Convex Hulls (2D)

O’Rourke, Chapter 3

Announcements

- For the assignments, polygon meshes (including triangulations) are represented using indirection.

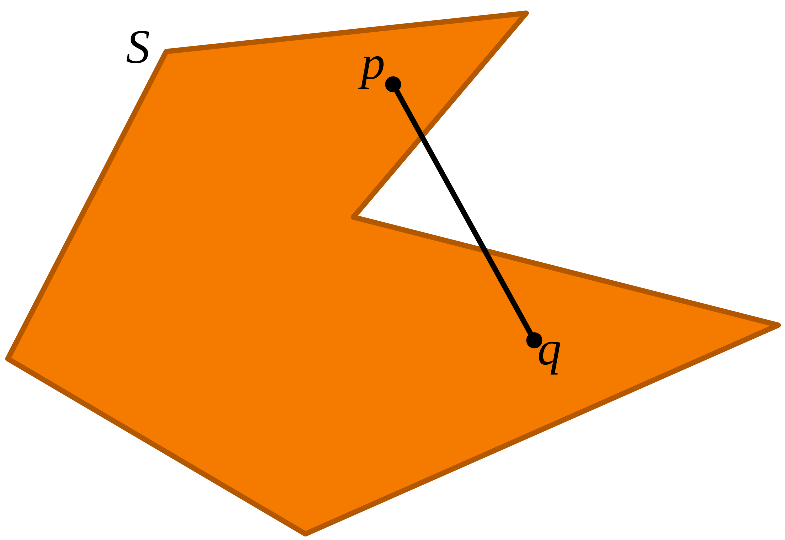
  That is, we store an array of vertices (coordinates in $\mathbb{R}^d$) and a polygon is represented as an array of indices into the vertex list.

Outline

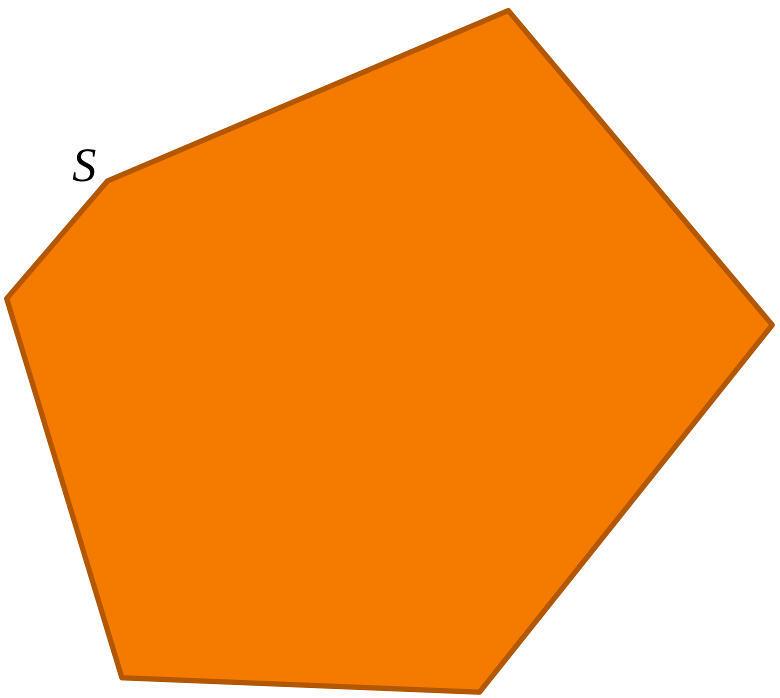
- Convex Hulls
- Algorithms
  - Naïve Implementation(s)
  - Gift Wrapping
  - Quick Hull
  - Graham’s Algorithm
  - Lower bound complexity

Convexity

A set S is *convex* if for any two points $p, q \in S$ the line segment $\overline{pq} \subset S$.



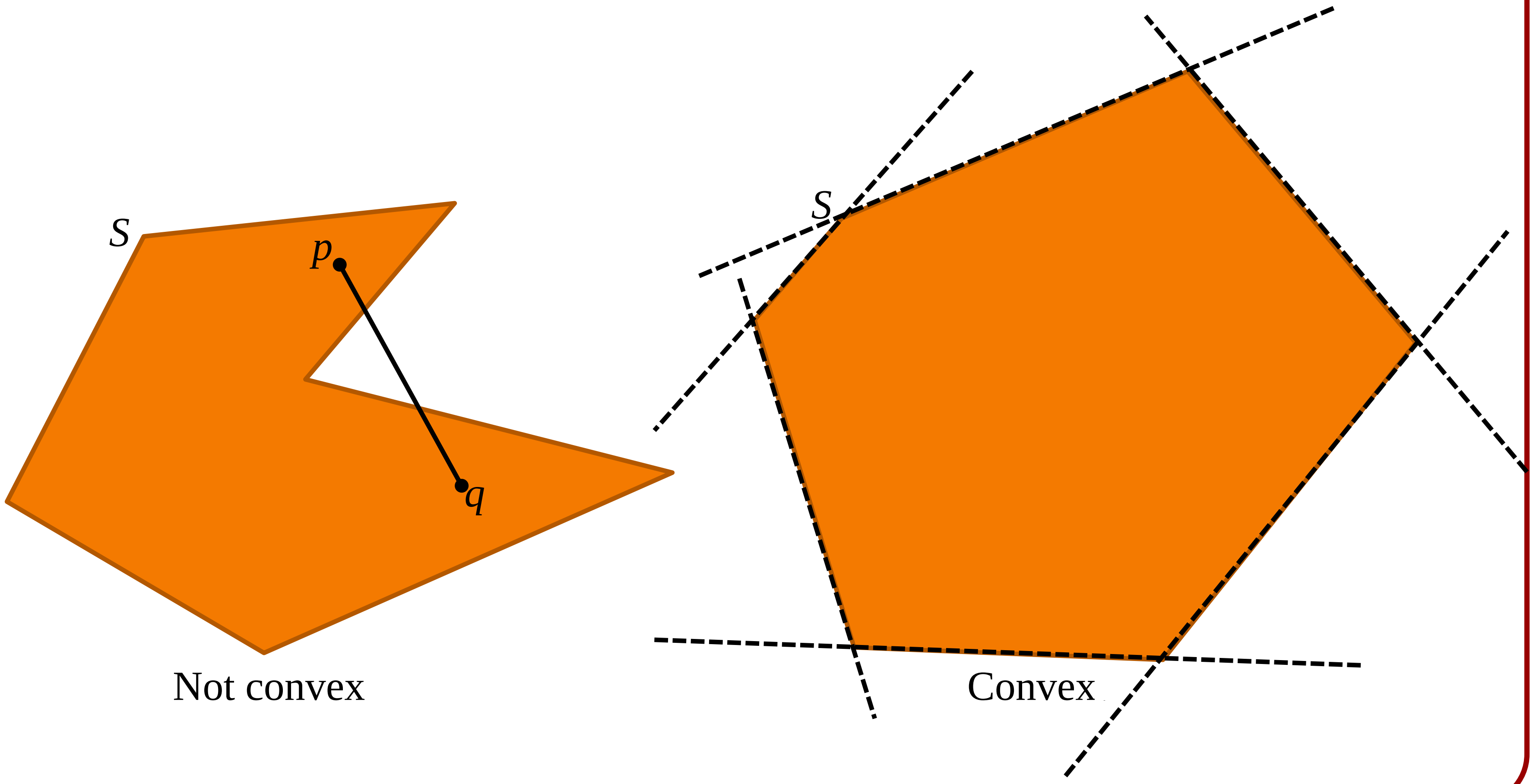
Not convex



Convex?

Convexity

A set S is *convex* if it is the intersection of (possibly infinitely many) half-spaces.



Convexity

Given points $\{p_1, \ldots, p_n\} \subset \mathbb{R}^d$, a point $q \in \mathbb{R}^d$ is a *convex combination* of the points if q can be expressed as the linear sum:

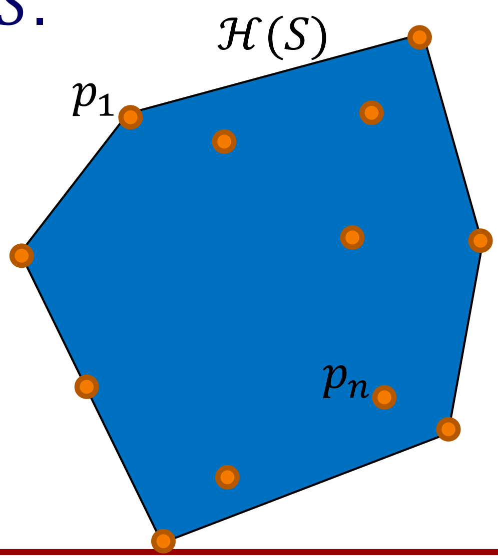
$$q = \sum_{i=1}^{n} \alpha_i \cdot p_i$$

with $\alpha_i \geq 0$ and $\alpha_1 + \cdots + \alpha_n = 1$.

Convex Hull

The *convex hull* of a set of points $S \subset \mathbb{R}^d$, denoted $\mathcal{H}(S)$, is the:

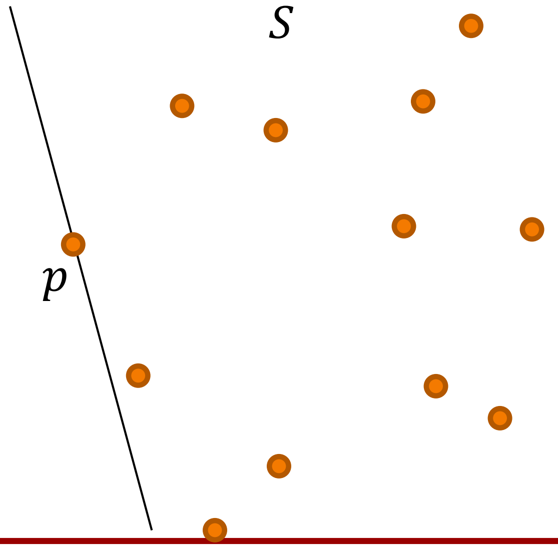
- set of all convex combinations of points in S,
- set of all convex combinations of $d + 1$ points in S,
- intersection of all convex sets C w/ $S \subset C$,
- intersection of all half-spaces H w/ $S \subset H$,
- smallest convex polygon containing S.



Convex Hull

Note:

If $p \in S \subset \mathbb{R}^2$ and p lies on a line with such that all points of S are to one side of the line, then p must be in on the boundary of the convex hull.



Convex Hull

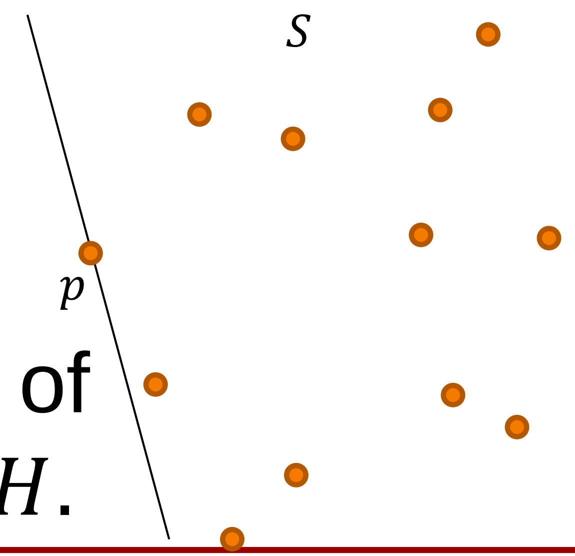
Proof:

The convex hull is the intersection of all half-spaces H w/ $S \subset H$.

