FFTs in Graphics and Vision

Representing Rotations

Outline

- Math Review
  - Polynomials
  - Eigenvectors
  - Orthogonal Transformations
  - Classifying the 2D Orthogonal Transformations
- Representing 3D Rotations

Math Review

Polynomials:

Let $P(x)$ be a polynomial of degree d:

$$P(x) = a_0 + a_1 \cdot x + \cdots + a_d \cdot x^d$$

Claim:

If d is odd, the polynomial $P(x)$ must have at least one real root.

Math Review

Polynomials:

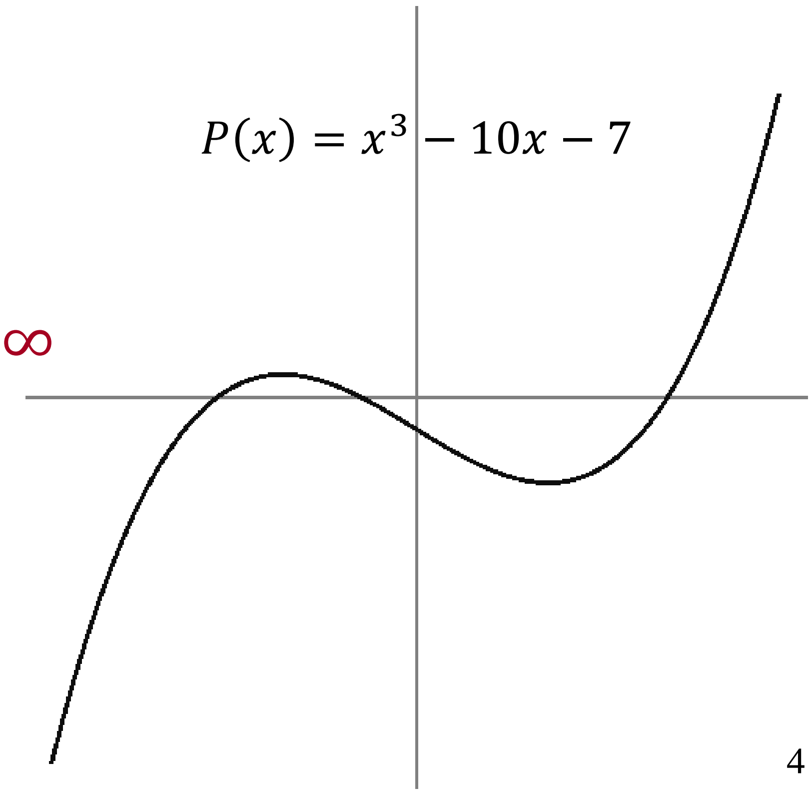
Let $P(x)$ be a polynomial of degree d:

$$P(x) = a_0 + a_1 \cdot x + \cdots + a_d \cdot x^d$$

Proof:

Consider the sign of a_d:

- If a_d is positive:
  - » As $x \to -\infty$: $P(x) \to -\infty$
  - » As $x \to \infty$: $P(x) \to \infty$



Math Review

Polynomials:

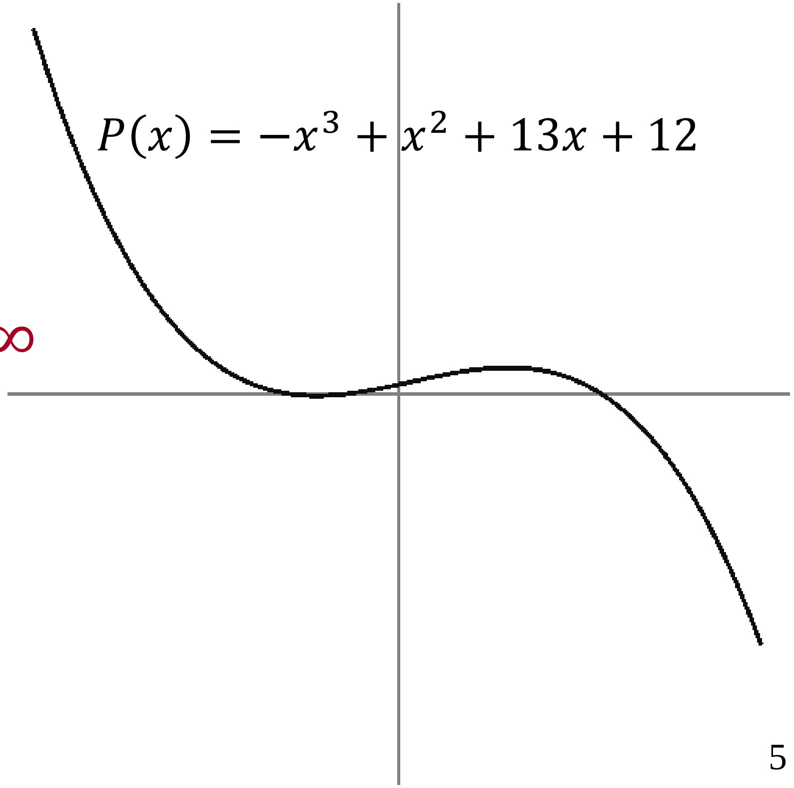
Let $P(x)$ be a polynomial of degree d:

$$P(x) = a_0 + a_1 \cdot x + \cdots + a_d \cdot x^d$$

Proof:

Consider the sign of a_d:

- If a_d is positive:
  - As $x \to -\infty$: $P(x) \to -\infty$
  - As $x \to \infty$: $P(x) \to \infty$
- If a_d is negative:
  - As $x \to -\infty$: $P(x) \to \infty$
  - As $x \to \infty$: $P(x) \to -\infty$



Math Review

Polynomials:

Let $P(x)$ be a polynomial of degree d:

$$P(x) = a_0 + a_1 \cdot x + \cdots + a_d \cdot x^d$$

Proof:

In either case, the value of $P(x)$ changes signs so it must have a zero-crossing somewhere.

Math Review

Eigenvectors:

Given a vector space V and invertible linear op. $A: V \to V$, if v is an e.vector of A with e.value λ then v is an e.vector of A^{-1} with e.value $1/\lambda$.

Since $A^{-1} \cdot A$ is the identity we have:

$$\begin{aligned} v &= A^{-1}(Av) \\ &= A^{-1}(\lambda v) \\ &= \lambda \cdot A^{-1} v \end{aligned}$$

$$\Downarrow$$

$$\frac{1}{\lambda} \cdot v = A^{-1} v$$

Math Review

Orthogonal Transformations:

For a real inner-product space V, a linear map R is orthogonal if for any $v, w \in V$, we have:

$$\langle v, w \rangle = \langle Rv, Rw \rangle$$

If the determinant of R is 1, the transformation is called a rotation.

Math Review

Orthogonal Transformations (Property 1):

The set of orthogonal transformations is a group.

Math Review

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The set of orthogonal transformations is a group.

To show this we need to show that if R and S are orthogonal transformations than:

- RS is orthogonal
- R^{-1} is orthogonal

Math Review

Orthogonal Transformations (Property 1):

If R and S are orthogonal transformations, then so is the transformation $R \cdot S$.

Since R is orthogonal:

$$\langle RSv, RSw \rangle = \langle Sv, Sw \rangle$$

Since S is orthogonal:

$$\langle Sv, Sw \rangle = \langle v, w \rangle$$

Thus, as desired, we get:

$$\langle RSv, RSw \rangle = \langle v, w \rangle$$

Math Review

Orthogonal Transformations (Property 1):

If R is an orthogonal transformation, then so is the transformation R^{-1}.

Starting with the identity:

$$\langle v, w\rangle = \langle RR^{-1}v, RR^{-1}w\rangle$$

Since R is orthogonal we get:

$$\langle RR^{-1}v, RR^{-1}w\rangle = \langle R^{-1}v, R^{-1}w\rangle$$

Thus, as desired, we get:

$$\langle v, w\rangle = \langle R^{-1}v, R^{-1}w\rangle$$

Math Review

Orthogonal Transformations (Property 2):

If R is an orthogonal transformation and v is an eigenvector of R with eigenvalue λ, then $\lambda = \pm 1$.

