Convex Hulls (3D)

O’Rourke, Chapter 4
Outline

• Correction

• Polyhedra
  ◦ Polytopes
  ◦ Euler Characteristic

• (Oriented) Mesh Representation
Correction

For implementing the trapezoidalization, we described using a sorting function which changes dynamically:

```cpp
float sweepHeight;

typedef function< bool ( const EKey &, const EKey & ) > EComparator;
EComparator eComparator = [&]( const EKey &k1, const EKey &k2 )
{
    // Compare the keys using the current value of sweepHeight
};
```
Correction

This is not necessary.

We could check if the $y$-spans of the two edges overlap.

- If they do not, call the lower edge “first”.

\[ e_2 < e_1 \]
Correction

This is not necessary.

We could check if the $y$-spans of the two edges overlap.

- If they do not, call the lower edge “first”.
- Otherwise, draw a horizontal line through some point on the overlap of the $y$-spans and sort based on that.

$e_1 < e_2$
Correction

However...

We can also use sweep-line algorithm to check if a closed (piecewise-linear) curve self-intersects by dynamically adding intersection events.
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(0,1) - (2,1) - (0,3) - (2,3)
Correction

However…

We can also use sweep-line algorithm to check if a closed (piecewise-linear) curve self-intersects by dynamically adding intersection events.

Note: Only need to check for intersections
1. Between adjacent edges in the active-edge list
2. Around newly added/removed edges

(0,1) - (2,1) - (0,3) - (2,3)
Polyhedra

Definition:

A polyhedron is a solid region in 3D space whose boundary is made up of planar polygonal faces comprising a connected 2D manifold.
Polyhedra

The boundary of a polyhedron can be expressed as a combination of vertices, edges, and faces:

- **Intersections are proper:**
  - Elements don’t overlap, or
  - They share a single vertex, or
  - They share an edge and the two vertices
Polyhedra

The boundary of a polyhedron can be expressed as a combination of vertices, edges, and faces:

- Intersections are proper
- Locally manifold:
  - Edges around a vertex can be sorted to match their incidence on adjacent faces.
Polyhedra

The boundary of a polyhedron can be expressed as a combination of vertices, edges, and faces:

- Intersections are proper
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Alternatively, the subgraph of the dual obtained by restricting to the adjacent faces (the link) is connected.
Polyhedra

The boundary of a polyhedron can be expressed as a combination of vertices, edges, and faces:

- Intersections are proper
- Locally manifold:
  - Edges around a vertex can be sorted to match their incidence on adjacent faces.
  - Exactly two faces meet at each edge.
Polyhedra

The boundary of a polyhedron can be expressed as a combination of vertices, edges, and faces:

- Intersections are proper
- Locally manifold
- Globally connected
Definition

Definition:

Given an edge on a polyhedron, the \textit{dihedral angle} of the edge is the internal angle between the two adjacent faces.

Aside:
The dihedral angle is a discrete measure of mean curvature.
Definition

Definition:

Given a vertex on a polyhedron, the \textit{deficit angle} at the vertex is $2\pi$ minus the sum of angles around the vertex.

$\Rightarrow \pi/2$

Aside:
The deficit angle is a discrete measure of Gauss curvature.
A convex polyhedron is a polytope:

- **Non-negative mean curvature:** All dihedral angles are less than or equal to $\pi$. (Necessary and sufficient.)
- **Non-negative Gaussian curvature:** Sum of angles around a vertex is at most $2\pi$. (Necessary but not sufficient.)
Platonic Solids

Definition:

A *regular polygon* is a polygon with equal sides and equal angles.
Platonic Solids

Definition:

A *regular polygon* is a polygon with equal sides and equal angles.

A *regular polyhedron* is a convex polyhedron, with all faces congruent regular polygons and vertices having the same valence.
Platonic Solids

Claim:
The five platonic solids are the only regular polyhedra.

[Images courtesy of Wikipedia]
Platonic Solids

Proof:

Assume each face is $p$-sided:

$\Rightarrow$ The sum of angles in a face is $\pi(p - 2)$

$\Rightarrow$ The angle at each vertex is $\pi(1 - 2/p)$

Assume each vertex has valence $\nu$:

$\Rightarrow$ The angle-sum at a vertex is $\nu\pi (1 - 2/p)$
Platonic Solids

Proof:

Since the polyhedron is convex:

\[ v \pi (1 - 2/p) < 2\pi \iff v(1 - 2/p) < 2 \]
\[ \iff v(p - 2) < 2p \]
\[ \iff vp - 2v - 2p < 0 \]
\[ \iff (p - 2)(v - 2) - 4 < 0 \]
Platonic Solids