Any such H must contain $p \in S$ either inside or on the boundary.

At least one such H has p on the boundary.

$\Rightarrow p$ must be on the boundary of the intersection of all such H.

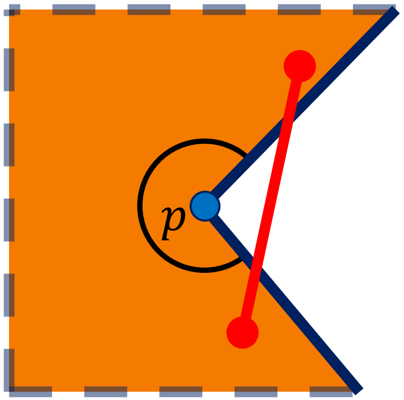


Convex Hull

Note:

If $p \in S \subset \mathbb{R}^2$ and p is a vertex of the convex hull then p must be a convex vertex.

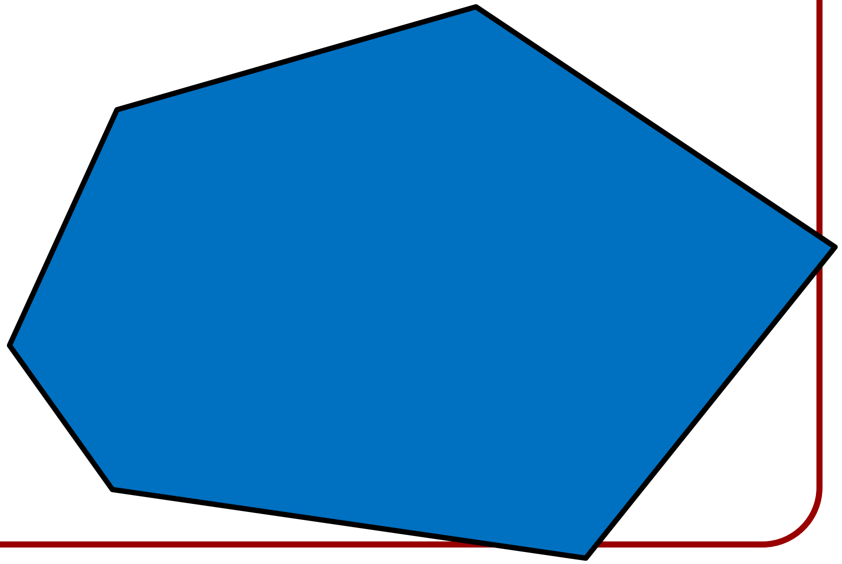
Otherwise, we could create a line segment with vertices inside of the hull but which isn't strictly interior.



Convex Hull

Claim:
If $P \subset \mathbb{R}^2$ is a polygon whose vertices are all convex, then P is convex.

We will show that if it is not convex, some vertex must be reflex.

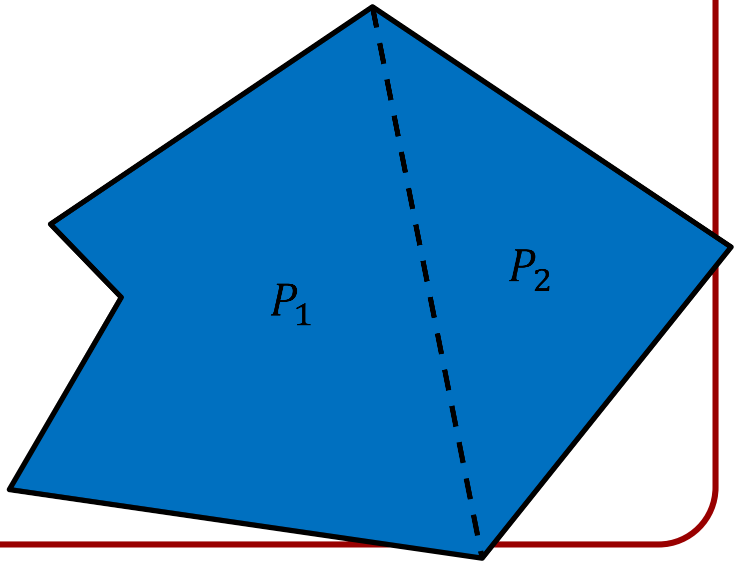


Convex Hull

Proof (by induction):

Otherwise, we could add a diagonal partitioning into sub-polygons P_1 and P_2.

⇒ By induction, each half is convex.



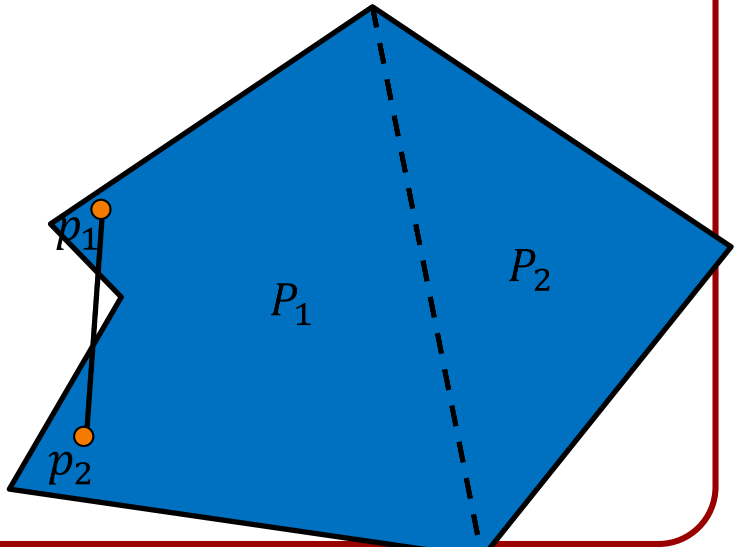
Convex Hull

Proof (by induction):

If P is not convex there must be a segment $\overline{p_1 p_2}$ between the two parts that exits P.

If p_1 and p_2 are in the same sub-polygon P_i, the line segment between them must be in the sub-polygon as well.

$\Rightarrow$ The line segment $\overline{p_1 p_2}$ cannot exit P.



Convex Hull

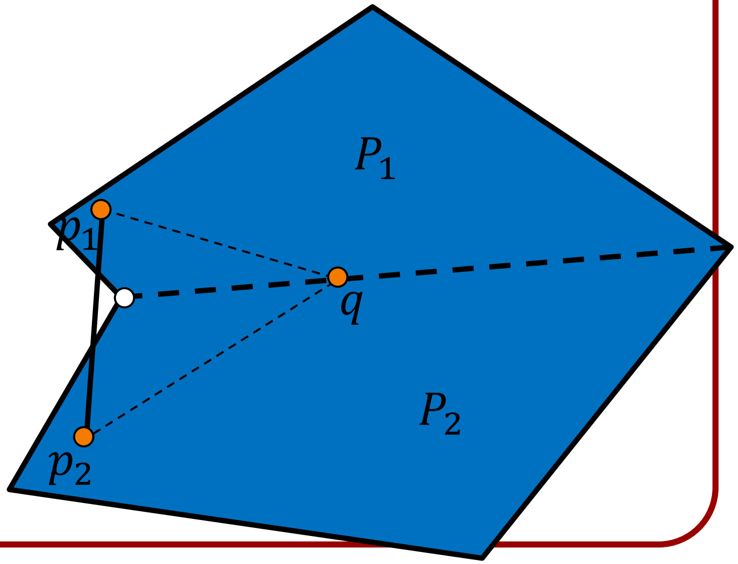
<u>Proof (by induction)</u>:

So, WLOG, $p_1 \in P_1$ and $p_2 \in P_2$.

Choose q on the diagonal.

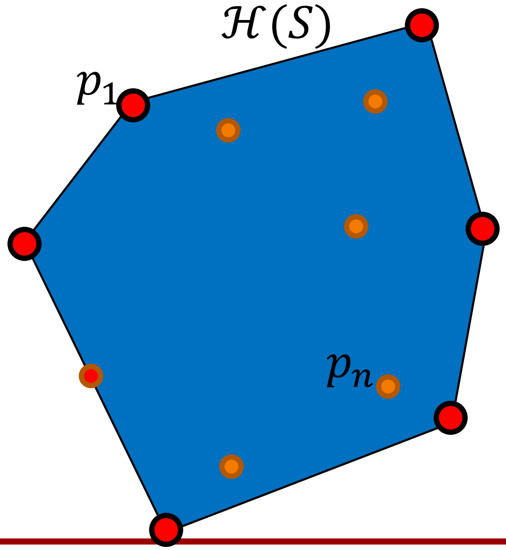
The edges $\overline{p_1 q}$ and $\overline{p_2 q}$ are entirely inside P.

$\Rightarrow$ There must be a reflex vertex inside triangle $\Delta q p_2 p_1$.



Convex Hull

The *extreme points* of a set of points $S \subset \mathbb{R}^2$ are the points which are on the convex hull and have interior angle strictly less than π.

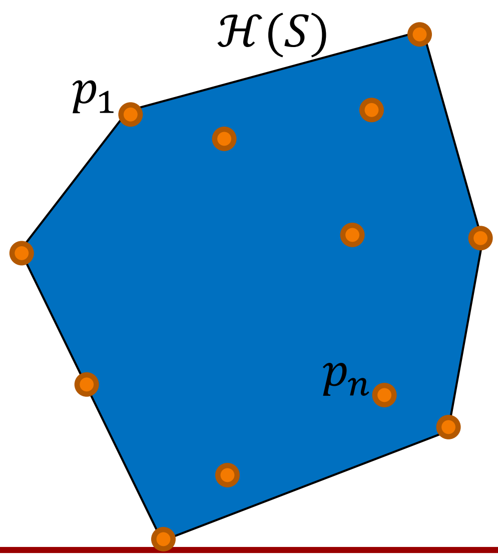


Convex Hull

Goal:

Given a set of points $S = \{p_1, \ldots, p_n\} \subset \mathbb{R}^d$, compute the convex hull $\mathcal{H}(S)$ efficiently.

- Do we want all points on the hull or just the extreme ones?
- Do the output vertices need to be sorted or is the set of (extreme) vertices sufficient?



Convex Hull

<u>Goal</u>:

Given a set of points $S = \{p_1, \dots, p_n\} \subset \mathbb{R}^d$, compute the convex hull $\mathcal{H}(S)$ efficiently.

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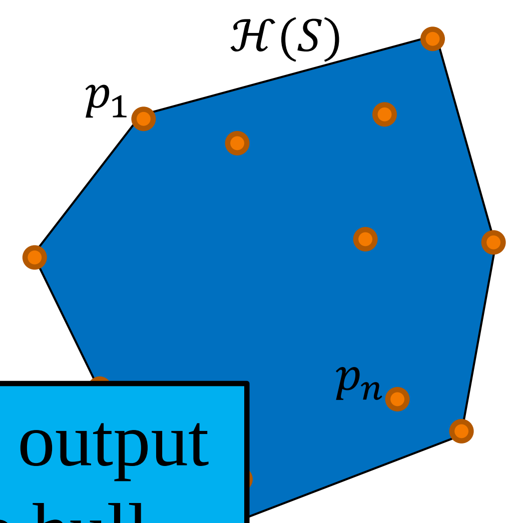


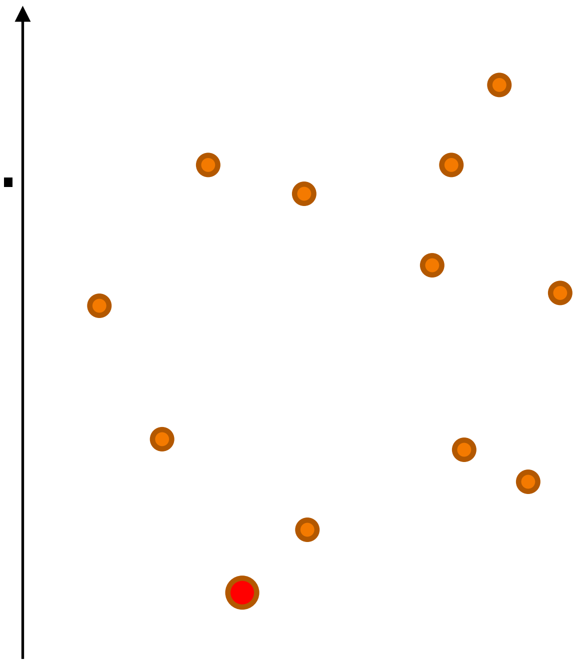
We will focus on the ordered output of the extreme points on the hull.

