FFTs in Graphics and Vision

Fast String Matching
and
Math Review

Fast Pattern Matching in Strings
Knuth *et al.*, 1977

Outline

Fast Substring Matching

Math Review

- Complex Numbers
- Vector Spaces
- Linear Operators

Fast Substring Matching

Challenge:

Given strings S and T, find all occurrences of T as a substring of S.

Fast Substring Matching

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Given strings S and T, find all occurrences of T as a substring of S.

Example:

S=A$\boxed{\text{CDB}}$EF$\boxed{\text{CDB}}$E T=CDB

Fast Substring Matching

Challenge:

Given strings S and T, find all occurrences of T as a substring of S.

Brute Force:

- For each position in S:
  - Test if the next $|T|$ letters in S match T

S=ACDBEFCDBE $\qquad$ T=CDB

CDB

$$\boxed{O(|S| \times |T|)}$$

Fast Substring Matching

<u>Challenge</u>:

Given strings S and T, find all occurrences of T as a substring of S.

<u>Brute Force</u>:

- For each position in S:
  - Test if the next $|T|$ letters in S match T

<u>Can we do this more efficiently</u>?

Fast Substring Matching

Challenge:

Given strings S and T, find all occurrences of T as a substring of S.

Observation:

On a failed match, we don't have to compare all $|T|$ letters in T:

S=A$\boxed{\text{CDB}}$EF$\boxed{\text{CDB}}$E

CDB

T=CDB

Comparisons: 3

Fast Substring Matching

<u>Challenge</u>:

Given strings S and T, find all occurrences of T as a substring of S.

<u>Observation</u>:

What if the situation is more complex?

S=AAAAAA$\boxed{\text{AAAB}}$ T=AAAB

AAABBBBBBAB Comparisons: 4

Fast Substring Matching

Challenge:

Given strings S and T, find all occurrences of T as a substring of S.

Knuth *et al*. (1977):

On a failed match, we don't have to re-start the matching.

S=AAAAAAAAAB T=AAAB

AAAB

$$\boxed{O(|S| + |T|)}$$

Fast Substring Matching

Challenge:

Given strings S and T, find all occurrences of T as a substring of S.

Knuth *et al*. (1977):

On a failed match, we don’t have to re-start the matching.

The key is to know where in T we have to start comparing.

Fast Substring Matching

<u>Challenge</u>:

Given strings S and T, find all occurrences of T as a substring of S.

<u>Knuth *et al*. (1977)</u>:

The size of the shift on a mismatch is determined by the repetitions in T, is independent of S, and can be computed in $O(|T|)$ time.

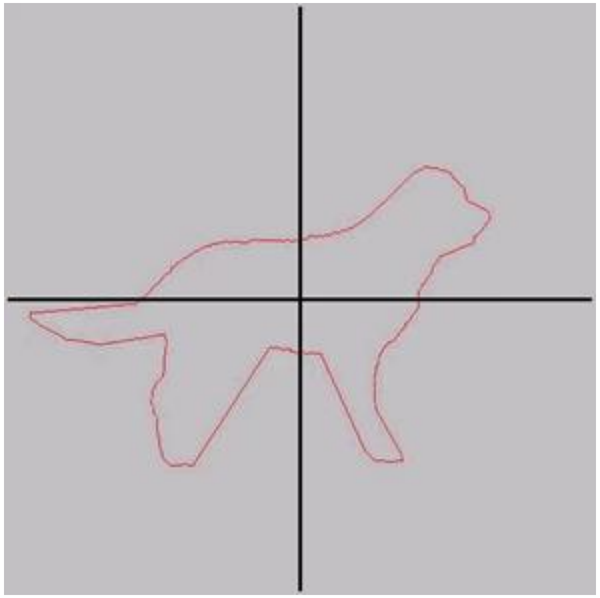
For more details, see:

Fast Pattern Matching in Strings.

Fast Substring Matching

Recall:

Our goal is to perform registration and symmetry detection on the circle.