Proof:

Since the polyhedron is convex:

\[(p - 2)(v - 2) - 4 < 0\]

Since \(p, v \geq 3\), valid options are \((p, v)\):

- (3,3)
- (3,4)
- (4,3)
- (3,5)
- (5,3)
Platonic Solids

The platonic solids come in dual pairs, where one solid is obtained from the other by replacing faces with vertices:

- Cube $\leftrightarrow$ Octahedron
- Icosahedron $\leftrightarrow$ Dodecahedron
- Tetrahedron $\leftrightarrow$ Tetrahedron

$(3,3)$ $(3,4)$ $(4,3)$ $(3,5)$ $(5,3)$
Topological Polyhedra

The boundary of a polyhedron can be expressed as a combination of vertices, edges, and faces:

- Intersections are proper
- Locally manifold
- Globally connected

\[ \{ \text{Geometric} \}
\[ \{ \text{Topological} \} \]
Topological Polyhedra

If we ignore the vertex positions, we get a combinatorial structure composed of faces (cells), edges, and vertices.*

[Nivoliers and Levy, 2013]

*These are CW complexes. (And, if faces are triangles, these are simplicial complexes).
Topological Polyhedra

Properties (CW Complex):

- Faces intersect at edges and vertices.
- Edges are topologically line segments and intersect at vertices.
- Interiors of faces have disk-topology and the boundary is a polygon made up of edges.
Topological Polyhedra

Properties (Manifold):

- Each vertex is on the boundary of some edge.
- Each edge is on the boundary of some face.
- Edges around a vertex can be sorted.
- An edge is on the boundary of two faces.
Topological Polyhedra

Note:

Given a topological polygon $P$, and given an edge $e \in P$ that only occurs once on $P$:

For any vertices $v_1, v_2 \in P$ there is a path from $v_1$ to $v_2$ that doesn’t pass through $e$. 

\[ v_1 \rightarrow \ldots \rightarrow v_2 \]
Topological Polyhedra

Claim:

If $f_1$ and $f_2$ are distinct faces of a topological polyhedron which share an edge $e$, then:

- replacing $f_1$ and $f_2$ with $f_1 \cup f_2$, and
- removing $e$ from the edge list,

we still have a valid topological polyhedron.
Topological Polyhedra

Proof (CW Complex):

The edges/vertices of \( f_1 \cup f_2 \) are in the complex.

*Since the intersection \( f_1 \cap f_2 \) is connected and the interiors of \( f_1 \) and \( f_2 \) have disk-topology, the interior of \( f_1 \cup f_2 \) also has disk-topology.

*This is just a sketch of the proof.
Topological Polyhedra

Proof (CW Complex):

The boundary of $f_1 \cup f_2$ is connected.

• Let $v \in e$ be an end-point.

• For $v_1, v_2 \in f_1 \cup f_2$, there is a curve connecting $v$ to each $v_i$ that does not contain the edge $e$.

• Concatenating the two curves we connect $v_1$ to $v_2$ along the boundary of $f_1 \cup f_2$. 
Topological Polyhedra

**Proof (Manifold):**

The smaller polyhedron still passes through all the vertices.

The edge $e$ is removed and all other edges remain adjacent to a face.
Topological Polyhedra

Proof (Manifold Edges):

The old edges still have only two faces on them (or one face twice).
Topological Polyhedra

Proof (Manifold Vertices):

If $v \notin e$, we can use the old edge ordering.
Topological Polyhedra

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Proof (Manifold Vertices):

If \( v \notin e \), we can use the old edge ordering.

If \( v \in e \) let \( \{e_1, e_2, ..., e_k\} \) be the old ordered edges around \( v \), shifted so that \( e_1 = e \).

Then \( e_k \) and \( e_2 \) are consecutive edges on \( f_1 \cup f_2 \) so \( \{e_2, ..., e_k\} \) is a valid ordering.
Topological Polyhedra

Proof (Manifold Vertices):

If \( v \notin e \), we can use the old edge ordering.

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Curves

A (connected) *curve* on a topological polyhedron is a list of edges such that the ending vertex of one edge is the starting vertex of the next.
Curves

A (connected) curve on a topological polyhedron is a list of edges such that the ending vertex of one edge is the starting vertex of the next.

A closed curve is a curve whose starting and ending points are the same.
Genus-0 Polyhedra

A polyhedron is *genus-0* (or *simply connected*) if every non-trivial closed curve disconnects the faces of the polyhedron.
Aside:

The definition can be extended to surfaces with boundary if curves that start and end at the boundary are also considered closed.
Genus-0 Polyhedra

Equivalently, given a topological polyhedron $P$, we can define the dual graph $P^* = (V^*, E^*)$.

