Arrangements

O'Rourke, Chapter 6

Outline

- Duality
- Generalizing Voronoi Diagrams
- Ham-Sandwich Cuts

Duality

Definition:

Given a point $p = (\alpha, \beta)$ in the plane, define the *dual line* to be the (non-vertical) line with equation:

$$D(p) = \{(x, y) | y = 2\alpha x - \beta\}$$

Note:

- The slope depends on the x-coordinate of p.
- The height depends on the y-coordinate of p. (Height decreases as the y-coordinate increases.)

Duality

<u>Definition</u>:

Given a point $p = (\alpha, \beta)$ in the plane, define the *dual line* to be the (non-vertical) line with equation:

$$D(p) = \{(x, y) | y = 2\alpha x - \beta\}$$

Given a (non-vertical) line $L = \{(x, y) | y = mx + b\}$, define the *dual point* to be the point with coordinates:

$$D(L) = \left(\frac{m}{2}, -b\right)$$

Duality

Claim (Inverse):

The dual of the dual is the identity.

Proof (Points):

$$p = (\alpha, \beta)$$
$$\Rightarrow D(p) = \{(x,y) | y = 2\alpha x - \beta\}$$
$$\Rightarrow D\big(D(p)\big) = \left(\left(\frac{2\alpha}{2}\right), \beta\right) = p$$

$$D((\alpha,\beta)) = \{(x,y) | y = 2\alpha x - \beta\} \qquad D(\{(x,y) | y = mx + b\}) = \left(\frac{m}{2}, -b\right)$$

Duality

Claim (Inverse):

The dual of the dual is the identity:

Alternate Proof (Lines):

$$L = \{(x,y) | y = mx + b\}$$

$$\Rightarrow D(L) = \left(\frac{m}{2}, -b\right)$$

$$\Rightarrow D\big(D(L)\big) = \left\{(x,y) \middle| y = \left(2\left(\frac{m}{2}\right)\right)x + b\right\} = L$$

$$D\big((\alpha,\beta)\big) = \{(x,y) | y = 2\alpha x - \beta\} \qquad D\big(\{(x,y)|y = mx + b\}\big) = \left(\frac{m}{2}, -b\right)$$

Duality

Claim (Incidence):

Given $p = (\alpha, \beta)$ and $L = \{(x, y) | y = mx + b\}$:

$$p \in L \quad \Leftrightarrow \quad D(L) \in D(p).$$

Proof:

$p \in L$

$\Leftrightarrow \beta = m\alpha + b$

$\Leftrightarrow -b = 2\alpha\left(\frac{m}{2}\right) - \beta$

$\Leftrightarrow D(L) \in D(p)$

$D((\alpha, \beta)) = \{(x, y) | y = 2\alpha x - \beta\}$ $\quad$ $D(\{(x, y) | y = mx + b\}) = \left(\frac{m}{2}, -b\right)$

Duality

Claim (Incidence):

Given $p = (\alpha, \beta)$ and $L = \{(x, y) | y = mx + b\}$:

$$p \in L \quad \Leftrightarrow \quad D(L) \in D(p).$$

Corollary:

$p \in L_1 \cap L_2$ if and only if $D(L_1), D(L_2) \in D(p)$.

$$D((\alpha, \beta)) = \{(x, y) | y = 2\alpha x - \beta\} \qquad D(\{(x, y) | y = mx + b\}) = \left(\frac{m}{2}, -b\right)$$

Duality

Claim (Ordering):

If line $L = \{(x, y) | y = mx + b\}$ is below/above point $p = (\alpha, \beta)$ then line $D(L)$ is above/below $D(p)$.

Proof:

L is below p

$\Leftrightarrow \beta > m\alpha + b$

$\Leftrightarrow -b > 2\alpha\left(\frac{m}{2}\right) - \beta$

$\Leftrightarrow \left(\frac{m}{2}, -b\right)$ is above $\{(x, y) | y = 2\alpha x - \beta\}$

$\Leftrightarrow D(L)$ is above $D(p)$

Duality

Claim (Parabola):

p is on the parabola if and only if $D(p)$ is the tangent to the parabola at p.