Since R orthogonal, we have:

$$\begin{aligned}\langle v, v\rangle &= \langle Rv, Rv\rangle \\ &= \langle \lambda v, \lambda v\rangle \\ &= \lambda^2 \langle v, v\rangle\end{aligned}$$

$$\Downarrow$$

$$\lambda^2 = 1$$

In particular, v is an eigenvector of R with eigenvalue: $\frac{1}{\lambda} = \lambda$.

Math Review

<u>Orthogonal Transformations (Property 3)</u>:

If R is an orthogonal transformation and v is an eigenvector of R, then if w is a vector perpendicular to v, Rw is also perpendicular to v.

Since R^{-1} is also an orthogonal transformation:

$$\begin{aligned}\langle v, Rw\rangle &= \langle R^{-1}v, R^{-1}Rw\rangle \\ &= \langle R^{-1}v, w\rangle \\ &= \frac{1}{\lambda}\langle v, w\rangle \\ &= 0\end{aligned}$$

Math Review

Orthogonal Transformations (Property 4):

If R is an orthogonal transformation and v_1 and v_2 are eigenvectors of R with eigenvalues λ_1 and λ_2, then if $\lambda_1 \neq \lambda_2$, v_1 and v_2 must be perpendicular.

Since R^{-1} is orthogonal we have:

$$\begin{aligned}\langle Rv_1, v_2\rangle &= \langle R^{-1}Rv_1, R^{-1}v_2\rangle \\ &= \langle v_1, R^{-1}v_2\rangle\end{aligned}$$

$$\Downarrow$$

$$\begin{aligned}\lambda_1\langle v_1, v_2\rangle &= 1/\lambda_2\langle v_1, v_2\rangle \\ &= \lambda_2\langle v_1, v_2\rangle\end{aligned}$$

$$\Downarrow$$

$$\langle v_1, v_2\rangle = 0$$

Math Review

Classifying the 2D Orthogonal Transformations:

Let V be the space of 2D arrays with the standard basis:

$$\{e_1 = (1,0), e_2 = (0,1)\}$$

with the standard inner product:

$$\langle e_i, e_j \rangle = \delta_{ij}$$

Math Review

Classifying the 2D Orthogonal Transformations:

In the basis $\{(1,0),(0,1)\}$ we can express a linear operator R as a matrix:

$$\mathbf{R} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}$$

R is orthogonal if $\mathbf{R}^t \cdot \mathbf{R}$ is the identity:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{a} & \mathbf{c} \\ \mathbf{b} & \mathbf{d} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{a}^2 + \mathbf{c}^2 & \mathbf{ab} + \mathbf{cd} \\ \mathbf{ab} + \mathbf{cd} & \mathbf{b}^2 + \mathbf{d}^2 \end{pmatrix}$$

Math Review

Classifying the 2D Orthogonal Transformations:

$$\begin{pmatrix} \boxed{1} & 0 \\ 0 & \boxed{1} \end{pmatrix} = \begin{pmatrix} \boxed{\mathbf{a}^2 + \mathbf{c}^2} & \mathbf{ab} + \mathbf{cd} \\ \mathbf{ab} + \mathbf{cd} & \boxed{\mathbf{b}^2 + \mathbf{d}^2} \end{pmatrix}$$

The diagonal entries give rise to the equations:

$$1 = \mathbf{a}^2 + \mathbf{c}^2$$
$$1 = \mathbf{b}^2 + \mathbf{d}^2$$

For appropriate θ and ϕ, this gives:

$$\mathbf{a} = \cos\theta \qquad \mathbf{c} = \sin\theta$$
$$\mathbf{b} = \cos\phi \qquad \mathbf{d} = \sin\phi$$

Math Review

Classifying the 2D Orthogonal Transformations:

$$\begin{pmatrix} 1 & \boxed{0} \\ \boxed{0} & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{a}^2 + \mathbf{c}^2 & \boxed{\mathbf{ab} + \mathbf{cd}} \\ \boxed{\mathbf{ab} + \mathbf{cd}} & \mathbf{b}^2 + \mathbf{d}^2 \end{pmatrix}$$

The other equations then become:

$$0 = \cos\theta \cdot \cos\phi + \sin\theta \cdot \sin\phi$$

Or equivalently:

$$0 = \cos(\theta - \phi)$$

Which implies that:

$$\phi = \theta + k\pi + \pi/2$$

Math Review

Classifying the 2D Orthogonal Transformations:

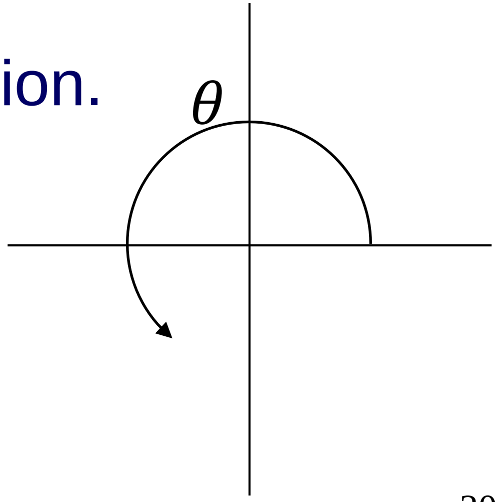
$$\phi = \theta + k\pi + \pi/2$$

If R is an orthogonal transformation, then in the basis $\{(1,0),(0,1)\}$ we have one of two cases:

- k is even:

$$\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

The determinant is 1 ⇒ this is a rotation.



Math Review

Classifying the 2D Orthogonal Transformations:

$$\phi = \theta + k\pi + \pi/2$$

If R is an orthogonal transformation, then in the basis $\{(1,0),(0,1)\}$ we have one of two cases:

- k is even:

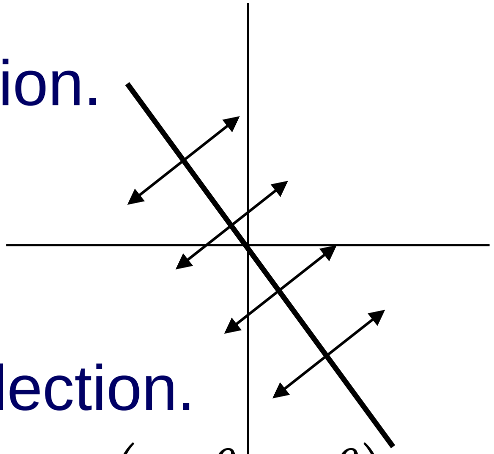
  $$\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

  The determinant is 1 $\Rightarrow$ this is a rotation.

- k is odd:

  $$\mathbf{R} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

  The determinant is -1 $\Rightarrow$ this is a reflection.



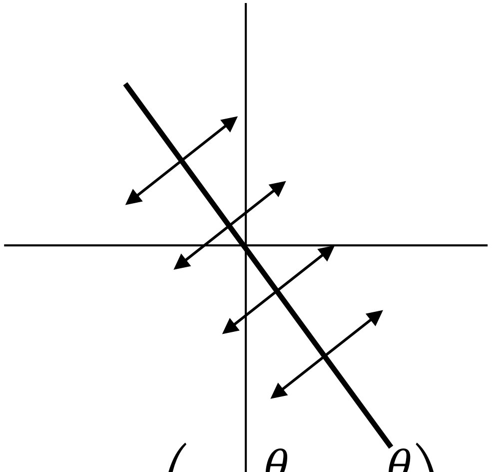
$$\left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\right)$$

Math Review

Classifying the 2D Orthogonal Transformations:

Claim:

In the case that k is odd, the orthogonal transformation has eigenvalues 1 and -1.



$\left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\right)$

Math Review

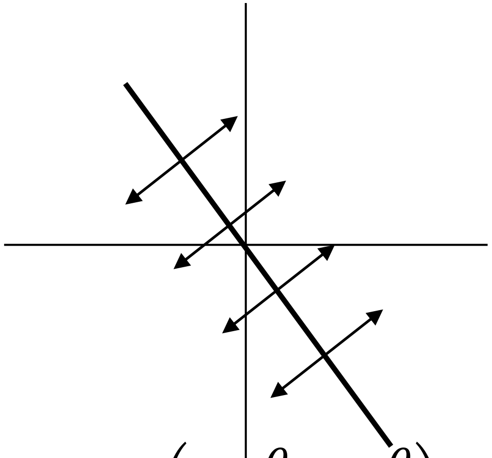
Classifying the 2D Orthogonal Transformations:

Claim:

In the case that k is odd, the orthogonal transformation has eigenvalues 1 and -1.