Fast Substring Matching

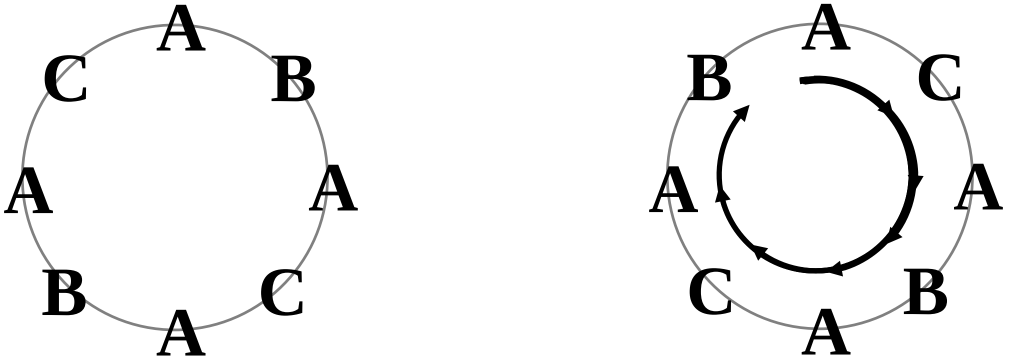
Applications:

If we think of a string as a signal on a circle:

- We can test if signal T is a rotation of S by testing if T is a substring of SS

S=ABACABAC

T=ACABACAB



SS=ABACABACABACABAC

T=ACABACAB

Fast Substring Matching

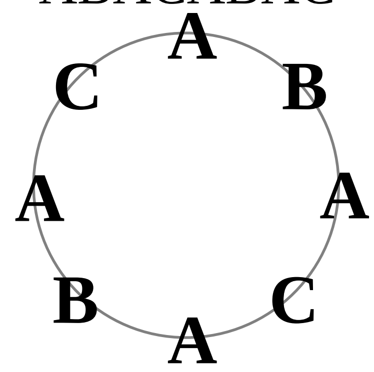
Applications:

If we think of a string as a signal on a circle:

- We can test if signal T is a rotation of S by testing if T is a substring of SS
- We can test if S has *rotational* symmetry by testing if S is a substring of SS

SS=ABACABACABACABAC
S= ABACABAC

S=ABACABAC



Fast Substring Matching

Applications:

If we think of a string as a signal on a circle:

- We can test if signal T is a rotation of S by testing if T is a substring of SS
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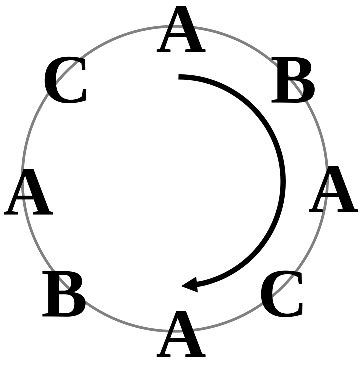
S=ABACABAC



SS=ABAC ABACABAC ABAC
S= ABAC ABAC
S= ABACABAC

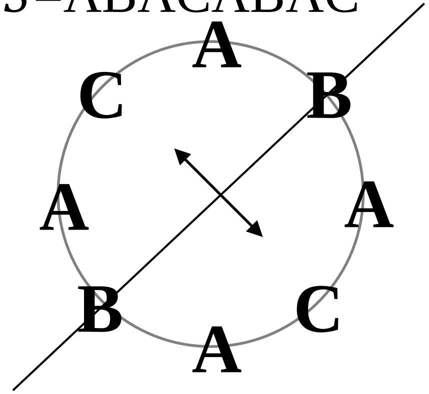
Fast Substring Matching

<u>Applications</u>:

If we think of a string as a signal on a circle:

- We can test if signal T is a rotation of S by testing if T is a substring of SS
- We can test if S has *rotational* symmetry by testing if S is a substring of SS
- We can test if S has *reflective* symmetry by testing if S is a substring of $(SS)^t$

S=ABACABAC



$$(SS)^t = \text{CABAC}\boxed{\text{ABACABAC}}\text{ABA}$$
$$S = \text{ABACABAC}$$

Fast Substring Matching

- ✓ A fast (linear time) algorithm for performing pattern detection on discrete signals
- ✗ Can only tell us if there is a perfect match
  - For real-world data, we need a continuous measure of similarity
- ✗ Only works for signals on a circle (or a line)
  - Hard to generalize to signals on other domains

Outline

Fast Substring Matching

Math Review

- Complex Numbers
- Vector Spaces
- Linear Operators

Complex Numbers

A complex number $c \in \mathbb{C}$ is any number that can be written as:

$$c = a + ib$$

with $a, b \in \mathbb{R}$ and i a square root of -1:

$$i^2 = -1$$

Complex Numbers

Given two complex numbers

$$c_1 = a_1 + ib_1 \quad \text{and} \quad c_2 = a_2 + ib_2$$

- The sum of the numbers is:

$$c_1 + c_2 = (a_1 + a_2) + i(b_1 + b_2)$$

- The product of the numbers is:

$$\begin{aligned} c_1 \cdot c_2 &= (a_1 + ib_1) \cdot (a_2 + ib_2) \\ &= a_1 \cdot a_2 + ib_1 \cdot ib_2 + a_1 \cdot ib_2 + ib_1 \cdot a_2 \\ &= (a_1 \cdot a_2 - b_1 \cdot b_2) + i(a_1 \cdot b_2 + b_1 \cdot a_2) \end{aligned}$$