Convex Hull (2D)

<u>Note</u>:

If a vertex is extremal with respect to some direction, it must be on the hull.

$\Rightarrow$ We can find a hull vertex in linear time.



Outline

- Convex Hulls
- Algorithms
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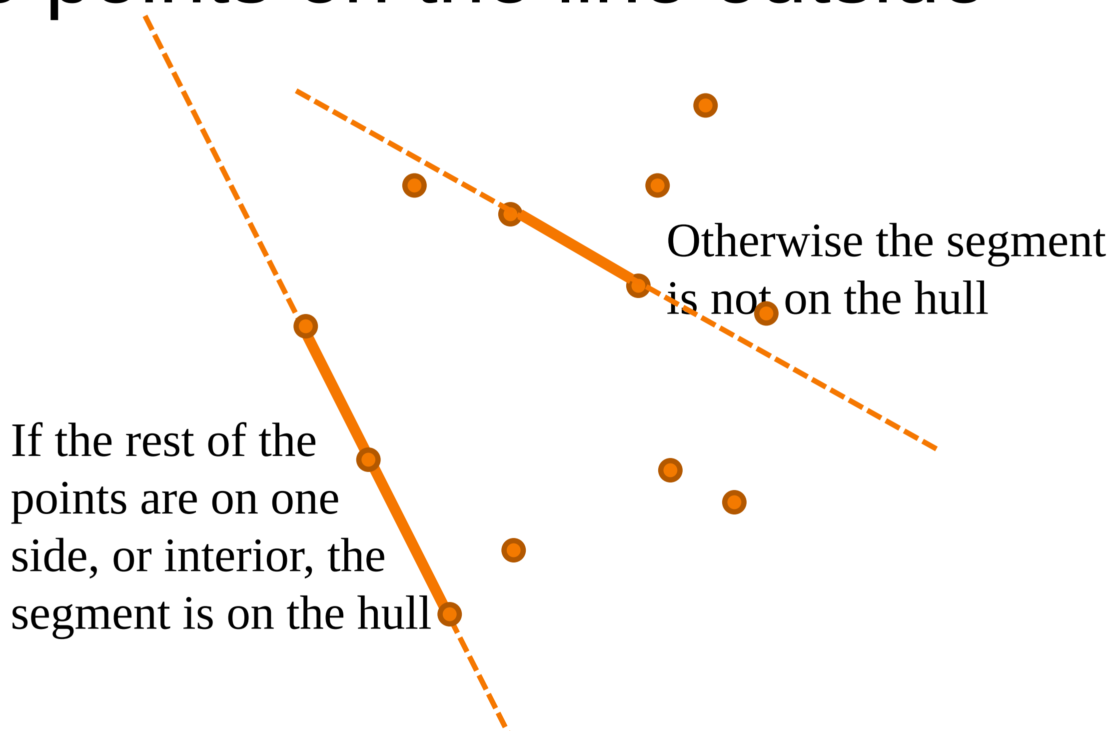
Convex Hull (2D)

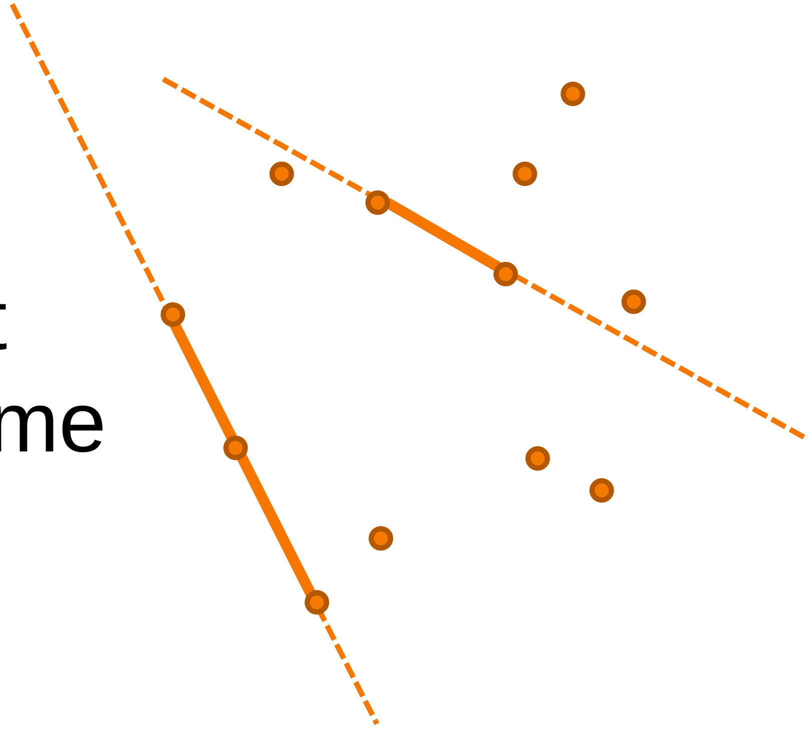
Naïve Algorithm:
For each directed edge $e \in S \times S$, check if the half-space to the right of e is empty of points (and there are no points on the line outside the segment).



Convex Hull (2D)

<u>Naïve Algorithm $O(n^3)$</u>:
For each directed edge $e \in S \times S$, check if the half-space to the right of e is empty of points (and there are no points on the line outside the segment).

<u>Note</u>:
The output is the set of (unordered) extreme points on the hull.



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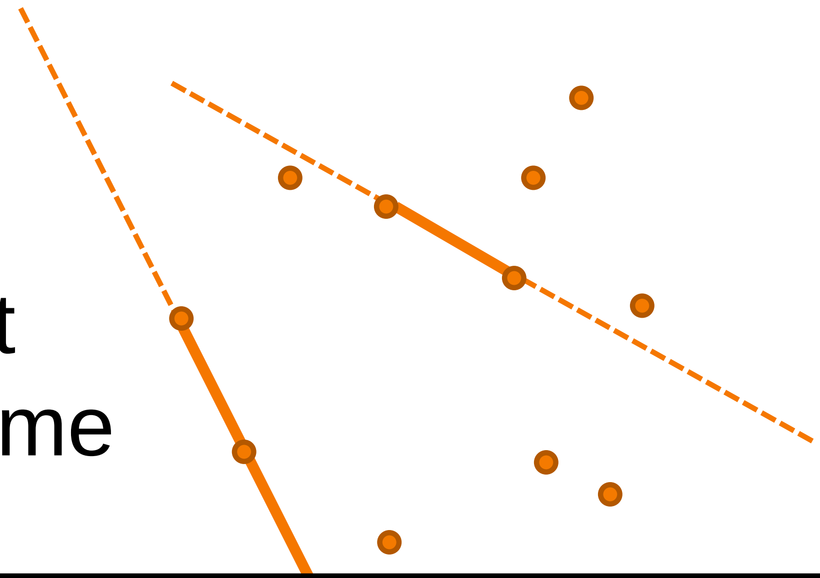


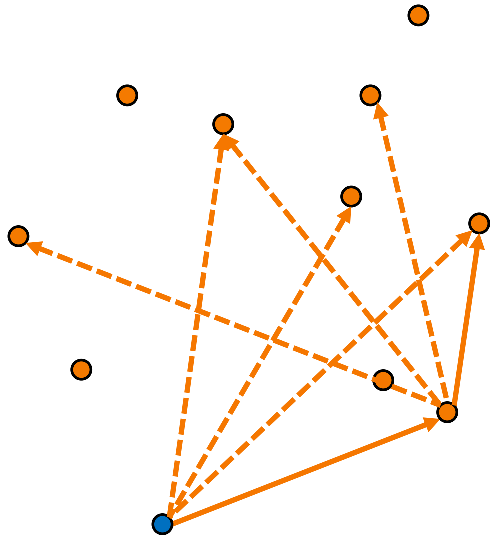
<u>Note</u>:
The output is the set of (unordered) extreme points on the hull.

If we want the ordered points, we can stitch the edges together in $\leq O(n^2)$ time in a post-processing step.

Convex Hull (2D)

Naïve Algorithm++:

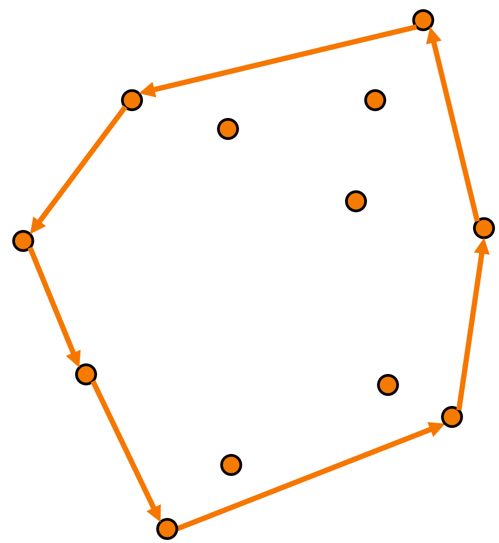
Grow the hull by starting at a hull vertex and searching for the next edge on the hull by trying all possible edges and testing if they are on the hull.



Convex Hull (2D)

Naïve Algorithm++ $O(n^2 h)^*$:
Grow the hull by starting at a hull vertex and searching for the next edge on the hull by trying all possible edges and testing if they are on the hull.

Note:
By explicitly forcing the output to be sorted, we end up with a faster algorithm.



$^*h$ is the number of points on the hull.

Convex Hull (2D)

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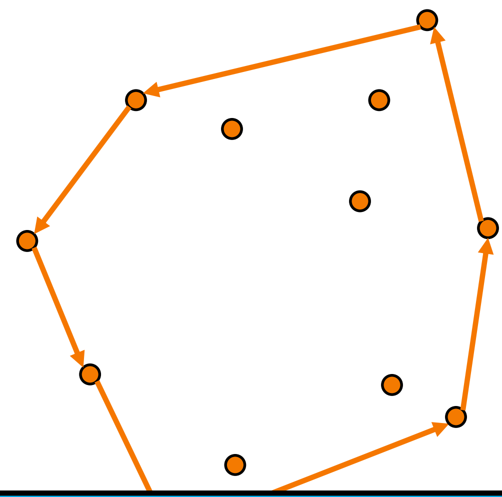

This implementation is *output sensitive*.