To compute the eigenvalues, we need to solve for the roots of the polynomial:

$$P_R(\lambda) = \det(R - \lambda \cdot \mathrm{Id})$$



$$\left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\right)$$

Math Review

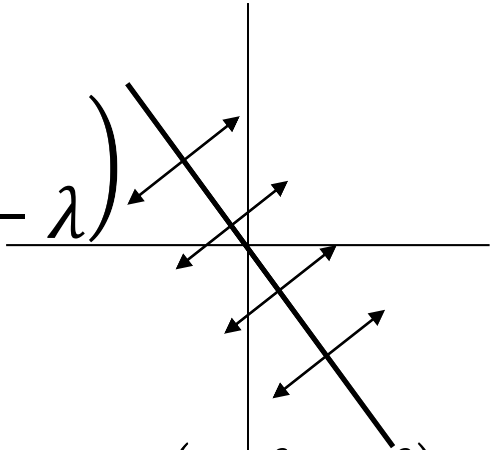
Classifying the 2D Orthogonal Transformations:

Claim:

In the case that k is odd, the orthogonal transformation has eigenvalues 1 and -1.

To compute the eigenvalues, we need to solve for the roots of the polynomial:

$$
\begin{aligned}
P_R(\lambda) &= \det\begin{pmatrix} \cos\theta - \lambda & \sin\theta \\ \sin\theta & -\cos\theta - \lambda \end{pmatrix} \\
&= \lambda^2 - \cos^2\theta - \sin^2\theta \\
&= \lambda^2 - 1
\end{aligned}
$$



$$\left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\right)$$

Math Review

Classifying the 2D Orthogonal Transformations:

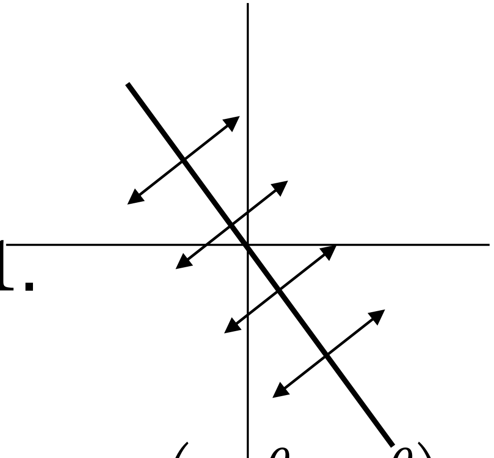
Claim:

In the case that k is odd, the orthogonal transformation has eigenvalues 1 and -1.

To compute the eigenvalues, we need to solve for the roots of the polynomial:

$$P_R(\lambda) = \det(R - \lambda \cdot \mathrm{Id})$$

This polynomial has two roots, $\lambda = \pm 1$.



$$\left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\right)$$

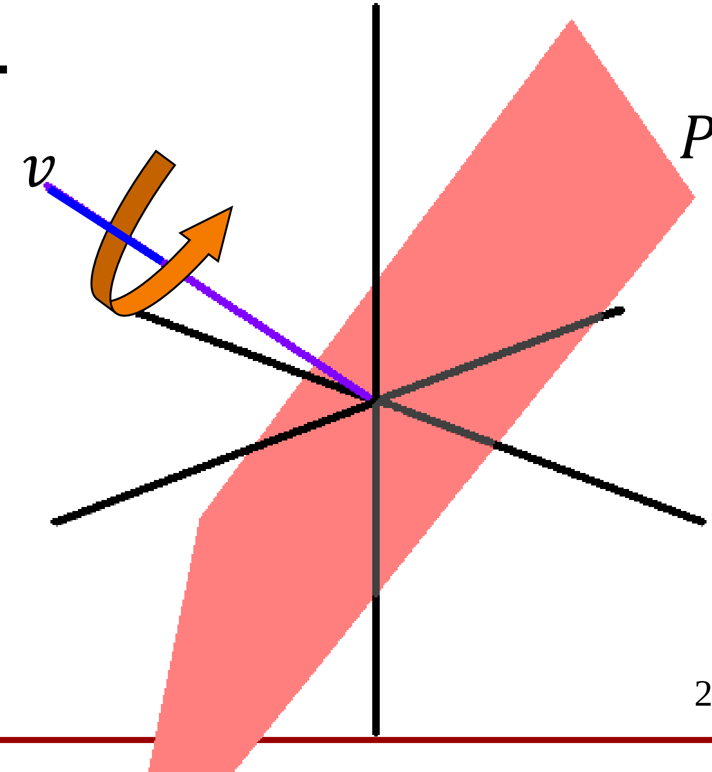
Outline

- Math Review
- Representing 3D Rotations
  - Axis-Angle
  - Euler Angles

Representing 3D Rotations (Axis-Angle)

We will show that any rotation R can be thought of as a rotation about some axis.

In particular, we need to show that every rotation R fixes some vector v and acts as a rotation in the plane P perpendicular to v.



Representing 3D Rotations (Axis-Angle)

Let V be the space of 3D arrays with the standard basis:

$$\{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$$

with the standard inner product:

$$\langle e_i, e_j \rangle = \delta_{ij}$$

Representing 3D Rotations (Axis-Angle)

In the basis $\{(1,0,0),(0,1,0),(0,0,1)\}$ we can express the linear operator R as a matrix:

$$\mathbf{R} = \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{d} & \mathbf{e} & \mathbf{f} \\ \mathbf{g} & \mathbf{h} & \mathbf{i} \end{pmatrix}$$

We can compute the eigenvalues of R by finding the roots of the determinant:

$$P_R(\lambda) = \det \begin{pmatrix} \mathbf{a}-\lambda & \mathbf{b} & \mathbf{c} \\ \mathbf{d} & \mathbf{e}-\lambda & \mathbf{f} \\ \mathbf{g} & \mathbf{h} & \mathbf{i}-\lambda \end{pmatrix}$$

Representing 3D Rotations (Axis-Angle)

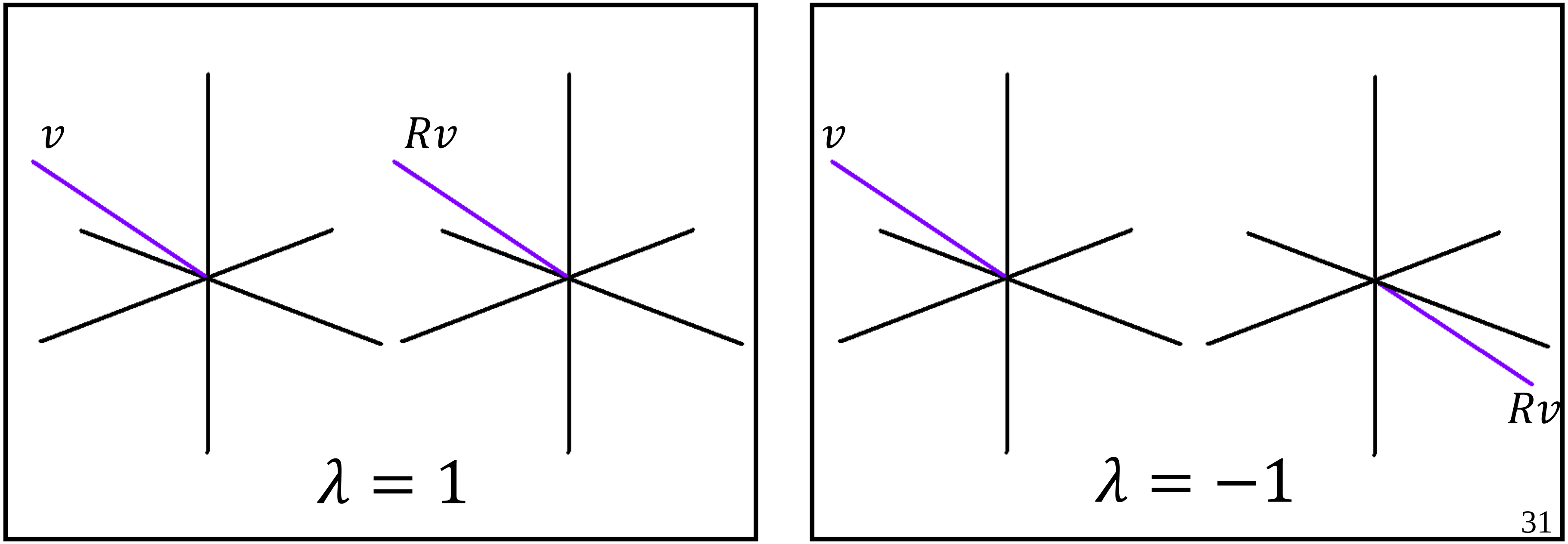
Since $P_R(\lambda)$ has odd degree ($d = 3$), it must have at least one root.