Complex Numbers

Given a complex numbers:

$$c = a + ib$$

- The negation of the number is:

$$-c = -a - ib$$

- The conjugate of the number is:

$$\bar{c} = a - ib$$

- The reciprocal of the number is:

$$\frac{1}{c} = \frac{1}{c} \cdot \frac{\bar{c}}{\bar{c}} = \frac{a - ib}{a^2 + b^2}$$

Complex Numbers

Why do we care?

Complex Numbers

Why do we care?

<u>Fundamental Theorem of Algebra</u>

Given any polynomial:

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

there always exists a complex number $c \in \mathbb{C}$ s.t.:

$$p(c) = 0$$

Vector Spaces

A (real/complex) vector space V is a <u>set</u> of elements $v \in V$, with:

- An <u>addition</u> operator "+", and
- A <u>scaling</u> operator "·"

(Adding two vectors together gives a vector and scaling a vector by a number gives a vector.)

Vector Spaces (Formal Properties 1)

For all $u, v, w \in V$:

- Associative addition:
$$(u + v) + w = u + (v + w)$$

- Commutative addition:
$$u + v = v + u$$

- Additive identity:
There exists a unique vector $0 \in V$ such that:
$$v + 0 = v$$

- Additive inverse:
There exists a vector $(-v) \in V$ such that:
$$v + (-v) = 0$$

Vector Spaces (Formal Properties 2)

For all $u, v \in V$, and $\alpha, \beta \in \mathbb{R}/\mathbb{C}$:

- Distributive over vector addition:
$$\alpha \cdot (u + v) = (\alpha \cdot u) + (\alpha \cdot v)$$
- Distributive over scalar addition:
$$(\alpha + \beta) \cdot u = (\alpha \cdot u) + (\beta \cdot u)$$
- Compatible scalar multiplication:
$$\alpha \cdot (\beta \cdot u) = (\alpha \cdot \beta) \cdot u$$
- Scalar Identity:
$$1 \cdot u = u$$

Vector Spaces: Examples

Real Vector Spaces:

- The real / complex numbers
- The space of n-dimensional arrays with real / complex entries
- The space of $m \times n$ matrices with real / complex entries
- The space of real / complex valued functions on a circle / line / plane / sphere / etc.

Complex Vector Spaces:

- The complex numbers
- The space of n-dimensional arrays with complex entries
- The space of $m \times n$ matrices with complex entries
- The space of complex valued functions on a circle / line / plane / sphere / etc.

Vector Spaces

Note:

With the exception of the very last corollary, we will be talking about real vector spaces. The discussion is identical for complex vector spaces (i.e. just replace $\mathbb{R}$ with $\mathbb{C}$).

Vector Space Basis

A basis of a vector-space V is a set $\{v^1, \ldots, v^n\} \subset V$ such that:

1. Any vector $v \in V$ can be expressed as:
$$v = \mathbf{a}_1 v^1 + \cdots + \mathbf{a}_n v^n$$
with $\mathbf{a} = (\mathbf{a}_1, \ldots, \mathbf{a}_n) \in \mathbb{R}^n$.
2. No basis vector v_i can be expressed as the linear sum of the other basis vectors.

The dimension of V is the number basis vectors and does not depend on the particular choice of basis.

Vector Space Basis

Given a vector space V, with basis $\{v^1, \ldots, v^n\}$, we can associate vectors in V with vectors in $\mathbb{R}^n$:

$$v \in V \leftrightarrow \mathbf{a} \in \mathbb{R}^n$$

by setting $\mathbf{a} = (\mathbf{a}_1, \ldots, \mathbf{a}_n)$ to be the coefficients of v with respect to the basis $\{v^1, \ldots, v^n\}$:

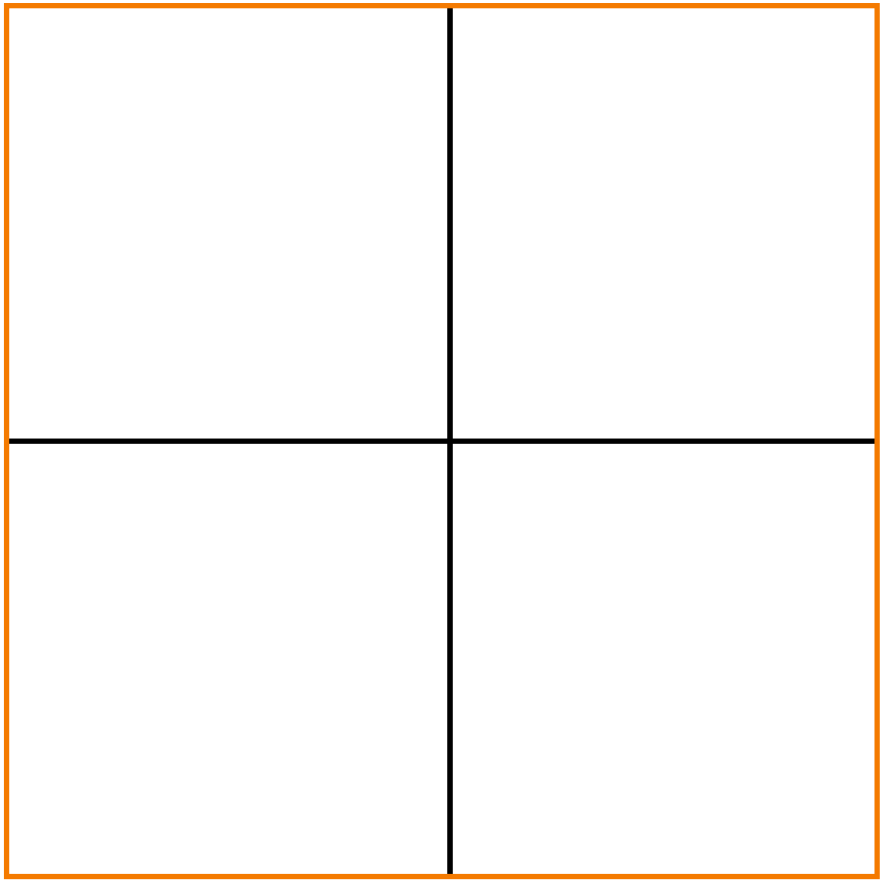
$$v = \mathbf{a}_1 v^1 + \cdots + \mathbf{a}_n v^n$$

Vector Space Basis

Many different bases can be used to represent the same vector space.

<u>Example</u>:

Consider the set of points in 2D Euclidean space.



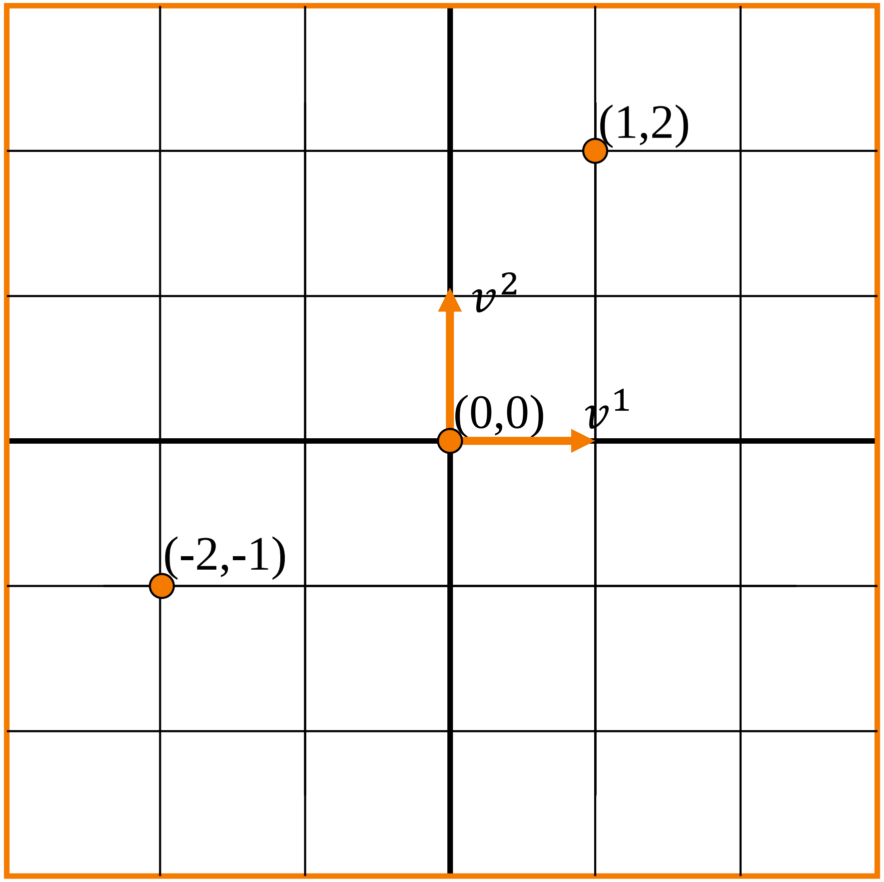
Vector Space Basis

Many different bases can be used to represent the same vector space.