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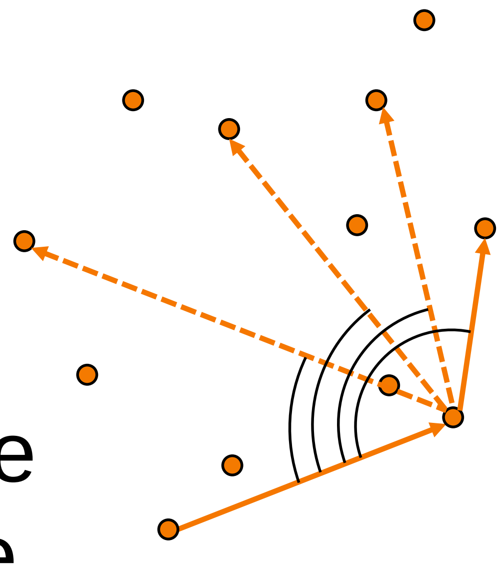
Convex Hull (2D)

Note:
The next edge on the hull is the one making the largest angle. (If two points make the same angle, ignore the closer one.)



Gift Wrapping:
Grow by finding the edge making the largest angle.

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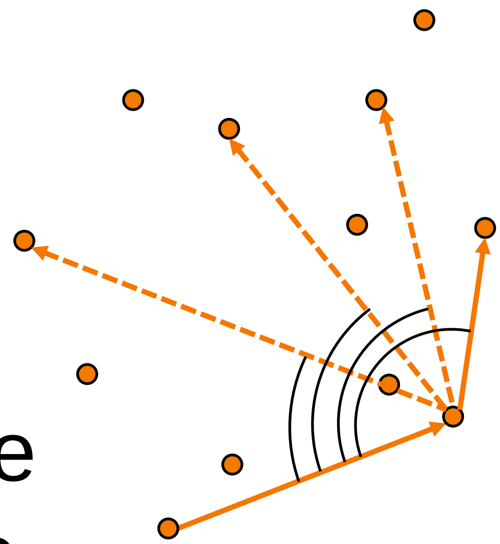

Gift Wrapping $O(nh)$:

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Convex Hull (2D)

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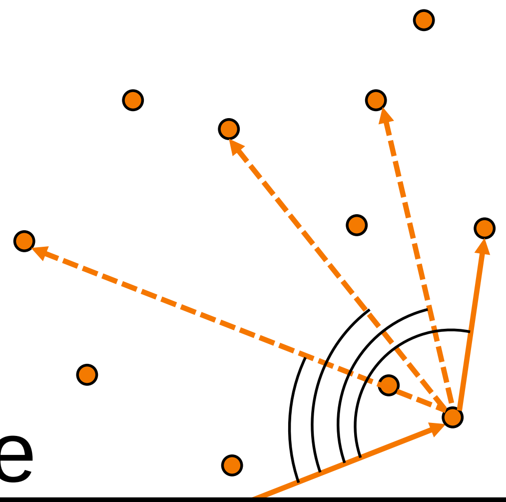

<u>Gift Wrapping $O(nh)$</u>:
Grow by finding the edge
mak

A similar approach makes it possible to find a hull edge in linear time.

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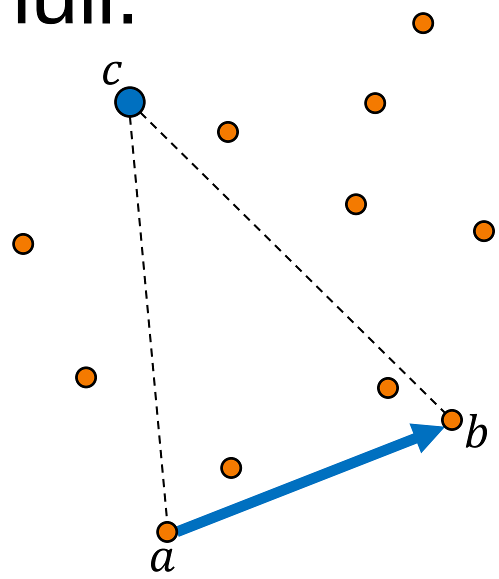
Convex Hull (2D)

Observation:

Given a hull edge (a, b), we can find the point c furthest from the edge in linear time.

1. The point c is on the hull.
2. The triangle Δabc partitions the input into three regions:
   I. Points inside Δabc.
   II. Points to the right of $\overrightarrow{bc}$.
   III. Points to the right of $\overrightarrow{ca}$.



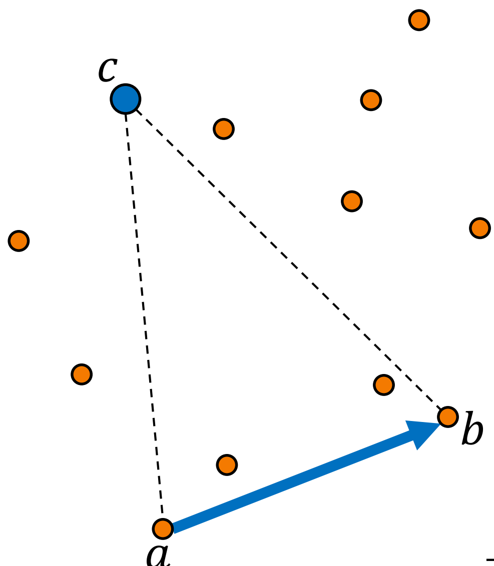
Convex Hull (2D)

Observation:

Given a hull edge (a, b), we can find the point c furthest from the edge in linear time.

$\Rightarrow$ Divide-and-conquer:

- Discard points inside Δabc
- Separately compute the half-hulls* to the right of $\overrightarrow{bc}$ and the right of $\overrightarrow{ca}$.
- Merge the two hulls.

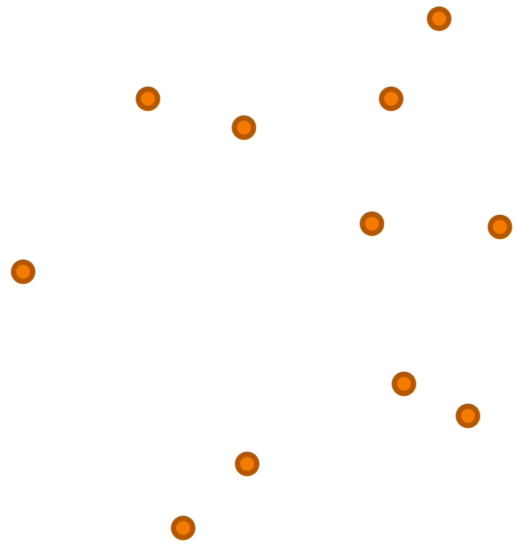


*The half hull to the right of $\overrightarrow{bc}$ is the hull of the points on or to the right of $\overrightarrow{bc}$, (i.e. including b and c) not including the edges touching either b or c.

Convex Hull (2D)

QuickHull($S \subset \mathbb{R}^2$)

- $(a, b) \leftarrow$ HorizontalExtrema(S)
- $A \leftarrow$ Right(S , $\overrightarrow{ab}$)
- $B \leftarrow$ Right(S , $\overrightarrow{ba}$)
- $Q_A \leftarrow$ QuickHalfHull(A , $\overrightarrow{ab}$)
- $Q_B \leftarrow$ QuickHalfHull(B , $\overrightarrow{ba}$)
- return $\{a\} \cup Q_A \cup \{b\} \cup Q_B$



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- $(\boldsymbol{a}, \boldsymbol{b}) \leftarrow$ **HorizontalExtrema(S)**
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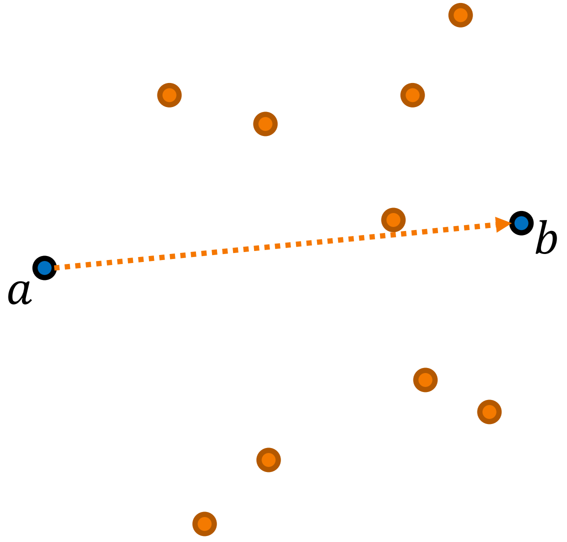

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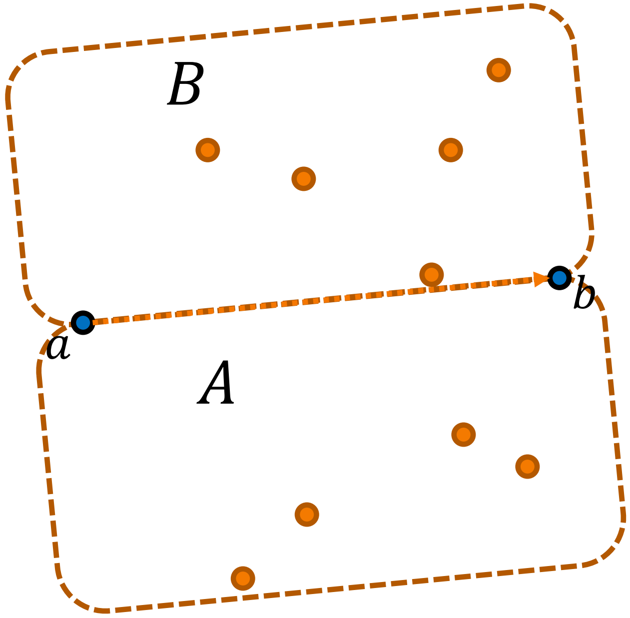

Convex Hull (2D)

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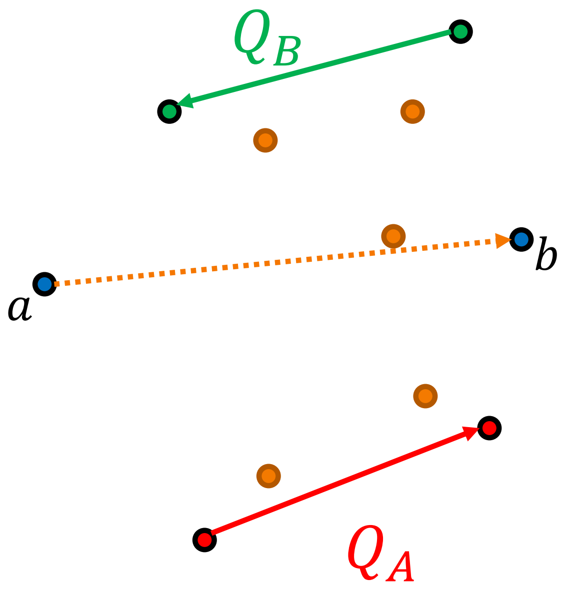

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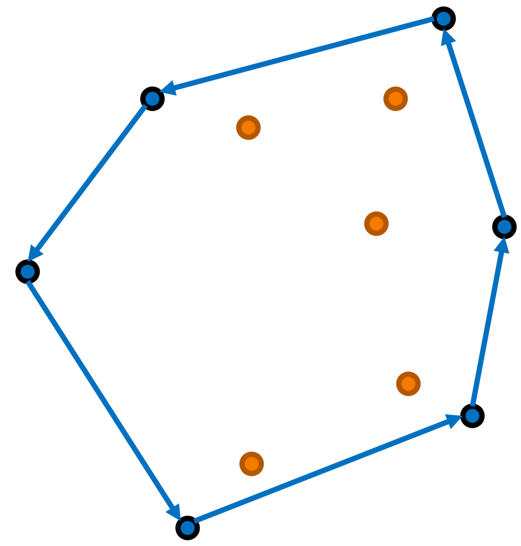