Proof:

$L = \{(x,y)|y = mx + b\}$ is tangent to the parabola

$$\Updownarrow$$

$$L = \{(x,y)|y = 2\alpha x - \alpha^2\}$$

$$\Updownarrow$$

$$D(L) = (\alpha, \alpha^2)$$

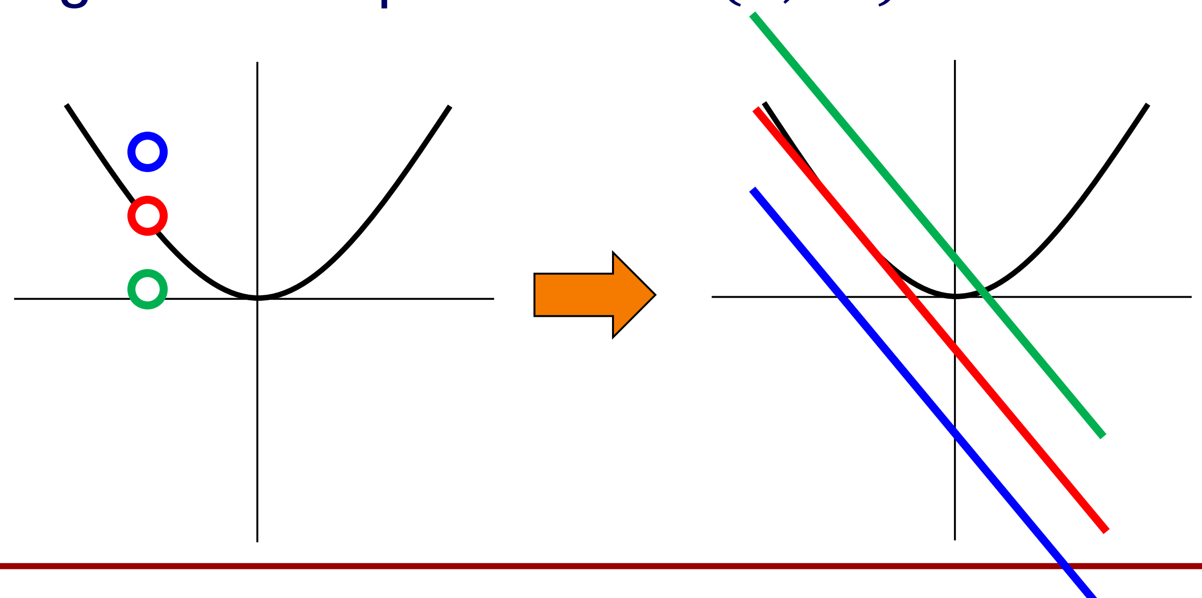
$$\Updownarrow$$

$D(L)$ is on the parabola

Duality

For a point $p = (\alpha, \beta)$:

- $\beta = \alpha^2$: If p is on the parabola, $D(p)$ is the tangent to the parabola at (α, α^2).
- $\beta < \alpha^2$: If p is below the parabola, $D(p)$ is parallel and above the tangent to the parabola at (α, α^2).
- $\beta > \alpha^2$: If p is above the parabola, $D(p)$ is parallel and below the tangent to the parabola at (α, α^2).



Outline

- Duality
- Generalizing Voronoi Diagrams
- Ham-Sandwich Cuts

Generalizing Voronoi Diagrams

Recall:

- Given a point $P(p) = (p, \|p\|^2)$ on the paraboloid, the tangent plane is given by:
$$z_p(r) = 2\langle p, r\rangle - \|p\|^2$$
- For any point q the (vertical) distance between the points on the parabola and the tangent plane are:
$$P(q) - z_p(q) = \|p - q\|^2$$
- The points in a Voronoi face are closer to the site associated to the face than to any other site.

Generalizing Voronoi Diagrams

Given a set of points in the plane $P = \{p_1, \dots, p_n\}$ if we draw the tangents to the paraboloid at the points $\{(p_i, \|p_i\|^2)\}$ and view from above, we "see" the Voronoi diagram.

Generalizing Voronoi Diagrams

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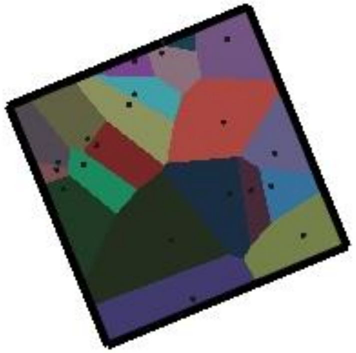


Generalizing Voronoi Diagrams

Given a set of points in the plane $P = \{p_1, \ldots, p_n\}$ if we draw the tangents to the paraboloid at the points $\{(p_i, \|p_i\|^2)\}$ and view from below, we "see" the furthest-point Voronoi diagram.