<u>Example</u>:

Consider the set of points in 2D Euclidean space.

We can represent each vector in terms of its (x, y)-coordinates.



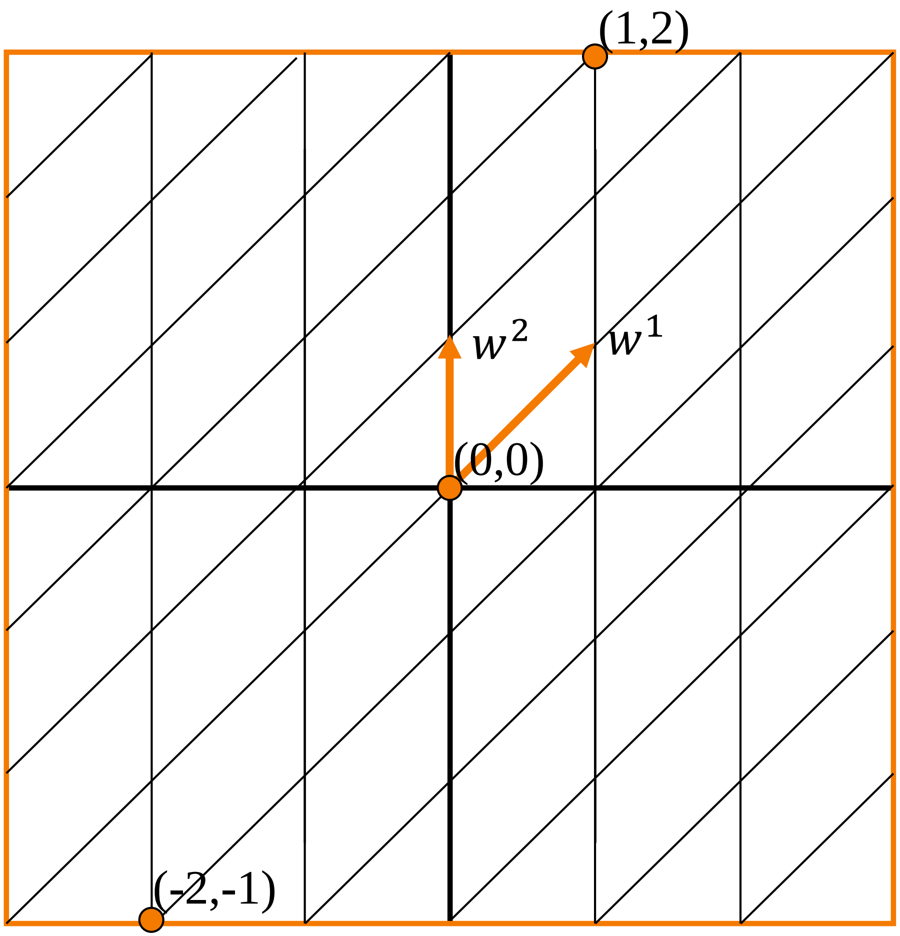
Vector Space Basis

Many different bases can be used to represent the same vector space.

<u>Example</u>:

Consider the set of points in 2D Euclidean space.

Or we could use a different basis…



Linear Maps

Given vector spaces V and W, the map $L: V \to W$, is linear if for all $v, w \in V$ and $\alpha, \beta \in \mathbb{R}$:

$$L(\alpha v + \beta w) = \alpha L(v) + \beta L(w)$$

If it exists, the inverse of a linear map L is the map L^{-1} with the property that:

$$L^{-1}\big(L(v)\big) = v$$

Linear Maps

If $L: V \rightarrow W$, is a linear map:

The set of vectors:

$$K = \{v \in V | L(v) = 0\}$$

is a vector subspace called the kernel.

The set of vectors:

$$I = \{w \in W | \exists v \in V \; s.t. \; w = L(v)\}$$

is a vector subspace called the image.