Thus, R has an eigenvector v with eigenvalue λ.

Since R is orthogonal $\lambda = \pm 1$.

Representing 3D Rotations (Axis-Angle)

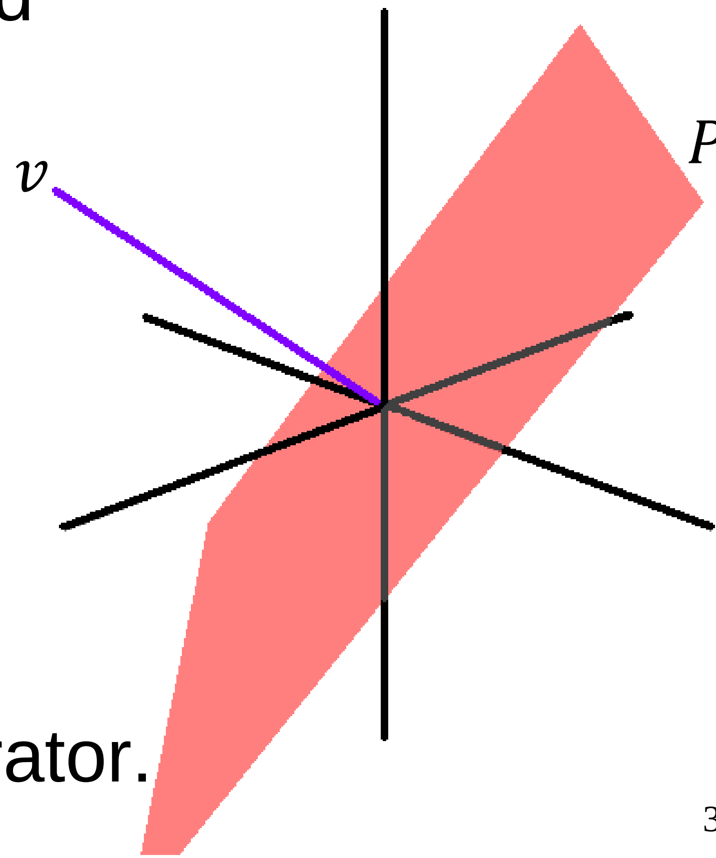
Thus, for every orthogonal transformation R, there must exist a vector v that is either fixed by R or mapped to its antipode.



Representing 3D Rotations (Axis-Angle)

What happens to the plane P that is orthogonal to the eigenvector v?

Since R maps the line spanned by v back into itself, and since R is orthogonal, R must map the plane P back into itself.

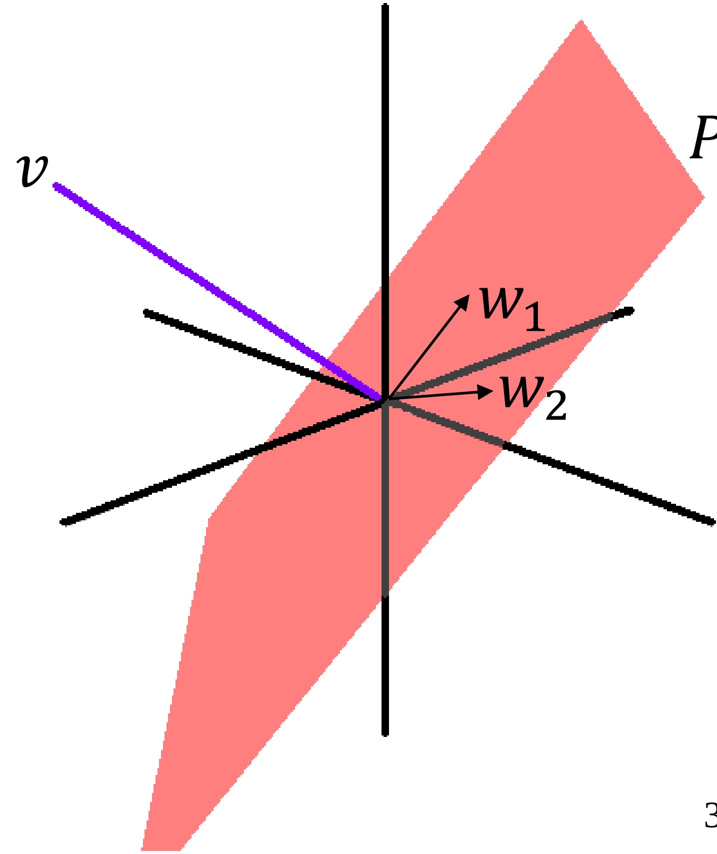


Since R preserves the inner-product and maps P to itself, the restriction of R to P is a 2D orthogonal operator.

Representing 3D Rotations (Axis-Angle)

Setting $\{w_1, w_2\}$ be an orthonormal basis for the plane P, with respect to the basis $\{v, w_1, w_2\}$, we can express R in matrix form as either:

$$\mathbf{R} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$



or:

$$\mathbf{R} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & \sin\theta & -\cos\theta \end{pmatrix}$$

Representing 3D Rotations (Axis-Angle)

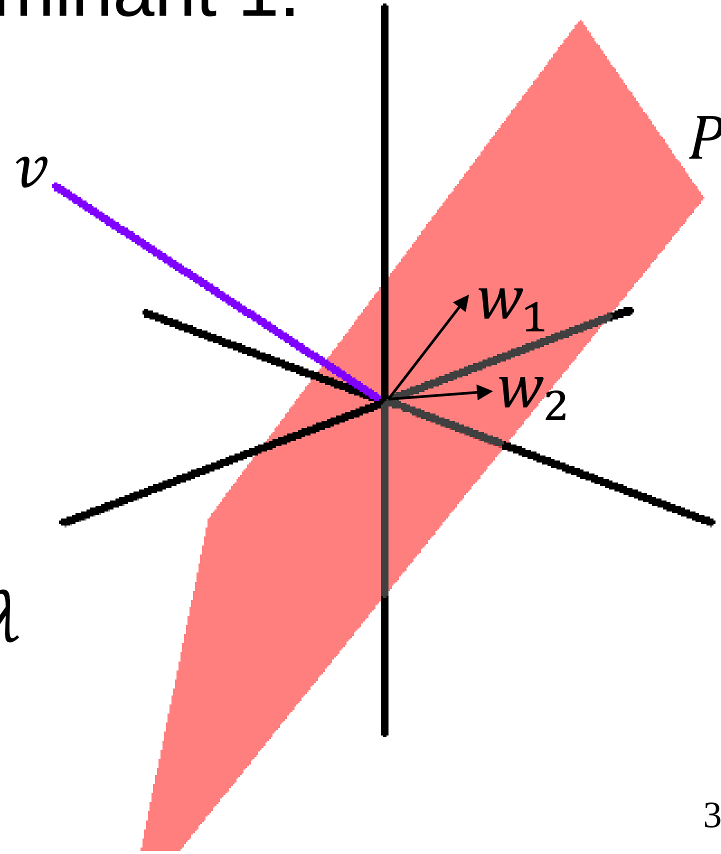
What happens in the case that R is a rotation?

If R is a rotation, then in addition to being orthogonal, it must have determinant 1.