Generalizing Voronoi Diagrams

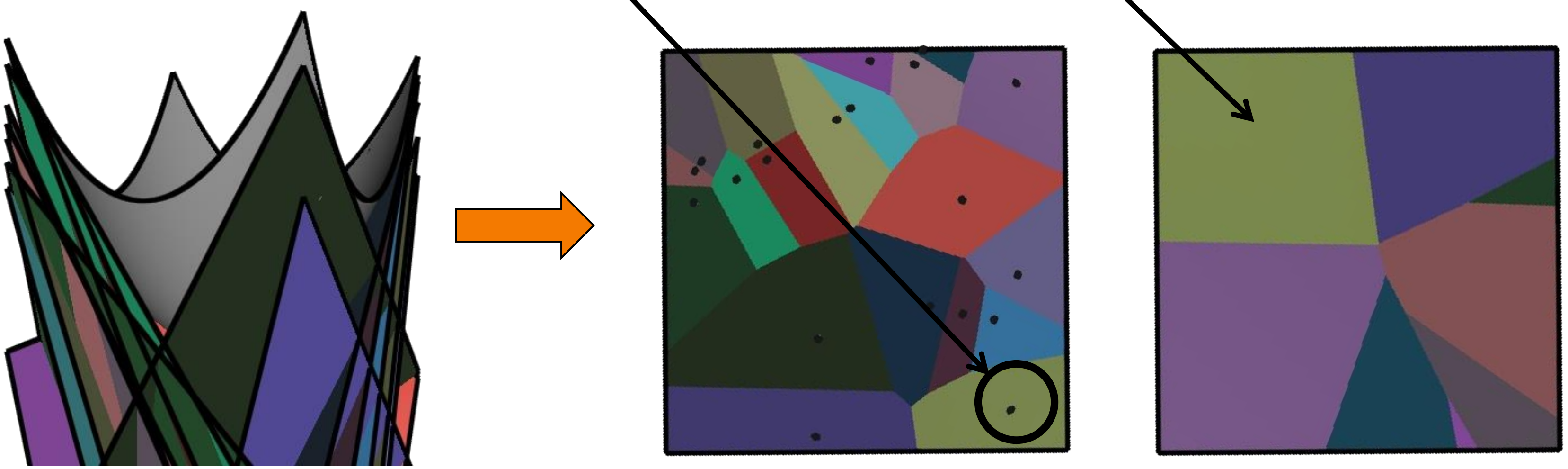
Given a set of points in the plane $P = \{p_1, \ldots, p_n\}$ if we draw the tangents to the paraboloid at the points $\{(p_i, \|p_i\|^2)\}$ and view from below, we "see" the furthest-point Voronoi diagram.

Generalizing Voronoi Diagrams

From Above: (Nearest-Point) Voronoi Diagram

From Below: Furthest-Point Voronoi Diagram

The points here are further from this site than from any other site.



Nearest Furthest

Generalizing Voronoi Diagrams

Definition:

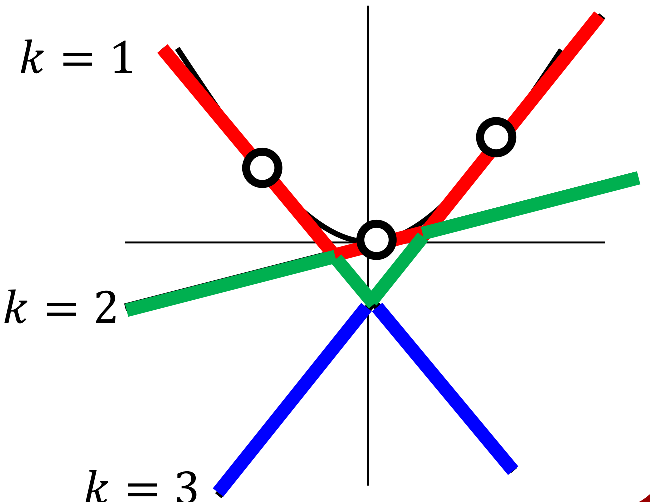
The k*-th order Voronoi Diagram* is a partition of space into convex cells, indexed by k-tuples of points $(p_{i_1},\dots,p_{i_k})$, with $i_j < i_{j+1}$, such that a point q is in cell $(p_{i_1},\dots,p_{i_k})$ iff. the k nearest neighbors of q are $\{p_{i_1},\dots,p_{i_k}\}$.

Generalizing Voronoi Diagrams

The set of tangent planes to the paraboloid form an arrangement.

<u>Definition</u>:

The k*-th level of the arrangement* is the set of faces in the arrangement which have exactly $k-1$ faces above them.