Matrices

Given a vector space V, with basis $\{v^1, \ldots, v^n\}$, a linear map $L: V \to V$ can be uniquely expressed by a <u>matrix</u> $\mathbf{L} \in \mathbb{R}^{n\times n}$ s.t.:

$$\mathbf{b} = \mathbf{L}\mathbf{a}$$

whenever:

$$\mathbf{b}_1 v^1 + \cdots + \mathbf{b}_n v^n = L(\mathbf{a}_1 v^1 + \cdots + \mathbf{a}_n v^n)$$

Change of Basis

Given a vector space V, and given two bases $\{v^1, \ldots, v^n\}$ and $\{w^1, \ldots, w^n\}$, the change of basis matrix $\mathbf{B} \in \mathbb{R}^{n\times n}$ is the matrix satisfying:

$$\mathbf{b} = \mathbf{Ba}$$

whenever:

$$\mathbf{b}_1 w^1 + \cdots + \mathbf{b}_n w^n = \mathbf{a}_1 v^1 + \cdots + \mathbf{a}_n v^n$$

Change of Basis

Given:

- A vector space V,
- Two bases $\{v^1, \cdots, v^n\}$ and $\{w^1, \cdots, w^n\}$,
- A linear operator L represented by the matrix $\mathbf{L}$ in terms of the basis $\{v^1, \cdots, v^n\}$.

The matrix representation for L in terms of the basis $\{w^1, \cdots, w^n\}$ is given by:

$$\mathbf{B}\mathbf{L}\mathbf{B}^{-1}$$

Change of Basis

Why do we care?

Change of Basis

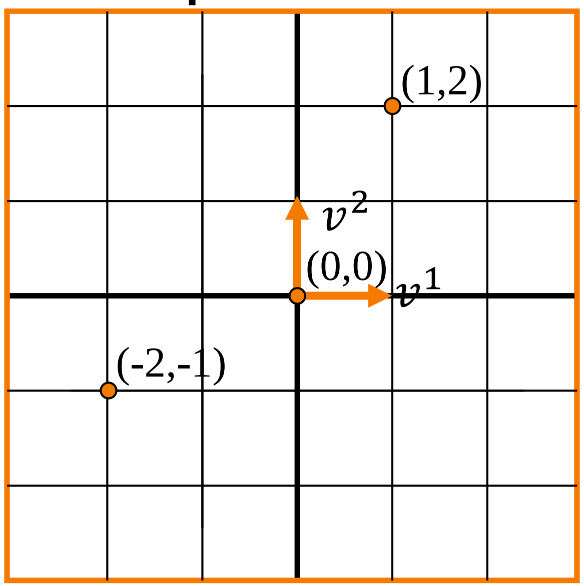
Why do we care?

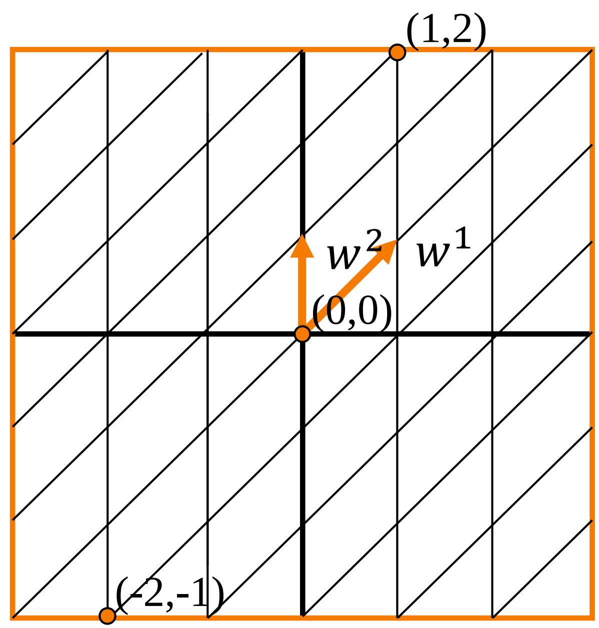
Choosing the appropriate basis can make it much easier to understand a linear operator.

Change of Basis

Why do we care?