For the two representations of R we get:

$$\det\begin{pmatrix}\lambda & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{pmatrix} = \lambda$$

$$\det\begin{pmatrix}\lambda & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & \sin\theta & -\cos\theta\end{pmatrix} = -\lambda$$



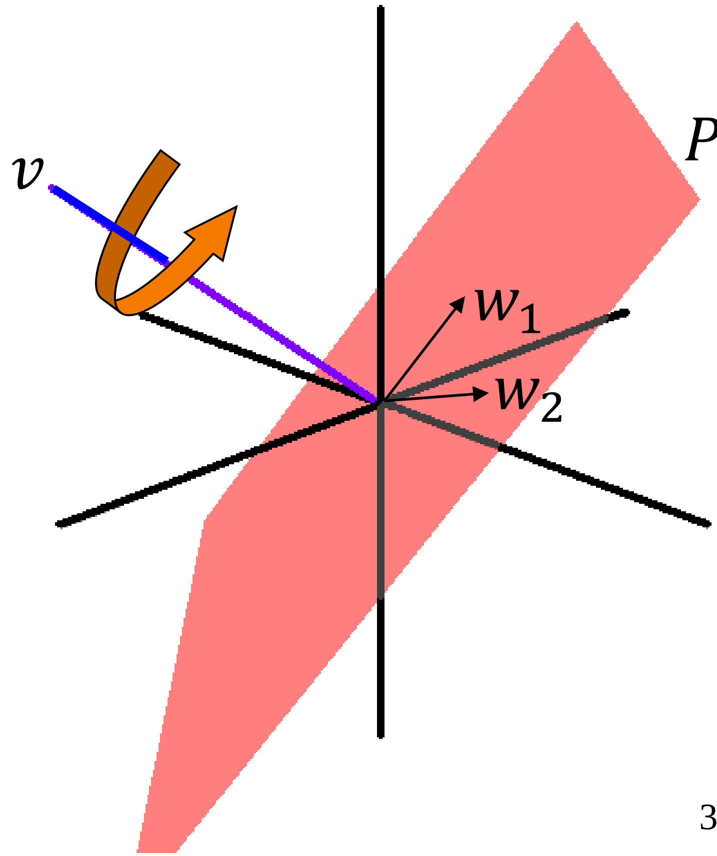
Representing 3D Rotations (Axis-Angle)

What happens in the case that R is a rotation?

If $\lambda = 1$, we have:

$$\mathbf{R} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

and R is a rotation in the plane P by angle θ.



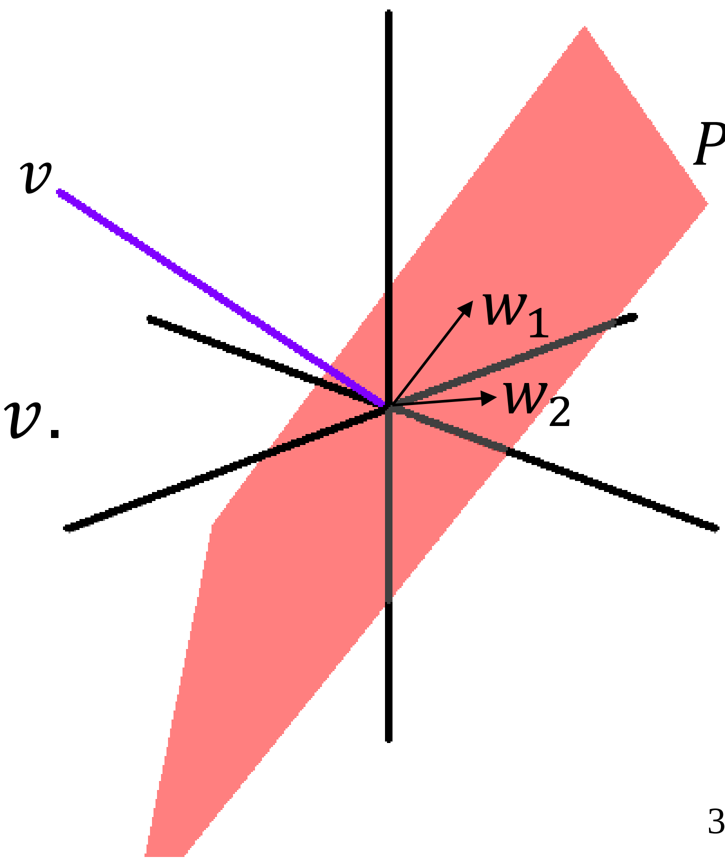
Representing 3D Rotations (Axis-Angle)

What happens in the case that R is a rotation?

If $\lambda = -1$, we get:

$$R \equiv \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & \sin\theta & -\cos\theta \end{pmatrix}$$

and R is the composition of a reflection in the plane P and a flip about the line spanned by v.



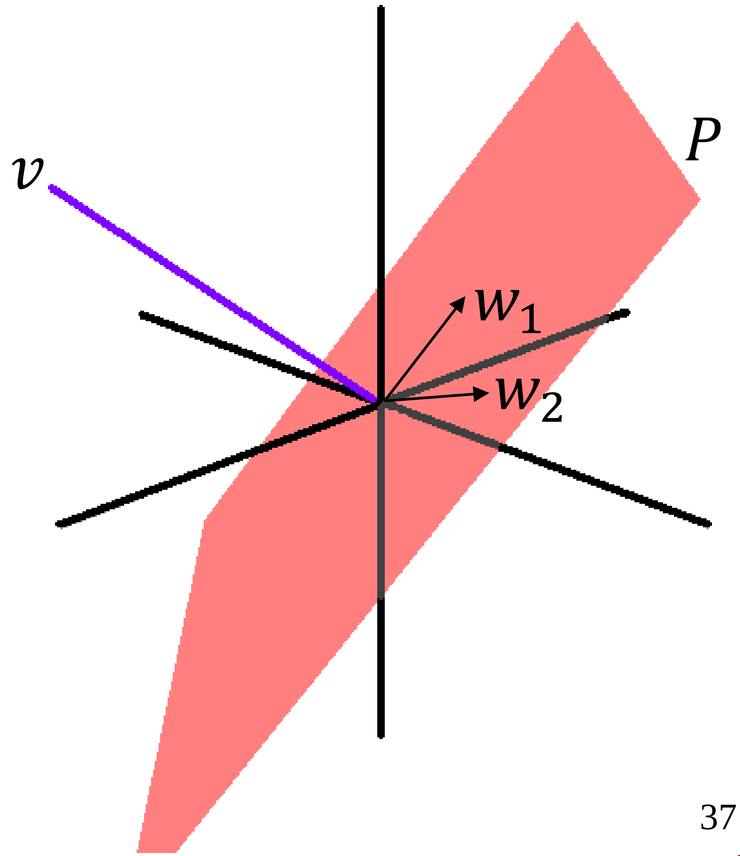
Representing 3D Rotations (Axis-Angle)

What happens in the case that R is a rotation?

Restricting R to the plane P, in the basis $\{w_1, w_2\}$ we get:

$$\mathbf{R}\Big|_P = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

which has eigenvalues -1 and 1.

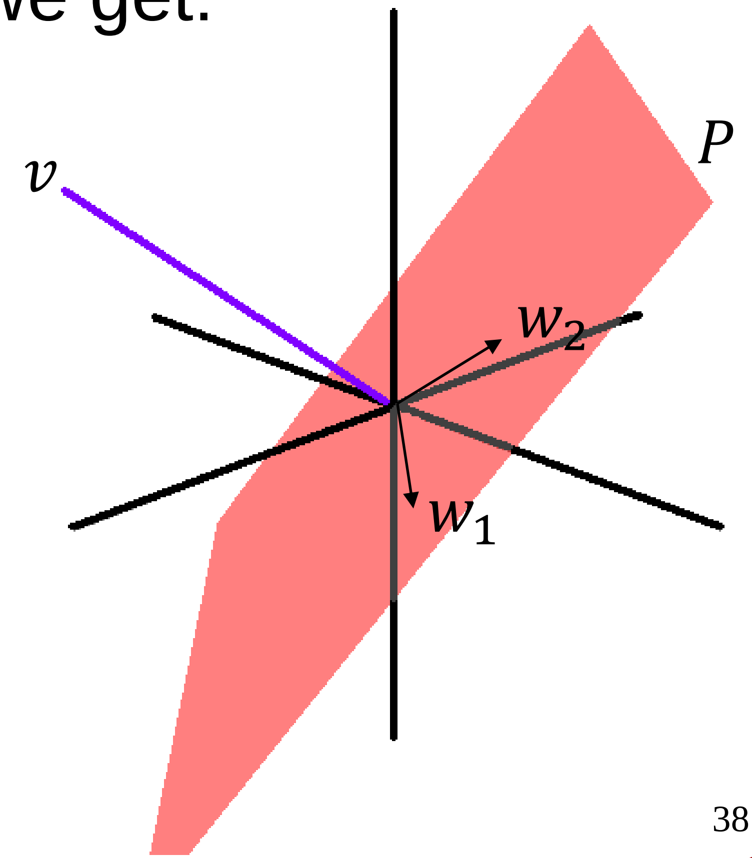


Representing 3D Rotations (Axis-Angle)

What happens in the case that R is a rotation?