Convex Hull (2D)

(Recursion Level 0)

QuickHalfHull($S \subset \mathbb{R}^2, \overrightarrow{ab} \in S \times S$)

- if($S == \{a, b\}$) return $\emptyset$
- else
  - » $c \leftarrow$ Furthest(S , $\overrightarrow{ab}$)
  - » $A \leftarrow$ RightOf(S , $\overrightarrow{ac}$)
  - » $B \leftarrow$ RightOf(S , $\overrightarrow{cb}$)
  - » $Q_A \leftarrow$ QuickHalfHull(A , $\overrightarrow{ac}$)
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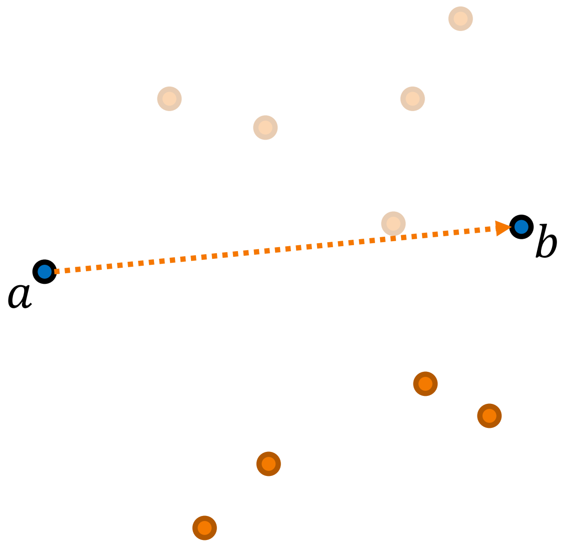
  - » return $Q_A \cup \{c\} \cup Q_B$



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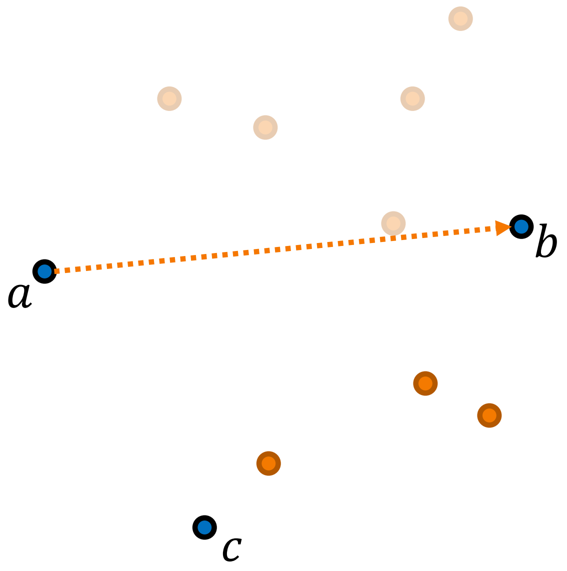

Convex Hull (2D)

(Recursion Level 0)

QuickHalfHull($S \subset \mathbb{R}^2, \overrightarrow{ab} \in S \times S$)

- if($S == \{a, b\}$) return ∅
- else
  - » $c \leftarrow$ Furthest(S , $\overrightarrow{ab}$)
  - » $\boldsymbol{A} \leftarrow$ **RightOn(** $\boldsymbol{S}$ **,** $\boldsymbol{\overrightarrow{ac}}$ **)**
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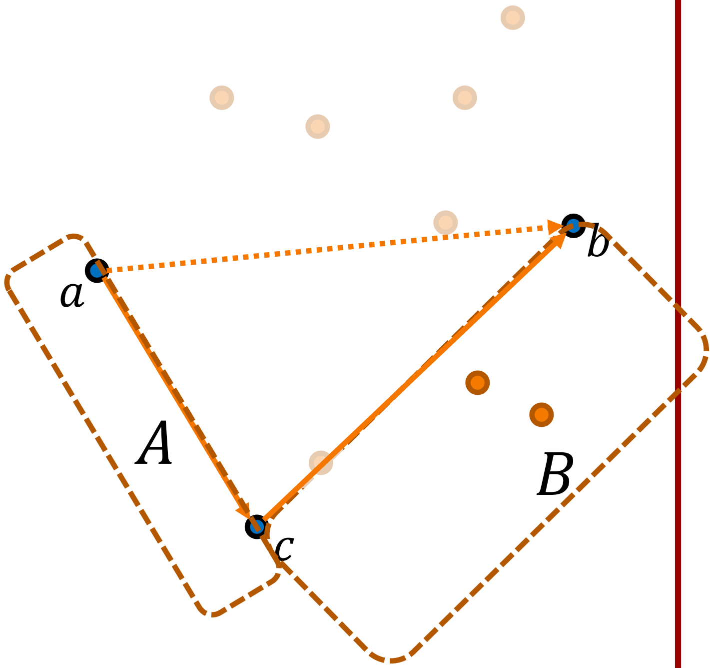

Convex Hull (2D)

(Recursion Level 0)

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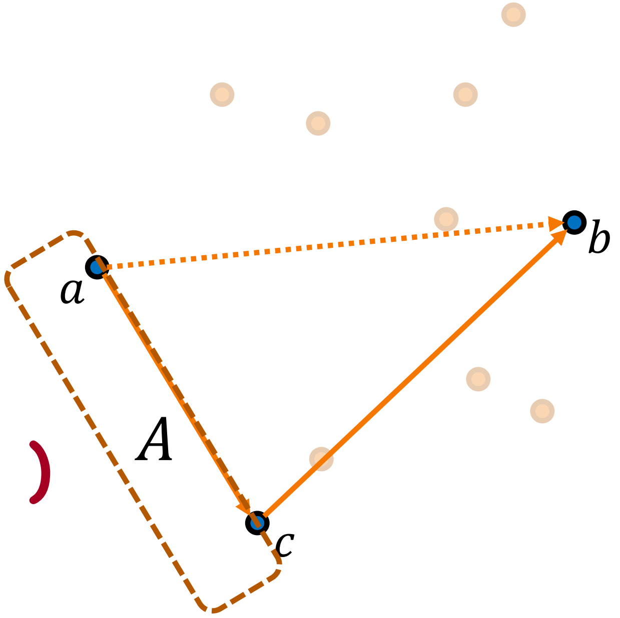

Convex Hull (2D)

(Recursion Level 1)

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- **if(** $S == \{a, b\}$ **) return** $\emptyset$
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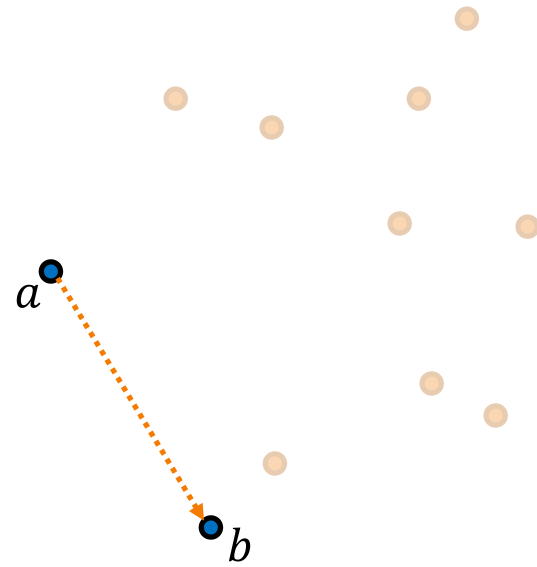

Convex Hull (2D)

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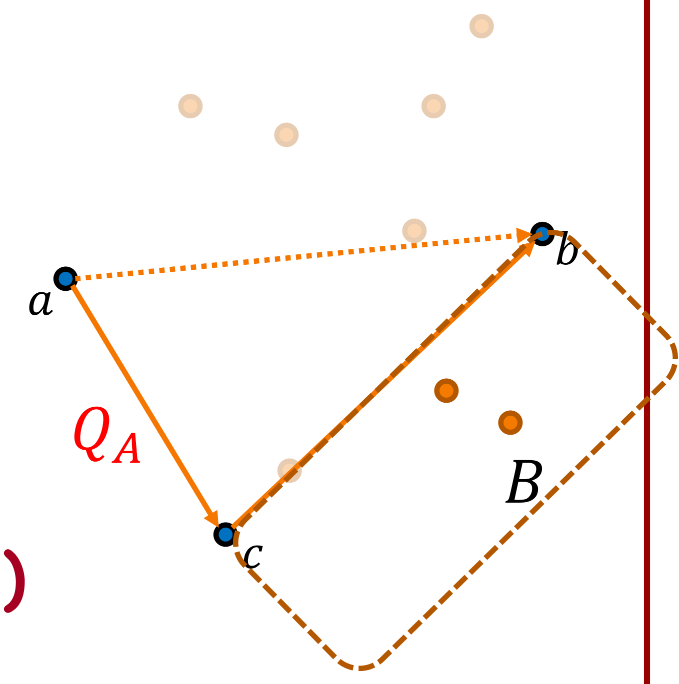
  - » $Q_A \leftarrow$ QuickHalfHull(A , $\overrightarrow{ac}$)
  - » **$Q_B \leftarrow$ QuickHalfHull(B , $\overrightarrow{cb}$)**
  - » return $Q_A \cup \{c\} \cup Q_B$



Convex Hull (2D)

(Recursion Level 1)

QuickHalfHull($S \subset \mathbb{R}^2$, $\overrightarrow{ab} \in S \times S$)

- if($S == \{a, b\}$) return $\emptyset$
- else
  - » $\boldsymbol{c}$ ← **Furthest(** $\boldsymbol{S}$ **,** $\boldsymbol{\overrightarrow{ab}}$ **)**
  - » A ← RightOn(S , $\overrightarrow{ac}$)
  - » B ← RightOn(S , $\overrightarrow{cb}$)
  - » Q_A ← QuickHalfHull(A , $\overrightarrow{ac}$)
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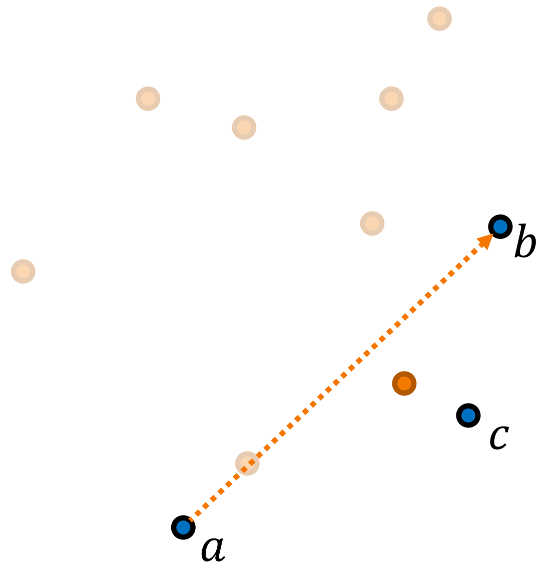

Convex Hull (2D)

(Recursion Level 1)

QuickHalfHull($S \subset \mathbb{R}^2, \overrightarrow{ab} \in S \times S$)