Example:





$$\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

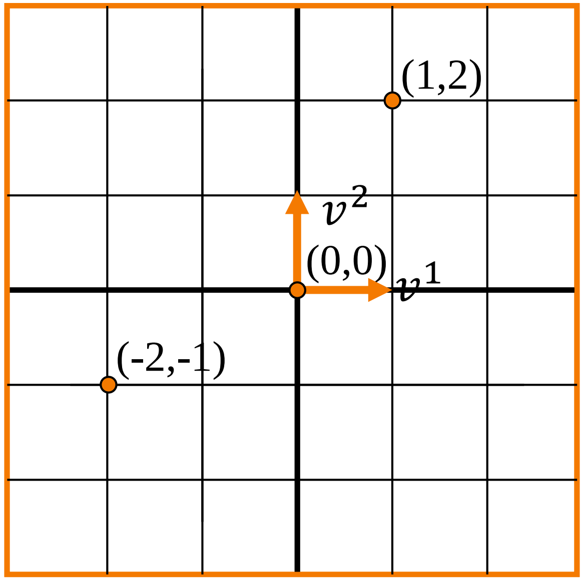
$$\mathbf{B}^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

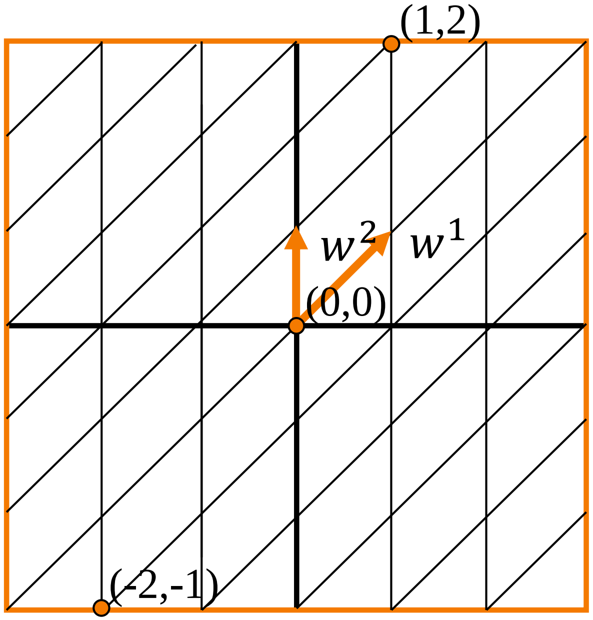
$$\mathbf{L} = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$$

Change of Basis

Why do we care?

Example:





$$\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

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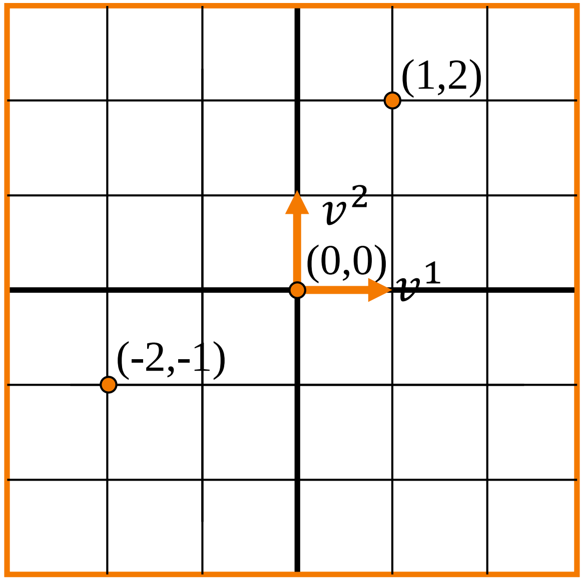
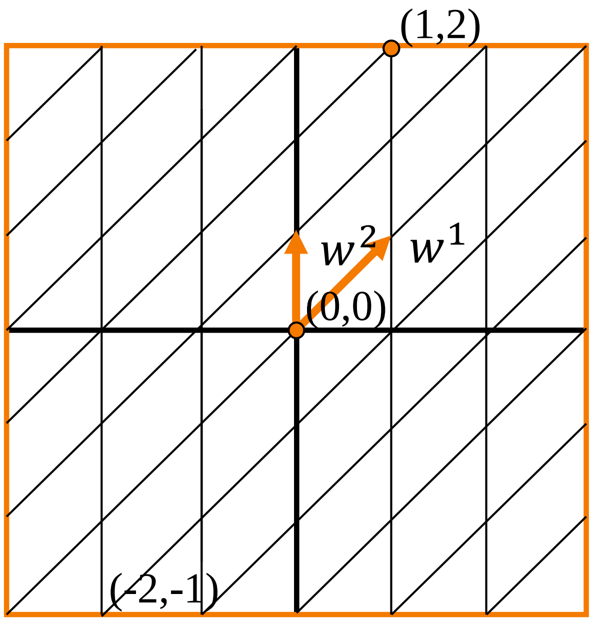
$$\mathbf{L} = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} \Longrightarrow \mathbf{BLB}^{-1}$$

Change of Basis

Why do we care?