In particular, if we set w_1 and w_2 to be the corresponding eigenvectors, we get:

$$\mathbf{R} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

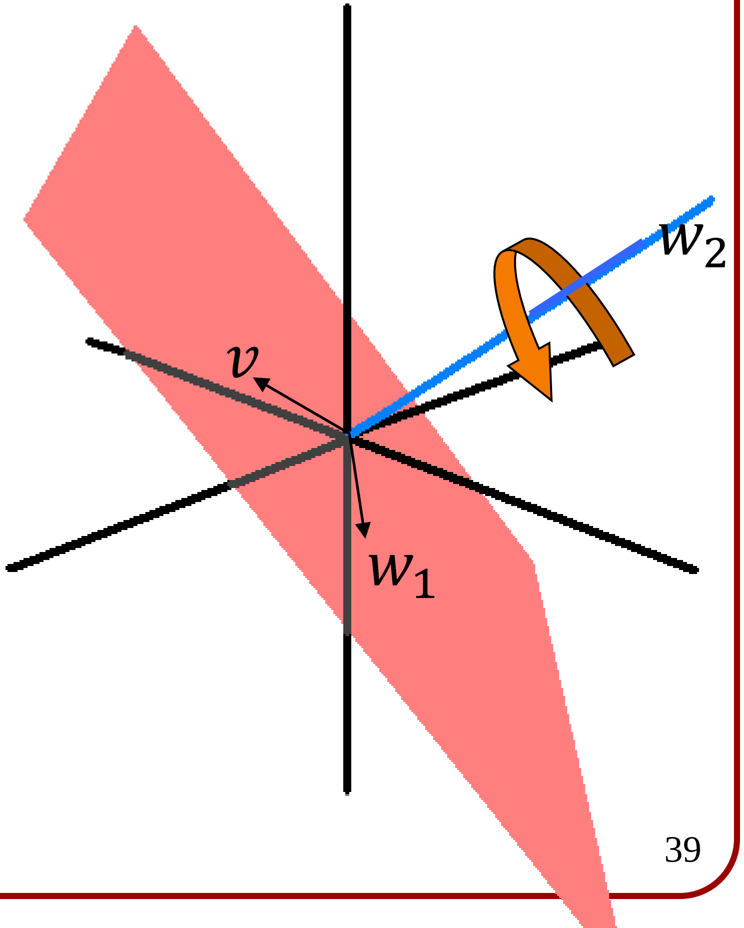


Representing 3D Rotations (Axis-Angle)

What happens in the case that R is a rotation?

$$\mathbf{R} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow$ R is a rotation by $180°$ in the plane spanned by v and w_1.

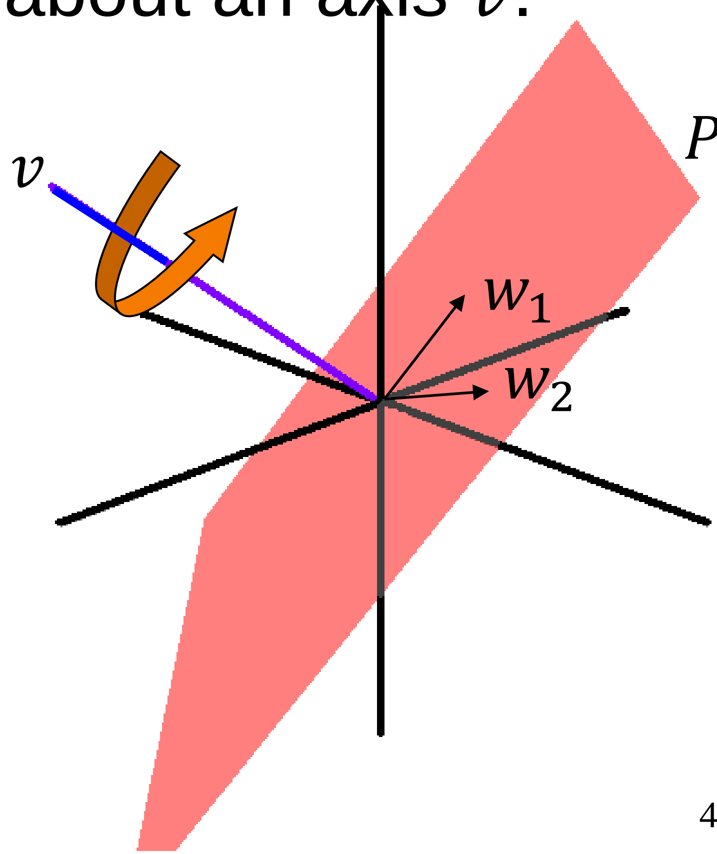


Representing 3D Rotations (Axis-Angle)

What happens in the case that R is a rotation?

So, in both cases, the rotation R can be realized as a rotation by some angle θ about an axis v.

That is, R sends the vector v back into itself and rotates vectors in the plane that is perpendicular to v by the angle θ.



Representing 3D Rotations (Axis-Angle)

What happens in the case that R is a rotation?

This motivates a representation of rotations by specifying the axis about which the rotation occurs and the angle of the 2D rotation.

Outline

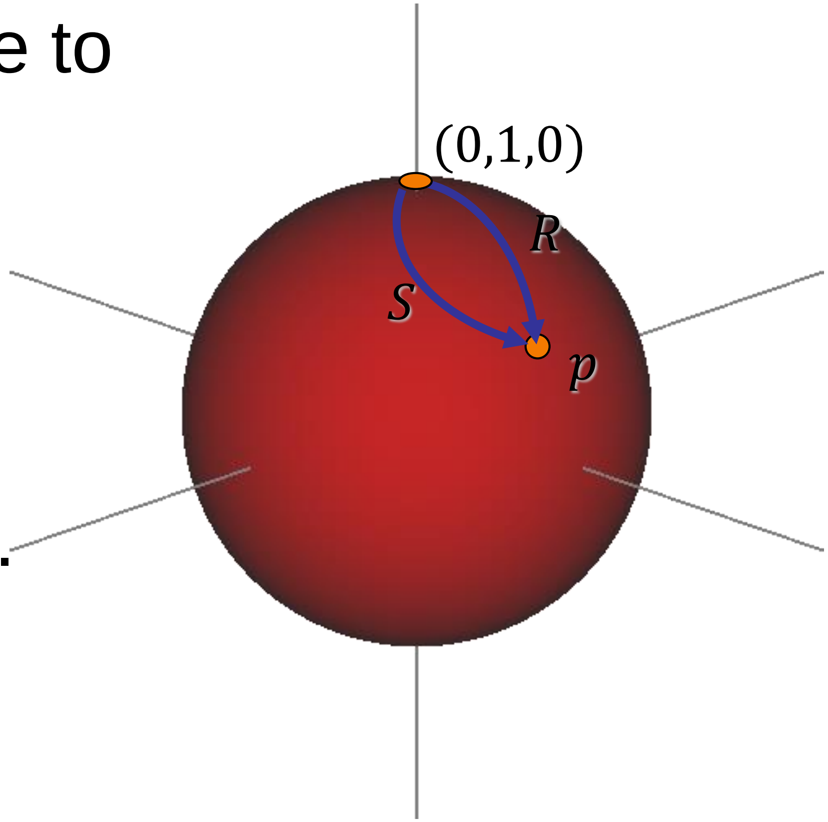
- Math Review
- Representing 3D Rotations
  - Axis-Angle
  - Euler Angles

Representing 3D Rotations (Euler)

We will consider a representation of rotations that describe what the rotation does to (0,1,0).