- if($S == \{a, b\}$) return ∅
- else
  - » $c \leftarrow$ Furthest(S , $\overrightarrow{ab}$)
  - » $\boldsymbol{A} \leftarrow$ **RightOn(** $\boldsymbol{S}$ **,** $\overrightarrow{\boldsymbol{ac}}$ **)**
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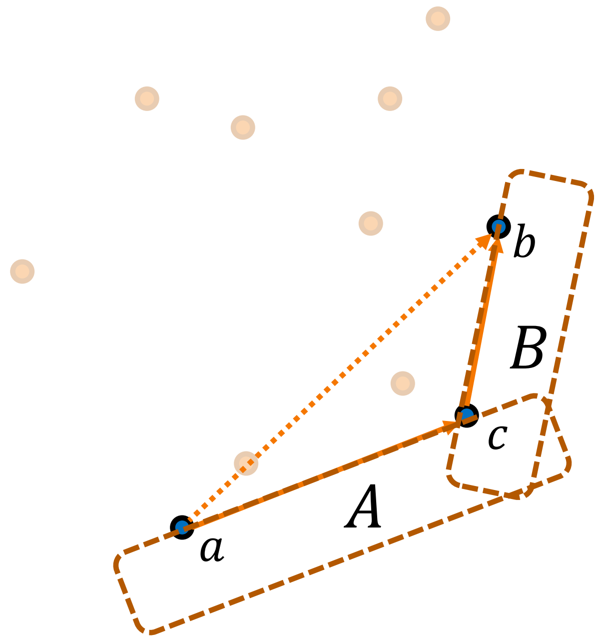

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  - » $\boldsymbol{Q_A} \leftarrow$ **QuickHalfHull(** $\boldsymbol{A}$ **,** $\boldsymbol{\overrightarrow{ac}}$ **)**
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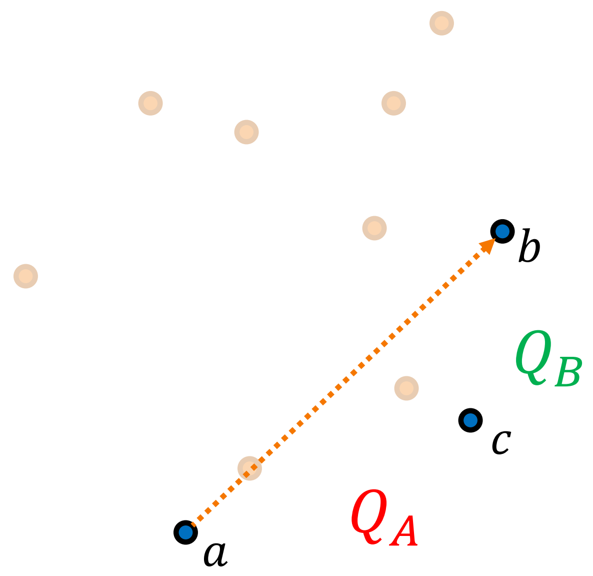

Convex Hull (2D)

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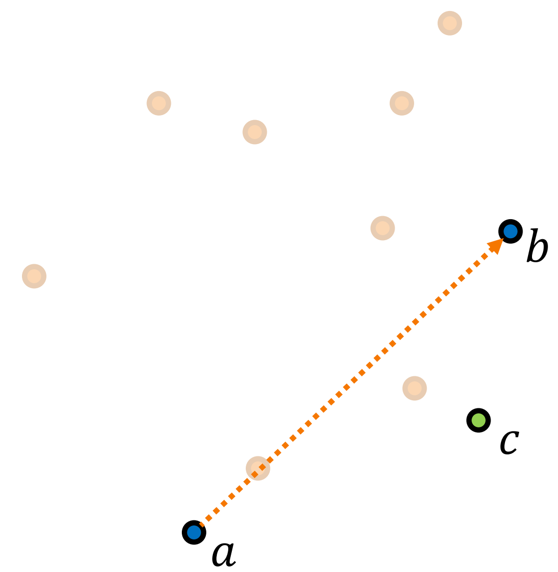

Convex Hull (2D)

(Recursion Level 0)

QuickHalfHull($S \subset \mathbb{R}^2$, $\overrightarrow{ab} \in S \times S$)

- if($S == \{a, b\}$) return $\emptyset$
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  - » **$Q_A \leftarrow$ QuickHalfHull(A , $\overrightarrow{ac}$)**
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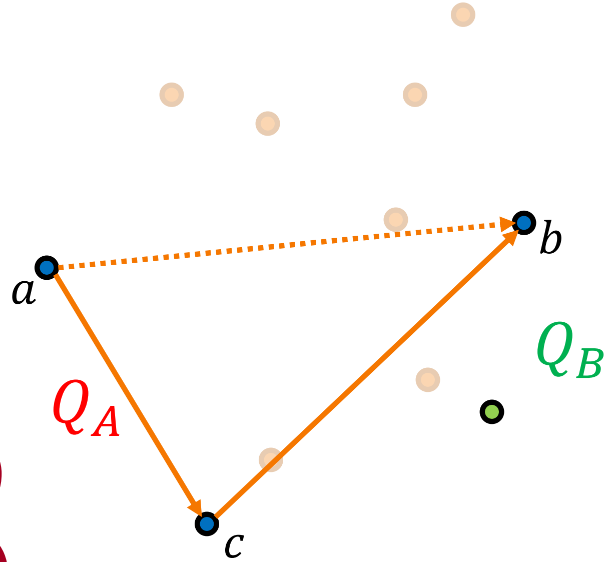

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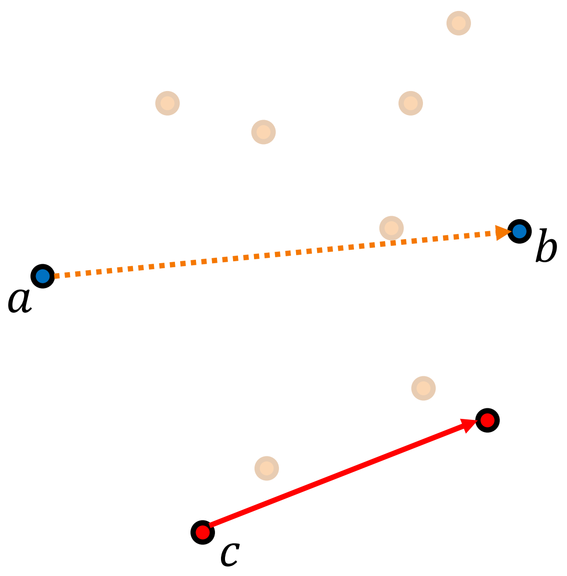
  - » **return** $\boldsymbol{Q_A \cup \{c\} \cup Q_B}$

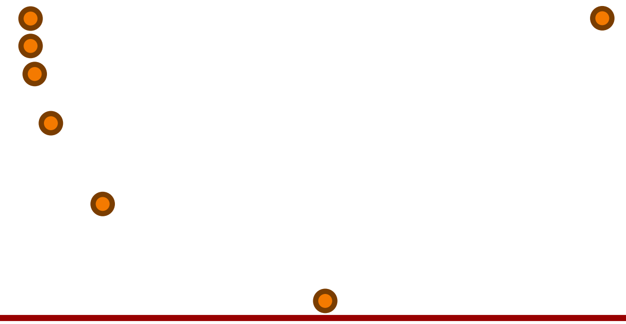


Convex Hull (2D)

QuickHull Complexity:

Like QuickSort:

- In the worst case, the complexity can be $O(n^2)$.



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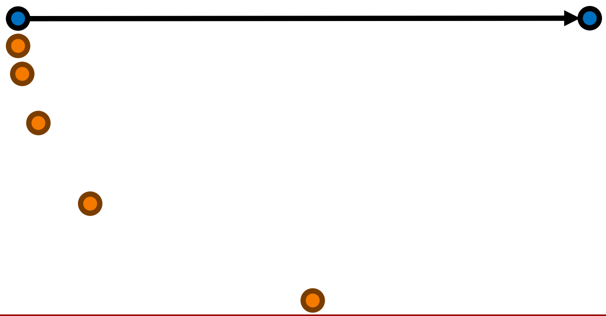

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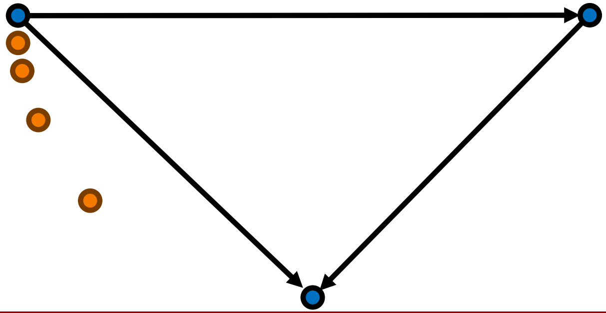

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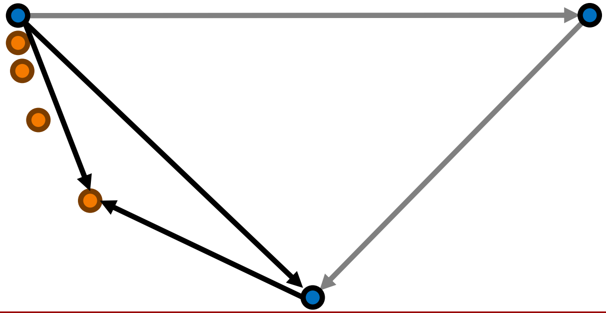

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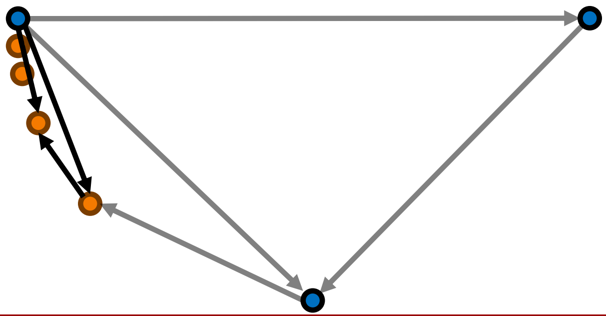


Convex Hull (2D)

QuickHull Complexity:

Like QuickSort:

- In the worst case, the complexity can be $O(n^2)$.
- In practice it is $O(n \log n)$.
- The implementation is output sensitive.

Does it extend to higher dimensions?

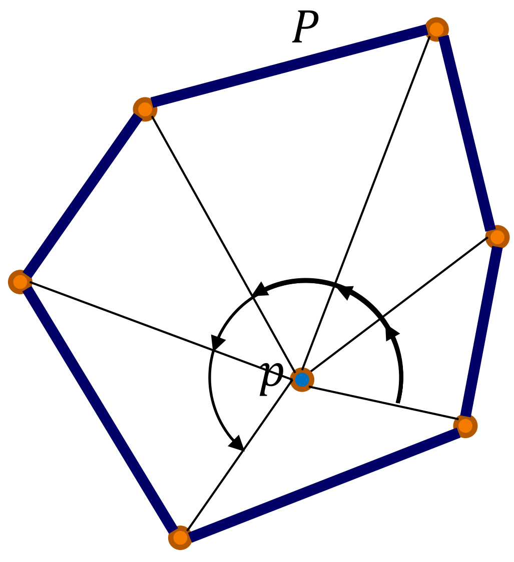
Outline

- Convex Hulls
- Algorithms
  - Naïve Implementation(s)
  - Gift Wrapping
  - Quick Hull
  - Graham's Algorithm
  - Lower bound complexity

Convex Hull (2D)

Graham's Observation:

If $P \subset \mathbb{R}^2$ is a convex polygon and $p \in P$ is a point in the interior of the polygon, then the angle of the line segments between p and the ordered vertices of P is monotonic.



Convex Hull (2D)

Graham's Observation:

WLOG assume p and v_i lie on a vertical line with p below v_i.

