arrangement is *simple* if all pairs of lines intersect, and no three lines intersect at the same point.
Combinatorics

Claim:
A simple arrangement of $n$ lines has

- $\binom{n}{2}$ vertices,
- $n^2$ edges, and
- $\binom{n}{2} + n + 1$ faces.
Combinatorics

**Proof (Vertices):**

Since each pair of lines intersects exactly once, the total number of vertices is the number of distinct line pairs, $\binom{n}{2}$. 
Combinatorics

Proof (Edges):

Since each line is intersected by $n - 1$ other lines, partitioning the lines into $n$ edges, the total number of edges is $n^2$. 
Combinatorics

Proof (Faces):

Using stereographic mapping, arrangements of lines in the plane can be thought of as polygonizations of the sphere.
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Using stereographic mapping, arrangements of lines in the plane can be thought of as polygonizations of the sphere.

Note:
The stereographic mapping of the lines intersect at the North Pole.
Combinatorics

Proof (Faces):
By Euler’s theorem the number of faces is:

\[ F = 2 - (V + 1) + E \]
\[ = 2 - \binom{n}{2} - 1 + n^2 \]
\[ = \binom{n}{2} + n + 1 \]
**Zone Theorem**

**Definition:**

Given an arrangement $A$ and a line $L$ (s.t. $A \cup \{L\}$ is simple) the *zone* of $L$ in $A$, $Z(L)$, is the set of faces of $A$ intersected by $L$. 

\[ A \ (n = 4) \]
Zone Theorem

Definition:

Given an arrangement $A$ and a line $L$ (s.t. $A \cup \{L\}$ is simple) the zone of $L$ in $A$, $Z(L)$, is the set of faces of $A$ intersected by $L$. 

$A (n = 4)$
Zone Theorem

Notation:

The number of edges in $Z(L)$ is denoted $z(L)$.
The max size of $z(L)$ over all lines is denoted $z_n$. 

$$z(L) = 11 \quad \text{and} \quad A(n = 4)$$
Zone Theorem

Note:

Assuming that no line in $A$ is horizontal, we mark an edge as *left* (resp. *right*) if it bounds a face of $Z(L)$ from the left (resp. right).*

*Note that an edge can be marked both *left* and *right*. 

$z(L) = 11 \quad A (n = 4)$
Zone Theorem

Note:

Assuming that no line in $A$ is horizontal, we mark an edge as *left* (resp. *right*) if it bounds a face of $Z(L)$ from the left (resp. right).

*Note:*
The number of edges in the zone is at most the number of edges marked *left* plus the number of edges marked *right*.

*Note that an edge can be marked both *left* and *right*. $z(L) = 11$ $A \ (n = 4)$
Zone Theorem

Theorem:

For an arrangement of \( n \) lines, \( z_n \leq 6n \).

In particular, the number of edges marked left (resp. right) at most \( 3n \).

\( z(L) = 11 \quad A(n = 4) \)
Zone Theorem

Proof:

Without loss of generality, assume that the line \( L \) is horizontal.

Proceed by induction.
Zone Theorem

Proof (base case):
Trivially true when $n = 0$. 
Zone Theorem

Proof (inductive case):

Remove the right-most line on $L$.

By induction, the number of left edges crossed is at most $3(n - 1)$.

Need to show that adding the line back generates at most 3 additional left edges.
Zone Theorem

Claim 1:

Adding the right-most line introduces exactly one new left edge.
Zone Theorem

Proof of Claim 1:

It introduces one because this will be a left edge of the right-most face.
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It introduces one because this will be a left edge of the right-most face.

It introduces exactly one because a right-most line cannot contribute more than one left edge.
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Proof of Claim 1:

It introduces one because this will be a left edge of the right-most face.

It introduces exactly one because a right-most line cannot contribute more than one left edge.

If it is split by a line from the left, only one of the two segments will be in the zone, (the one containing $L$.)
Proof of Claim 1:

It introduces one because this will be a left edge of the right-most face.

It introduces exactly one because a right-most line cannot contribute more than one left edge.

If it is split by a line from the right, then it wasn’t right-most.
Zone Theorem

**Claim 2:**

Adding the right-most line splits at most two existing left edges.
Zone Theorem

Proof of Claim 2:
If the right-most line splits a left edge in two, the edge must be on the right-most face.
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Consider the segment of the right-most line from $L$ to the left edge.
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Proof of Claim 2:

If the right-most line splits a left edge in two, the edge must be on the right-most face.

Consider the segment of the right-most line from $L$ to the left edge.

If it is not split by another line, the left edge must have been on the right-most face.
Zone Theorem

Proof of Claim 2:

If the right-most line splits a left edge in two, the edge must be on the right-most face.

Consider the segment of the right-most line from $L$ to the left edge.

If it is split by another line, only one of the two sides of the left edge will be in the zone.
Zone Theorem

Proof of Claim 2:

If the right-most line splits a left edge in two, the edge must be on the right-most face.

Since faces are convex, the line splits at most two edges on the right-most face.

These must be left edges because otherwise the line was not right-most.
Zone Theorem

Corollary:

We can construct a (simple) arrangement of $n$ lines in $O(n^2)$ time.
Zone Theorem

Proof:
Iteratively add the $k$-th line.
Zone Theorem

**Proof:**

Iteratively add the $k$-th line.
- Find an intersection with an existing edge.
Zone Theorem

Proof:

Iteratively add the $k$-th line.

• Find an intersection with an existing edge.
• Cycle around faces to the left
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- Cycle around faces to the right
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Proof:

Iteratively add the $k$-th line. $\mathcal{O}(n)$ iterations

- Find an intersection with an existing edge. $\mathcal{O}(k)$
- Cycle around faces to the left
- Cycle around faces to the right
Zone Theorem

Proof:

Iteratively add the $k$-th line. $O(n)$ iterations

- Find an intersection with an existing edge. $O(k)$
- Cycle around faces to the left $O(k)$
- Cycle around faces to the right $O(k)$
Zone Theorem

Proof:

Iteratively add the $k$-th line. $O(n)$ iterations

- Find an intersection with an existing edge. $O(k)$
- Cycle around faces to the left $O(k)$
- Cycle around faces to the right

The total complexity is $O(n^2)$. 
Zone Theorem

Generalizations:

In $d$-dimensional space:

- The number of faces of any dimension of an arrangement is $O(n^d)$.
- The number of faces in the zone of a hyper-plane is bounded by $O(n^{d-1})$.
- The arrangement can be computed in $O(n^d)$ time.