Given a rotation R, if we know that R maps the North pole to the point p, is that enough information to define R?

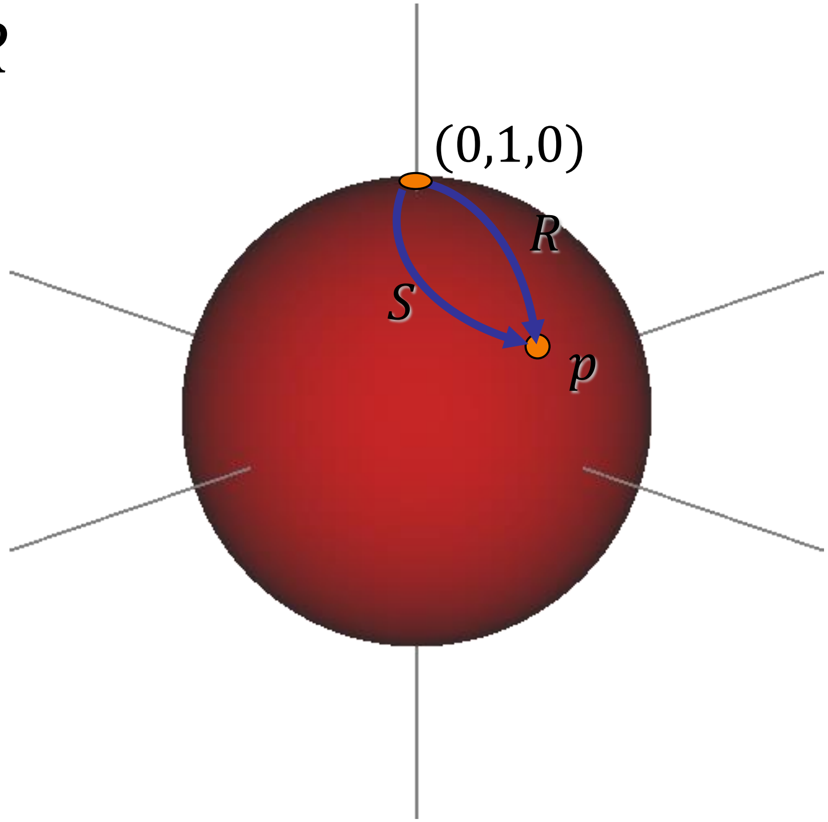
No. There can be many different rotations that all send (0,1,0) to the point p.



Representing 3D Rotations (Euler)

In particular, if R and S are two rotations mapping the North pole to the point p, then $R^{-1} \cdot S$ must map the North pole back to itself.

That is, for any rotations R and S mapping the North pole to the point p, $R^{-1} \cdot S$ must be a rotation about the North pole.



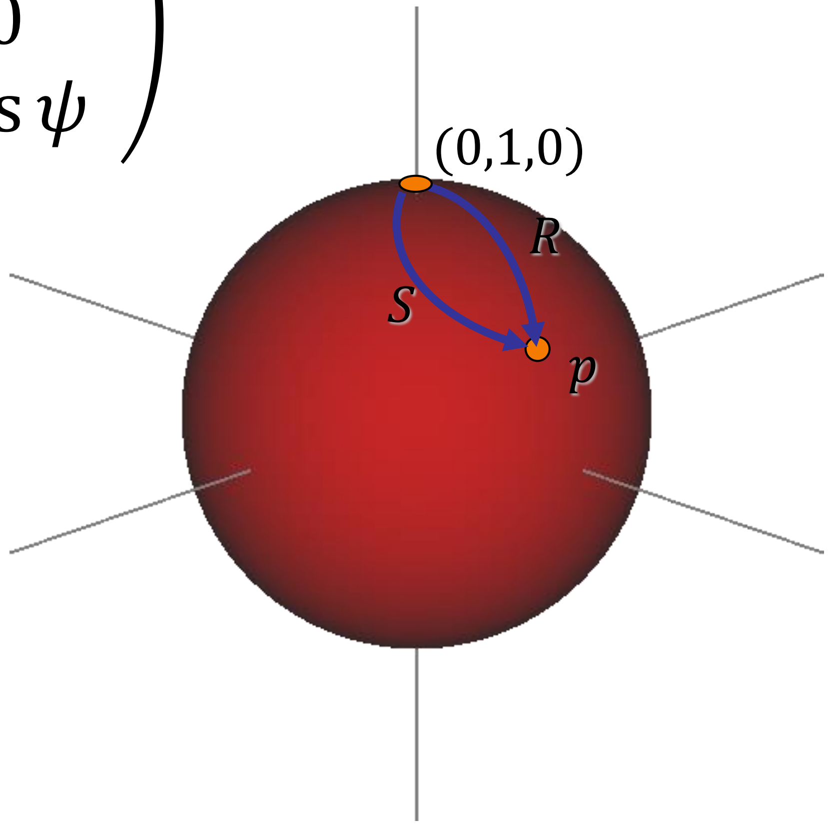
Representing 3D Rotations (Euler)

Denote by $R_y(\psi)$ the rotation about the y-axis (North pole) by ψ degrees:

$$R_y(\psi) = \begin{pmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{pmatrix}$$

If R and S are two rotations sending the North pole to p, then:

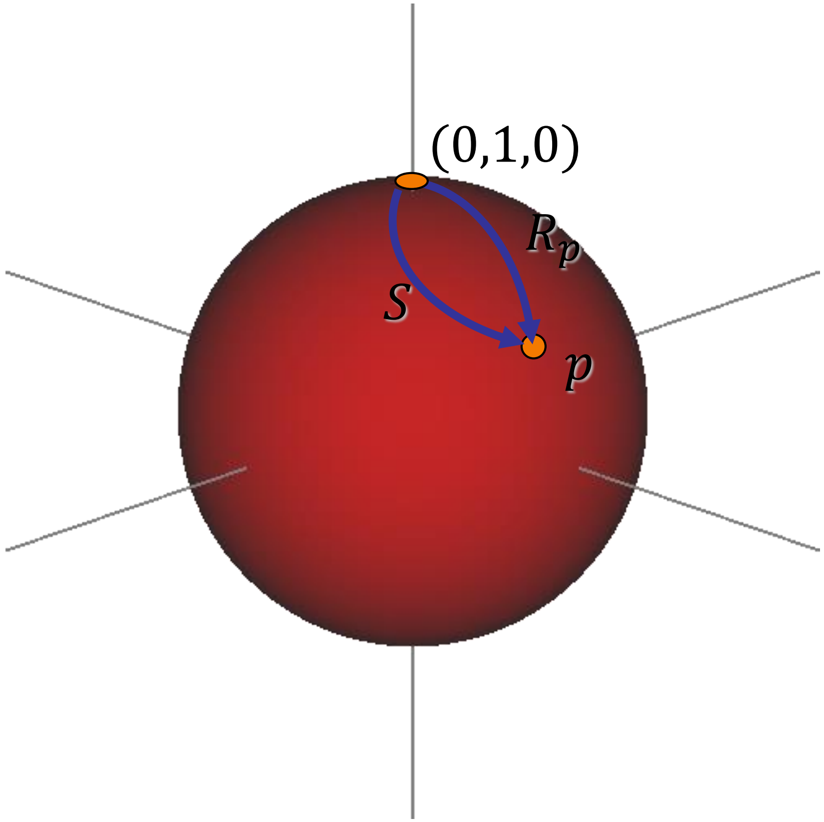
$$R^{-1} \cdot S = R_y(\psi)$$



Representing 3D Rotations (Euler)

In particular, this implies that if R_p is some rotation sending the North pole to p, then any other rotation S that sends the North pole to p must be of the form:

$$S = R_p \cdot R_y(\psi)$$



Representing 3D Rotations (Euler)