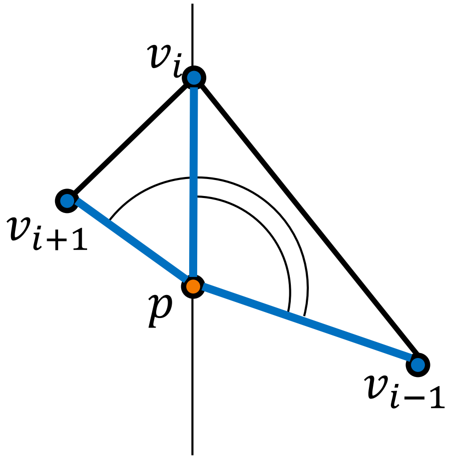
Since the polygon is convex, p is to the left of $\overrightarrow{v_i v_{i+1}}$.

$\Rightarrow v_{i+1}$ is to the left of the vertical.

Since the polygon is convex, p is to the left of $\overrightarrow{v_{i-1} v_i}$.

$\Rightarrow v_{i-1}$ is to the right of the vertical.

$\Rightarrow$ The angles $\angle p v_{i-1}$, $\angle p v_i$, $\angle p v_{i+1}$ increase monotonically.



Convex Hull (2D)

GrahamScan($S \subset \mathbb{R}^2$)

- $p \leftarrow$ PointOnHull(S)
- $H \leftarrow$ SortByAngle(S , p)
- while(RemoveReflexVertex(H)){ }
- return H

Convex Hull (2D)

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- $p \leftarrow$ PointOnHull(S)
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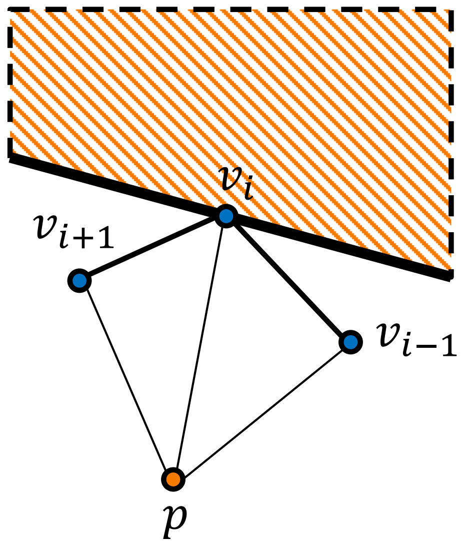


Note:

At every iteration, the vertices of H are sorted by angle relative to p.

$\Rightarrow$ Hull vertices can never be removed because the angle between the previous and next vertex is always convex.

Convex Hull (2D)

GraphamScan($S \subset \mathbb{R}^2$)

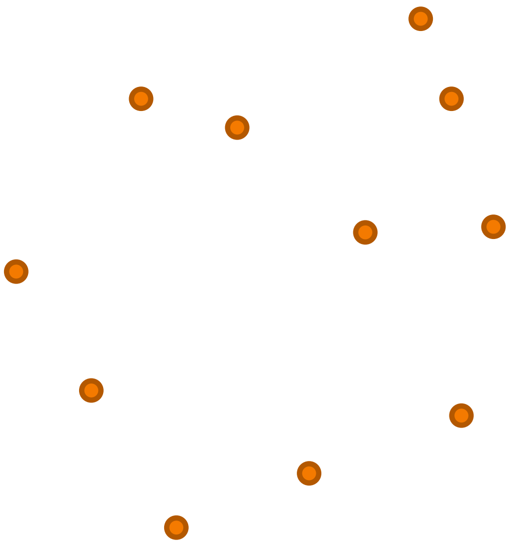
- $p \leftarrow$ PointOnHull(S)
- $H \leftarrow$ SortByAngle(S , p)
- while(RemoveReflexVertex(H)){ }
- return H

Correctness:

- The output polygon has only convex vertices.
  $\Rightarrow$ It's convex.
  $\Rightarrow H \subset \mathcal{H}(S)$.
- All hull vertices are in H.
  $\Rightarrow \mathcal{H}(S) \subset H$.

Convex Hull (2D)

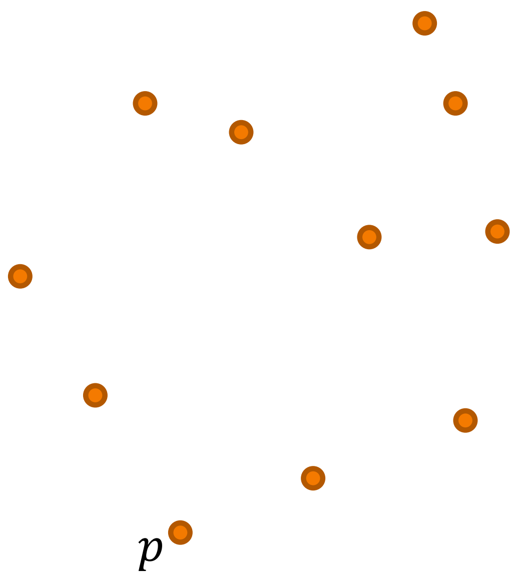
GrahamScan($S \subset \mathbb{R}^2$)



Convex Hull (2D)

GrahamScan($S \subset \mathbb{R}^2$)

- $p \leftarrow$ BottommostRightmost(S)

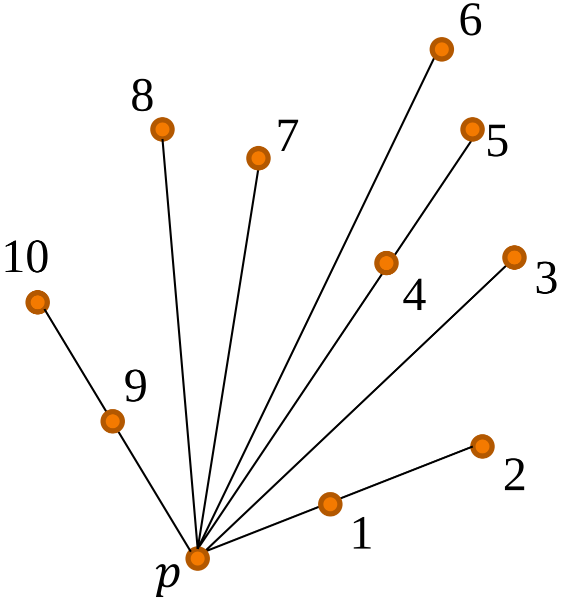


Convex Hull (2D)

GrahamScan($S \subset \mathbb{R}^2$)

- $p \leftarrow$ BottommostRightmost(S)
- $\tilde{S} \leftarrow$ SortByAngleAndLength(p , $S - \{p\}$)

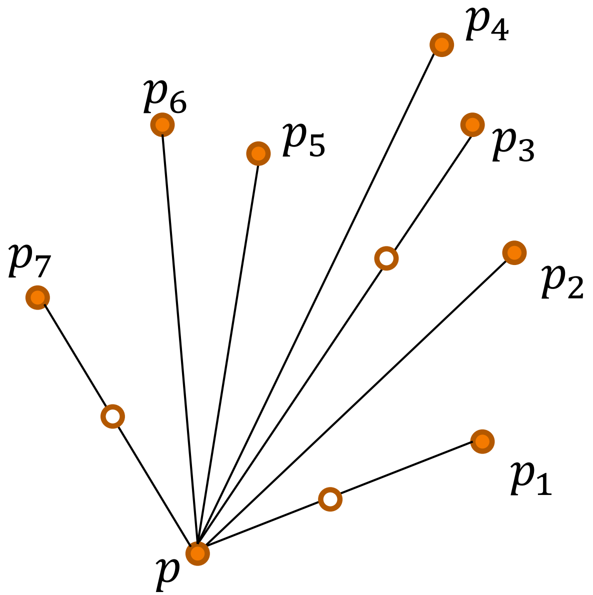
<u>Note</u>:
Since p is bottom-most, angles are in the range $[0, \pi)$ and points p_i and p_j can be sorted by testing if p_j is left of $\overrightarrow{pp_i}$.



Convex Hull (2D)

GrahamScan($S \subset \mathbb{R}^2$)

- $p \leftarrow$ BottommostRightmost(S)
- $\tilde{S} \leftarrow$ SortByAngleAndLength(p , $S - \{p\}$)
- if(angle(p_i) == angle(p_{i+1})): $\tilde{S} \leftarrow \tilde{S} - \{p_i\}$



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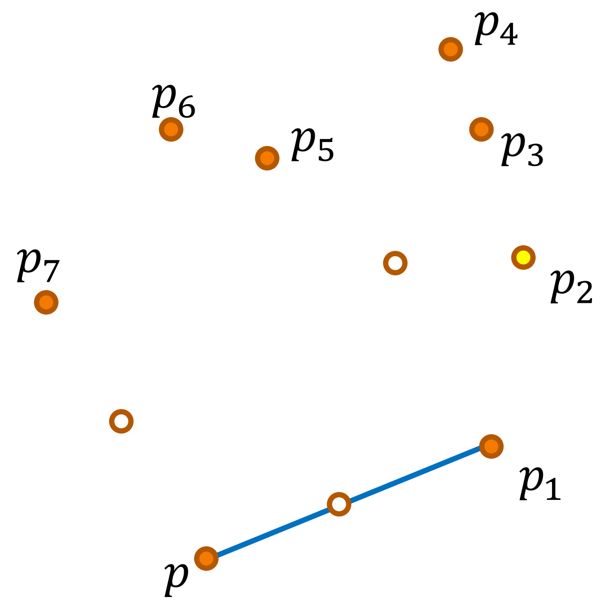
- if(angle(p_i) == angle(p_{i+1})): $\tilde{S} \leftarrow \tilde{S} - \{p_i\}$
- $i \leftarrow 2$
- $Q \leftarrow \{p, p_1\}$

<u>Note</u>:
Since p is bottom-(right)-most, vertices are sorted by angle in $(0, \pi]$, and non-extreme points are removed, $\overline{pp_1}$ is on the hull.