Example:





$$\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

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$$\mathbf{L} = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} \Longrightarrow \mathbf{BLB}^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

Change of Basis

Why do we care?

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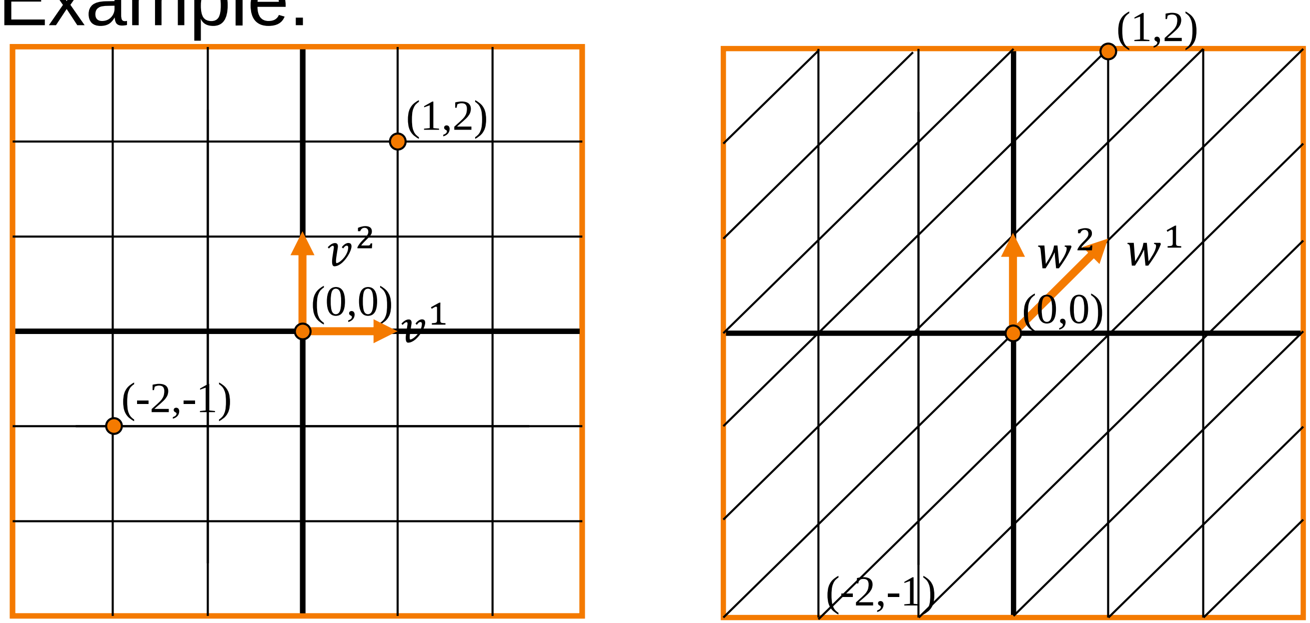

$$\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

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$$\mathbf{L} = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} \Longrightarrow \mathbf{BLB}^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

In this basis the linear operator becomes anisotropic scaling!

Determinants

The <u>determinant</u> is a function that associates a scalar value to every linear map $L: V \to V$.

The determinant of $L: V \to V$ is the (signed) volume of the image of (any) unit cube in V.

Determinants

The determinant of a linear map $L: V \to V$ equals zero if and only if there exists $v \in V$, with $v \neq 0$, such that $L(v) = 0$.

Eigenvalues and Eigenvectors

The scalar $\lambda \in \mathbb{R}$ is an eigenvalue of a linear operator $L: V \to V$ if there exists $v \in V$ such that:

$$\lambda \cdot v = L(v).$$

In this case, v is an eigenvector of L.

Eigenvalues and Eigenvectors

If $L: V \to V$ has an eigenpair $(\lambda, v) \in \mathbb{R} \times V$, then:

$$0 = (L - \lambda \cdot \mathrm{Id.})(v)$$

$\Rightarrow$ The linear operator

$$(L - \lambda \cdot \mathrm{Id.}): V \to V$$

has zero determinant.

Characteristic Polynomials

If we treat λ as a variable, the determinant:

$$\chi_L(\lambda) = \det(L - \lambda \cdot \mathrm{Id.})$$

is a polynomial of degree n in λ.

This is the characteristic polynomial of L.

Characteristic Polynomials

The roots of the characteristic polynomial of L:

$$\chi_L(\lambda) = \det(L - \lambda \cdot \mathrm{Id.})$$

are precisely the eigenvalues of L.

Corollary:

If $L: V \to V$ is a linear map on a **complex** vector space, L always has at least one eigenvalue.