$\Rightarrow$ A curve $C \subset E$ corresponds to a set of dual edges $C^* \subset E^*$ of the dual.

$\Rightarrow$ $P$ is genus-0 if removing $C^*$ disconnects $P^*$. 

Genus-0 Polyhedra

1. There is a continuous map from a polytope to a sphere.
   (e.g. Put the center of mass at the origin and normalize the positions.)

2. By the Jordan Curve Theorem the sphere is genus-zero.

One Can Show:

⇒ The polytope must also be genus-0.
Euler’s Formula

For a genus-0 polyhedron $P$, the number of vertices, $|V|$, the number of edges, $|E|$, and the number of faces, $|F|$, satisfy:

$$|V| - |E| + |F| = 2$$
Euler’s Formula (by Induction on $|F|$)

**Base case:** $|F| = 1$

We have:

- $V = \{v_1, ..., v_n\}$

*The edges on the boundary of the face form a connected tree (otherwise there is a closed loop and the interior of the face is disconnected).

Then there are $n - 1$ edges:

$$|V| - |E| + |F| = n - (n - 1) + 1 = 2$$

*This is just a sketch of the proof.*
Euler’s Formula (by Induction on $|F|$)

**Induction**: Assume true for $|F| = n - 1$

Find $e \in E$ shared by two distinct faces.

If no such $e$ exists, then all faces are adjacent to themselves, which contradicts the assumption that the polyhedron is connected.
Euler’s Formula (by Induction on $|F|$)

**Induction**: Assume true for $|F| = n - 1$

Find $e \in E$ shared by two distinct faces.

Remove $e$ and merge the two adjoining faces, $f_1$ and $f_2$.

**Claim**:

The new polyhedron, $P'$, is still genus-0.
Euler’s Formula (by Induction on $|F|$)

Proof ($P'$ is genus-zero):

Let $C$ be a non-trivial curve on $P'$.

$\Rightarrow$ $C$ is a non-trivial curve on $P$ with $e \notin C$.

$\Rightarrow f_1$ and $f_2$ are in the same component.

$\Rightarrow C$ disconnects $f_1 \cup f_2$ from a face $g$ on $P$.

$\Rightarrow C$ disconnects $f_1 \cup f_2$ from $g$ in $P'$. 
**Euler’s Formula (by Induction on $|F|$)**

**Induction:** Assume true for $|F| = n - 1$

Find $e \in E$ shared by two distinct faces.

Remove $e$ and merge the two adjoining faces.

$P'$ is genus-0 with $|E| - 1$ edges, $|F| - 1$ faces, and $|V|$ vertices.

By the induction hypothesis we have:

$$|V| - (|E| - 1) + (|F| - 1) = 2$$

$\uparrow$

$$|V| - |E| + |F| = 2$$
Euler’s Formula

\[ |V| - |E| + |F| = 2 \]

More Generally:

If a polygon mesh is genus-\(g\) (\(g\) is the number of handles) then:

\[ |V| - |E| + |F| = 2 - 2g. \]

\( |V| = 24, \ |E| = 48, \ |F| = 24 \)

[Wikipedia: Toroidal Polyhedron]
Euler’s Formula

Implication:
The number of faces and edges is linear in the number of vertices.
Euler’s Formula

**Proof:**

Assume all faces are triangles. (Triangulating only increase $|F|$ and $|E|$.)

Since each edge is shared by two triangles:

$$|E| = 3|F|/2$$

Using Euler’s Formula:

$$|V| - |E| + |F| = 2$$

$\Updownarrow$

$$|F| = 2|V| - 4 \quad \text{and} \quad |E| = 3|V| - 6$$
Outline

• Polyhedra

• (Oriented) Mesh Representation
  ○ Face-vertex data-structure
  ○ Winged-edge data-structure
(Oriented) Mesh Representation

Face-Vertex Lists:

Most often (e.g. ply, obj, etc. formats) polygon meshes are represented using vertex and face lists:

- **Vertex Entry**: $(x, y, z)$ coordinates.
- **Face Entry**: Count and CCW indices of the vertices.
(Oriented) Mesh Representation

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### Face List

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Limitation:
- Variable sized rows
- No explicit connectivity
(Oriented) Mesh Representation

Winged-Edge List:

Common representation for connectivity querying, represented using vertex, half-edge, and face lists:

- **Vertex Entry:**
  - (x, y, z) coordinates
  - Outgoing h.e. index

- **Face Entry:**
  - h.e. index

- **Half-Edge Entry:**
  - in/out wing h.e. indices
  - opposite h.e. index
  - end vertex index
  - face index
Mesh Representation