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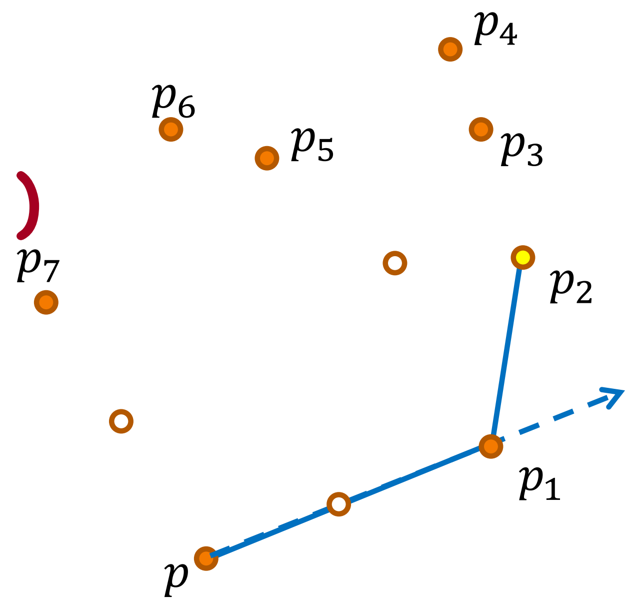
- $i \leftarrow 2$
- $Q \leftarrow \{p, p_1\}$
- while($i < |\tilde{S}|$)
  - » if(Left(p_i , LastEdge(Q))
    - Push(p_i , Q)
    - $i \leftarrow i + 1$
  - » else
    - Pop(Q)



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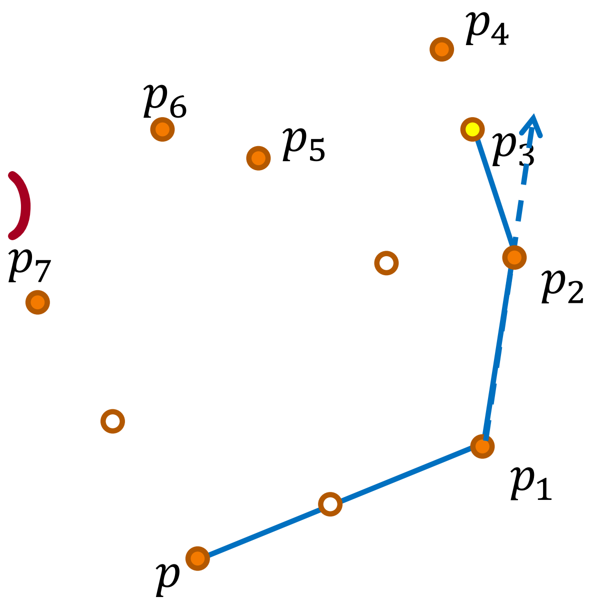

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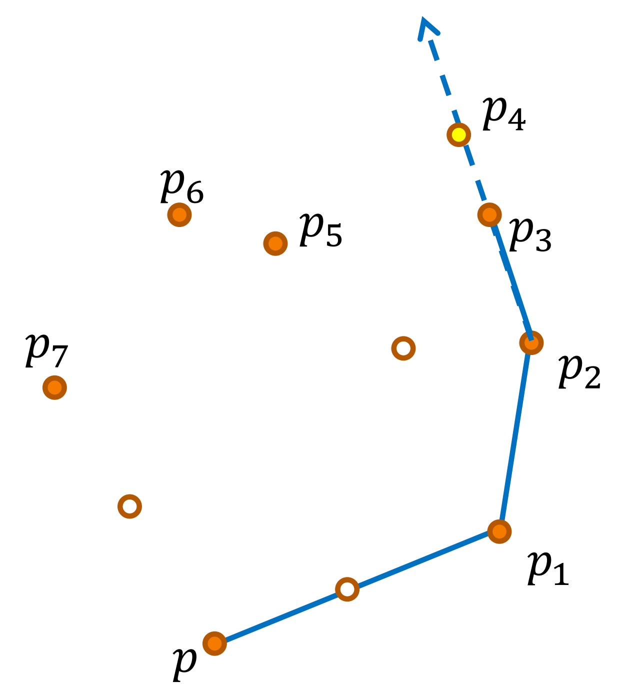

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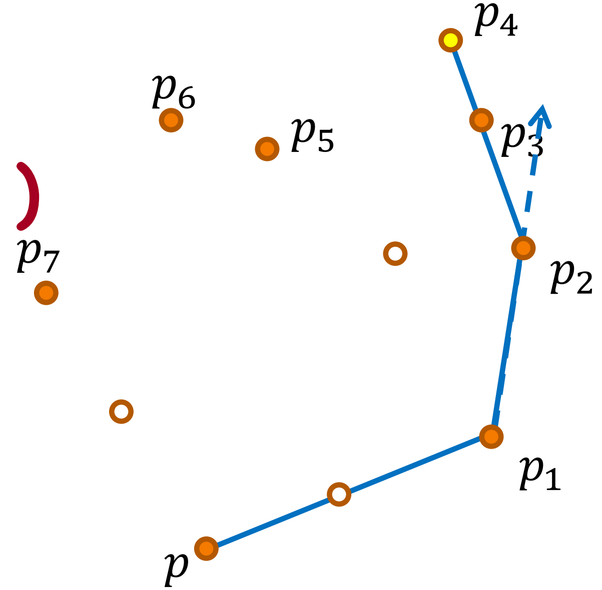

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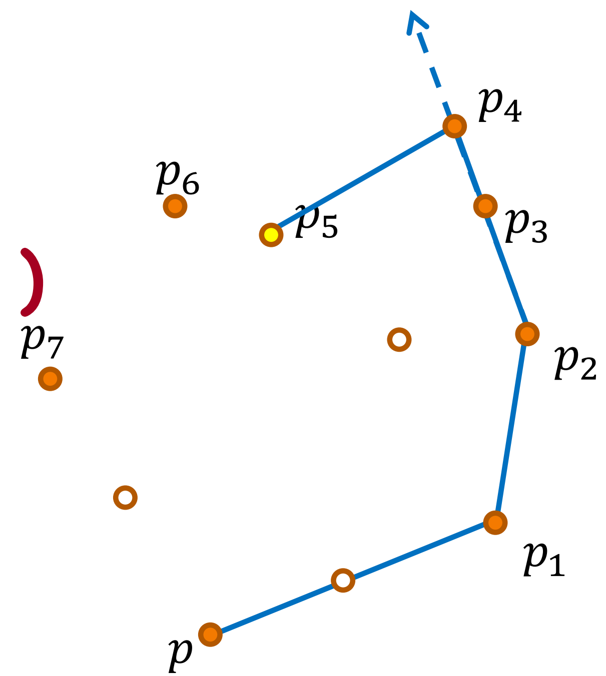

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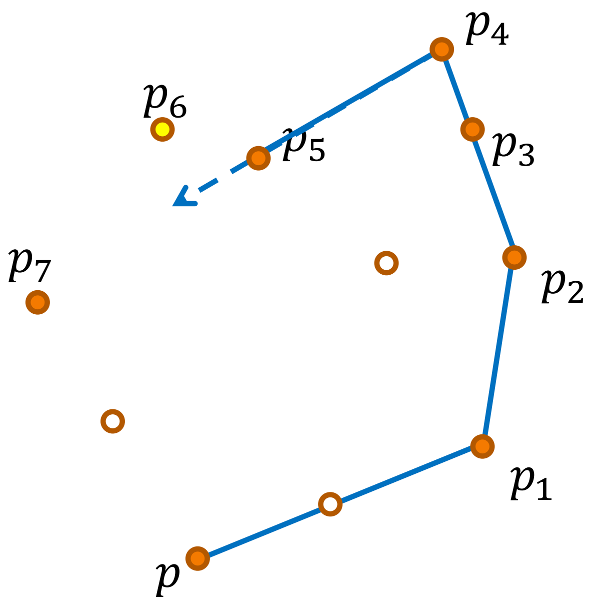

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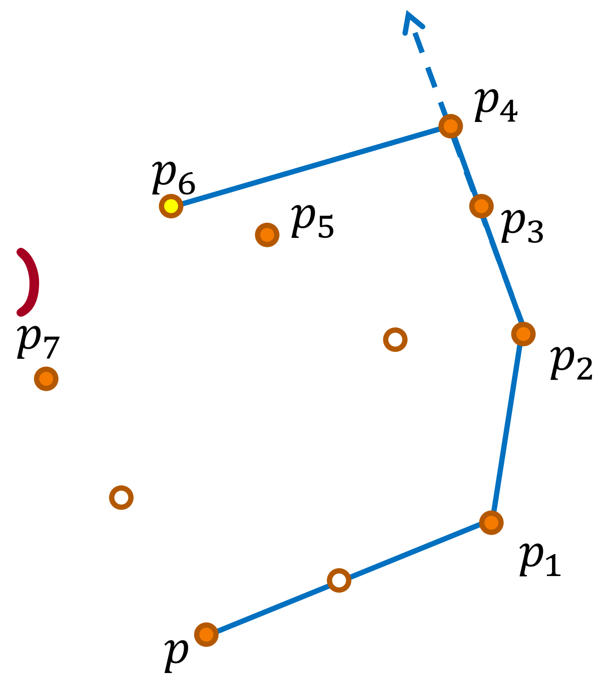

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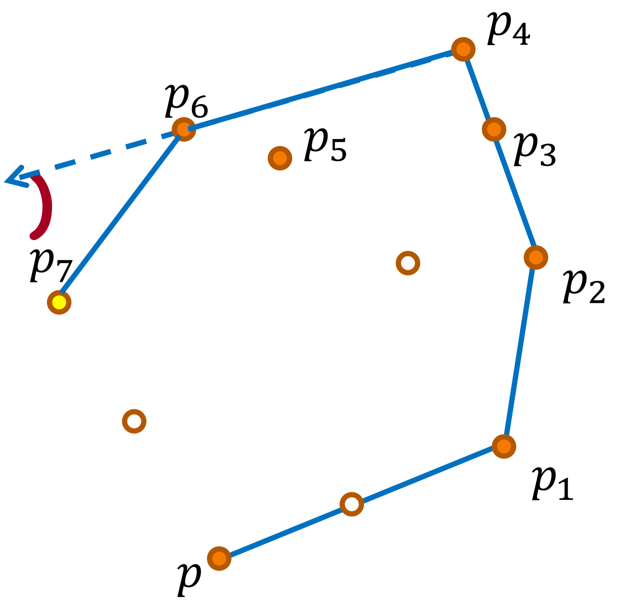

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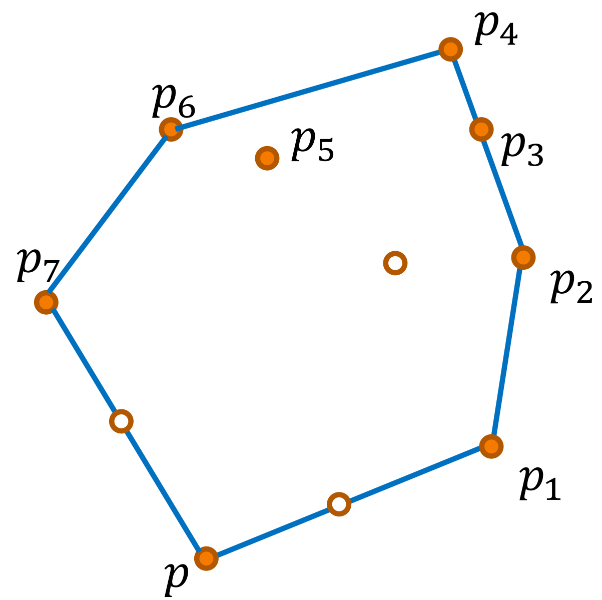

Convex Hull (2D)

GrahamScan($S \subset \mathbb{R}^2$)

- $p \leftarrow$ BottommostRightmost(S) $O(n)$
- $\tilde{S} \leftarrow$ SortByAngleAndLength(p , $S - \{p\}$) $O(n \log n)$
- if(angle(p_i) == angle(p_{i+1})): $\tilde{S} \leftarrow \tilde{S} - \{p_i\}$ $O(n)$
- $i \leftarrow 2$
- $Q \leftarrow \{p, p_1\}$
- while($i < |\tilde{S}|$)
  - » if(Left(p_i , LastEdge(Q)))
    - Push(p_i , Q)
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  - » else
    - Pop(Q)

$O(n)$

Outline

- Convex Hulls
- Algorithms
  - Naïve Implementation(s)
  - Gift Wrapping
  - Quick Hull
  - Graham’s Algorithm
  - Lower bound complexity

Lower Bound Complexity

Recall:

Sorting n numbers has lower bound complexity $O(n \log n)$.

Approach:

We will show that computing the 2D hull has the same complexity by reducing sorting to the problem of computing the convex hull.

Lower Bound Complexity

Sorting → Convex Hull Reduction (Shamos):

Given a set of points $\{x_i\} \subset \mathbb{R}$:

- Choose a function $f(x)$ w/ $f''(x) > 0$
- Lift the points onto the curve
- Compute the convex hull
- Return the points on the lower hull, starting w/ the left-most.