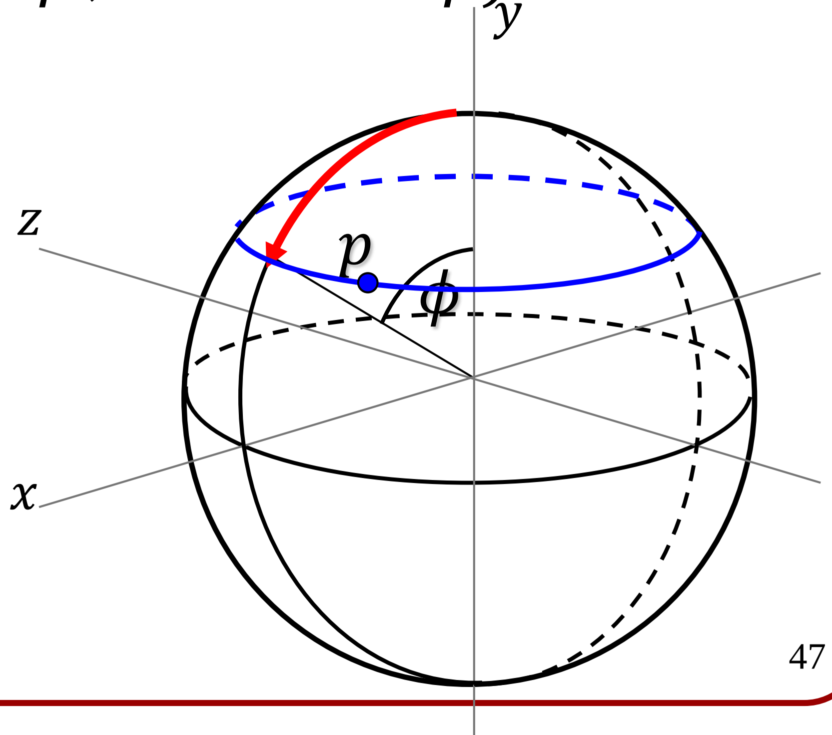
In order to represent all rotations, we need to find an expression for the map that sends the North pole to the point p.

Let (θ, ϕ) be the spherical coordinates of p:

$$p = (\cos\theta \cdot \sin\phi, \cos\phi, \sin\theta \cdot \sin\phi)$$

The point p must lie on the circle about the y-axis with height $\cos\phi$.

We can get $(0,1,0)$ to this circle with a rotation by an angle of ϕ about the z-axis.



Representing 3D Rotations (Euler)

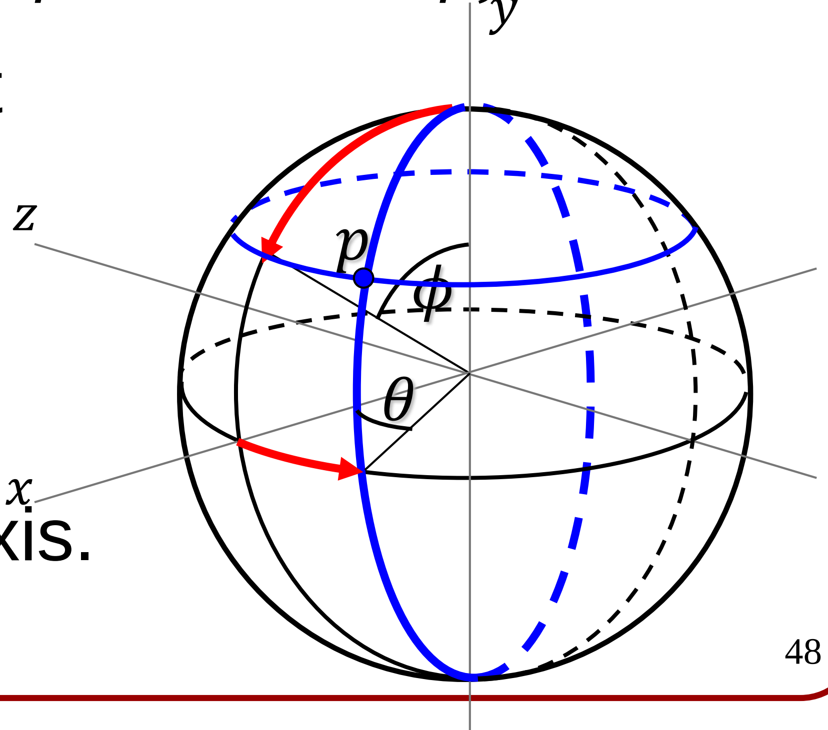
In order to represent all rotations, we need to find an expression for the map that sends the North pole to the point p.

Let (θ, ϕ) be the spherical coordinates of p:

$$p = (\cos\theta \cdot \sin\phi\,, \cos\phi\,, \sin\theta \cdot \sin\phi)$$

We also know that the point p makes an angle of θ with the xy-plane.

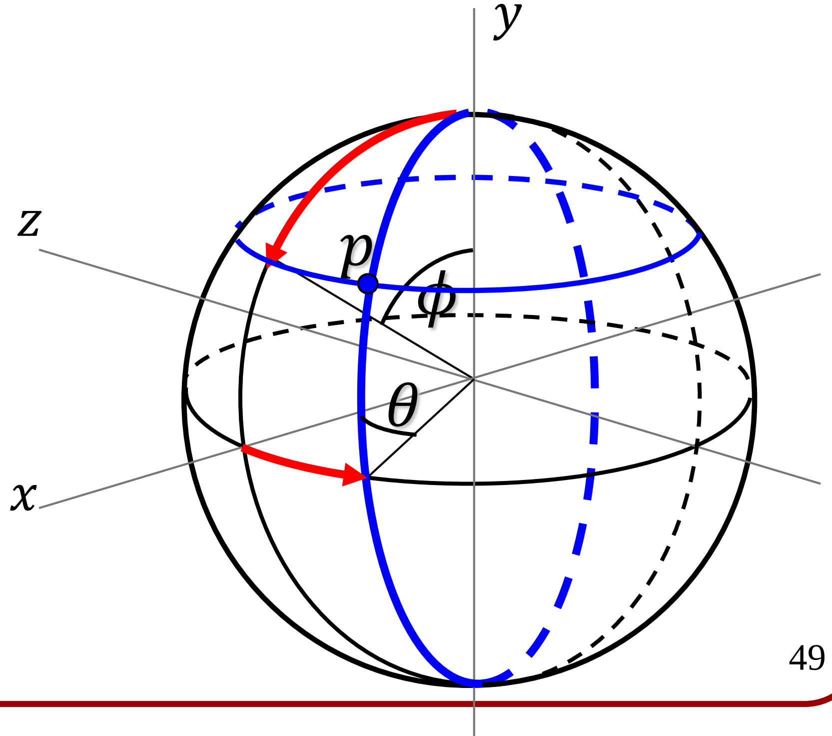
We can get the rotation of $(0,1,0)$ to p by rotating by an angle of θ about the y-axis.



Representing 3D Rotations (Euler)

Thus, when the spherical coordinates of the point p are (θ,ϕ), we can rotate $(0,1,0)$ to p by:

- First rotating by ϕ degrees about the z-axis, and
- Then rotating by θ degrees about the y-axis.



Representing 3D Rotations (Euler)

Since every rotation R can be described by a rotation about the y-axis, followed by a rotation that maps $(0,1,0)$ to $p = \Phi(\theta, \phi)$, we have:

$$R = R_y(\theta) \cdot R_z(\phi) \cdot R_y(\psi)$$

where $R_y(\alpha)$ is the rotation about the y-axis by α, and $R_z(\beta)$ is the rotation about the z-axis by β.

Representing 3D Rotations (Euler)

In matrix form, the triplet of angles (θ, ϕ, ψ) represents the rotation:

$$\mathbf{R}(\theta,\phi,\psi) = \underbrace{\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{Rotation sending } (0,1,0) \to p = \Phi(\theta,\phi)} \underbrace{\begin{pmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{pmatrix}}_{\text{Rotation about the } y\text{-axis by } \psi}$$

Rotation sending $(0,1,0) \rightarrow p = \Phi(\theta, \phi)$

Rotation about the y-axis by ψ

This is the Euler Angle parameterization of 3D rotations.