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Oriented Mesh Representation

A common representation for connectivity querying, represented using vertex, half-edge, and face lists:

- **Vertex Entry**: 
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- **Outgoing h.e. index**
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- **Half-Edge Entry:**
  - in/out wing h.e. indices
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Oriented Mesh Representation

Winged Edge List: Common representation for connectivity querying, represented using vertex, half-edge, and face lists:

- **Vertex Entry:**
  - \((x, y, z)\) coordinates
  - Outgoing h.e. index

- **Face Entry:**
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- **Half-Edge Entry:**
  - Example:

```
Find CCW vertices around \(v_1\):
```
### Oriented Mesh Representation

**Winged-Edge List:**

Common representation for connectivity querying, represented using vertex, half-edge, and face lists:

- **Vertex Entry:**
  - \((x, y, z)\) coordinates
  - Outgoing half-edge index

- **Face Entry:**
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- **Half-Edge Entry:**
  - Example:

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</tbody>
</table>

- **Outgoing half-edge index**
- **Face Entry:**
  - h.e. index
  - **Half-Edge Entry:**

**Example:**

Find CCW vertices around \(v_1\):
Oriented Mesh Representation

Winged Edge List:

- Common representation for connectivity querying, represented using vertex, half-edge, and face lists:

  - **Vertex Entry**:
    - \((x, y, z)\) coordinates
  - **Face Entry**:
    - Half-edge index
  - **Half-Edge Entry**:
    - Example:

Find CCW vertices around \(v_1: v_3\)
**Oriented** Mesh Representation

**Winged-Edge List:**
Common representation for connectivity querying, represented using vertex, half-edge, and face lists:

**Vertex Entry:**
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- Outgoing h.e. index

**Face Entry:**
- h.e. index

**Half-Edge Entry:**

Example:
Find CCW vertices around \(v_1: v_3\)
Common representation for connectivity querying, represented using vertex, half-edge, and face lists:

- **Vertex Entry:**
  - \((x, y, z)\) coordinates
  - Outgoing h.e. index

- **Face Entry:**
  - h.e. index

- **Half-Edge Entry:**
  - \(w_1\), \(w_0\), \(v\), \(f\)

**Example:**

Find CCW vertices around \(v_1 : v_3\)
**Oriented Mesh Representation**

Common representation for connectivity querying, represented using vertex, half-edge, and face lists:

- **Vertex Entry**:
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- **Face Entry**:
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- **Half-Edge Entry**:
  - Example:

Find CCW vertices around \(v_1\): \(v_3, v_4\)
Winged-Edge List: Common representation for connectivity querying, represented using vertex, half-edge, and face lists:

- **Vertex Entry:**
  - \((x, y, z)\) coordinates
  - Outgoing h.e. index

- **Face Entry:**
  - h.e. index

- **Half-Edge Entry:**
  - Example:

**Example:**
Find CCW vertices around \(v_1: v_3, v_4\)
Winged-Edge List: Common representation for connectivity querying, represented using vertex, half-edge, and face lists:

- **Vertex Entry:**
  - \((x, y, z)\) coordinates
  - Outgoing h.e. index

- **Face Entry:**
  - h.e. index

- **Half-Edge Entry:**
  - Example:

Find CCW vertices around \(v_1: v_3, v_4\)
**Oriented Mesh Representation**

**Winged-Edge List:**
- Common representation for connectivity querying, represented using vertex, half-edge, and face lists:

- **Vertex Entry:**
  - \((x, y, z)\) coordinates
  - Outgoing half-edge index

- **Face Entry:**
  - Half-edge index

- **Half-Edge Entry:**
  - Example:

<table>
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<th>(w_o)</th>
<th>v</th>
<th>f</th>
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- **Outgoing half-edge index**
- **Face Entry:**
  - h.e. index
- **Half-Edge Entry:**

**Example:**

Find CCW vertices around \(v_1\): \(v_3, v_4, v_2\)
Oriented Mesh Representation

### Winged-Edge List:

Common representation for connectivity querying, represented using vertex, half-edge, and face lists:

- **Vertex Entry**
  
  » \((x, y, z)\) coordinates

- **Outgoing h.e. index**

- **Face Entry**
  
  » h.e. index

- **Half-Edge Entry**

Example:

Find CCW vertices around \(v_1: v_3, v_4, v_2\)
Oriented Mesh Representation

**Winged-Edge List:**
Common representation for connectivity querying, represented using vertex, half-edge, and face lists:

- **Vertex Entry:**
  - \((x, y, z)\) coordinates
  - Outgoing half-edge index

- **Face Entry:**
  - Half-edge index

- **Half-Edge Entry:**
  - Example:
    - Find CCW vertices around \(v_1\): \(v_3, v_4, v_2\)

<table>
<thead>
<tr>
<th>Vertex List</th>
<th>Face List</th>
<th>Half-Edge List</th>
</tr>
</thead>
<tbody>
<tr>
<td>Id</td>
<td>x</td>
<td>y</td>
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</table>

- **Outgoing half-edge index**
- **Face Entry:**
  - **h.e. index**
- **Half-Edge Entry:**

![Diagram](image-url)
Oriented Mesh Representation

Winged-Edge List:

Common representation for connectivity querying, represented using vertex, half-edge, and face lists:

- **Vertex Entry**:
  - \((x, y, z)\) coordinates
  - Outgoing h.e. index

- **Face Entry**:
  - h.e. index

- **Half-Edge Entry**:

Example:

Find CCW vertices around \(v_1\): \(v_3, v_4, v_2\)

- **Vertex List**
  - | Id | x | y | z | h |
  - |----|---|---|---|---|
  - | 1  | -1| -1| 0 | 4 |
  - | 2  | 0 | 0 |-1| 2 |
  - | 3  | 1 |-1| 0 | 3 |
  - | 4  | -1| 1| 0 | 6 |
  - | 5  | 1 | 1| 0 |...

- **Face List**
  - | Id | h |
  - |----|---|
  - | 1  | 4 |
  - | 2  | 3 |
  - | 3  | 5 |
  - | 4  |...
  - | 5  |...

- **Half-Edge List**
  - | Id | o | w_i | w_o | v | f |
  - |----|---|------|------|---|---|
  - | 1  | 2 | 3    | ...  | 2 | 2 |
  - | 2  | 1 | ...  | 5    | 1 | 3 |
  - | 3  | 4 | ...  | 1    | 1 | 2 |
  - | 4  | 3 | 6    | ...  | 3 | 1 |
  - | 5  | 6 | 2    | ...  | 4 | 3 |
  - | 6  | 5 | ...  | 4    | 1 | 1 |

Computational complexity is linear in output size.

- Outgoing h.e. index
  - **Face Entry**:
    - h.e. index
  - **Half-Edge Entry**:

Example:

Find CCW vertices around \(v_1\): \(v_3, v_4, v_2\)
(Oriented) Mesh Representation

Goal:

Given a face-vertex representation of a mesh $(V,F)$, convert it to a winged-edge representation $(V,E,F)$. 
(Oriented) Mesh Representation

Goal:

Given a face-vertex representation of a mesh \((V,F)\), convert it to a winged-edge representation \((V,E,F)\).

Warning:

The following discussion assumes that in a mesh, a (directed) edge is uniquely determined by its starting and ending vertices.

This does not have to be true.
(Oriented) Mesh Representation

GenerateHalfEdge(V, F, _V, _E, _F)

_V.resize(v.size()) , _F.resize(F.size())
for( i=0 ; i<_V.size() ; i++ ) _V[i].p = V[i]

unordered_map<VertexPair, int> eMap
ConstructEdgeToFaceMap(F, eMap)

_E.resize(eMap.size())

SetHalfEdgeIndices(eMap, _V, _E, _F)

Assuming that:
• The VertexPair object defines a hashing function
(Oriented) Mesh Representation

ConstructEdgeToFaceMap( F, eMap )
  for( f=0 ; f<F.size() ; f++ )
    for( v=0 ; v<F[f].size() ; v++ )
      VertexPair key( F[f][v] , F[f][v+1] )
      eMap[key] = f

Assuming that:
- Indexing is modulo the face size
(Oriented) Mesh Representation

SetHalfEdgeIndices( eMap , _V , _E , _F )

int e = 0
for( iter i=eMap.begin() ; i!=eMap.end() ; i++ , e++ )
    int v1 = i.key.first , v2 = i.key.second , f = i.value
    _E[e].v = v2 , _E[e].f = f
    _V[v1].he = _F[f].he = i.value = e
for( f=0 ; f<F.size() ; f++ ) for( v=0 ; v<F[f].size() ; v++ )
    VertexPair key( F[f][v ] , F[f][v+1] )
    VertexPair oKey( F[f][v+1] , F[f][v ] )
    VertexPair nKey( F[f][v+1] , F[f][v+2] )
    _E[ eMap[ key ] ].opposite = eMap[ oKey ]
    _E[ eMap[ key ] ].next = eMap[ nKey ]
    _E[ eMap[ nKey ] ].previous = eMap[ key ]