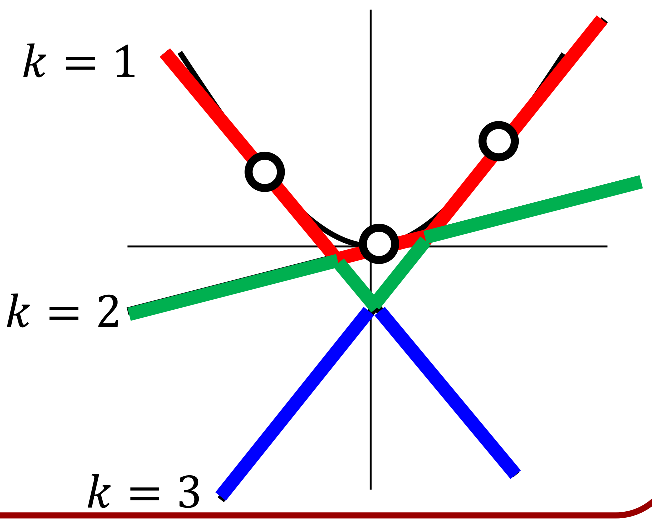
Generalizing Voronoi Diagrams

Note:

Every point on a k-th level edge of the arrangement have the same set of lines above them.

⇒ The projection of the duals of those lines are the sites closest to the projection of the point.

⇒ The projection of the line segment is a connected component of the k-th level Voronoi diagram.



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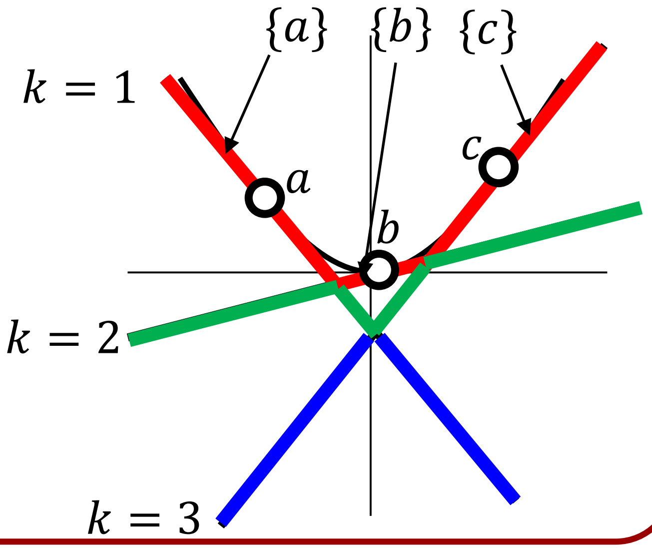

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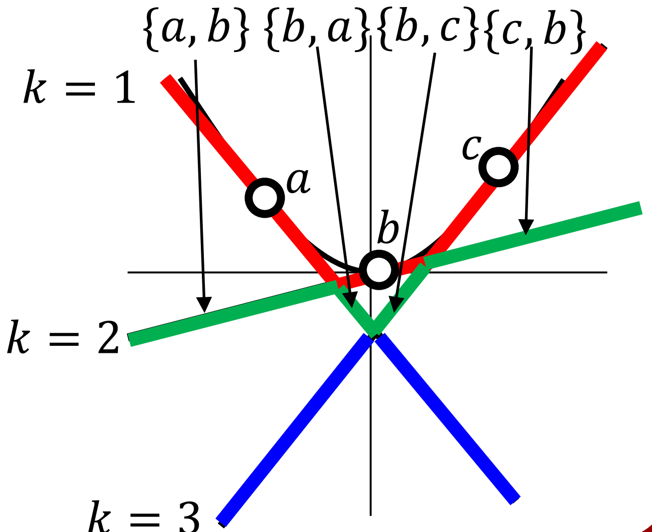

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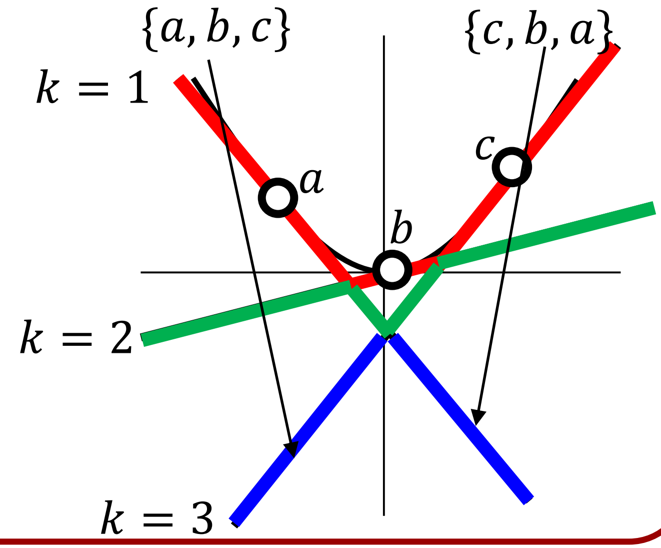
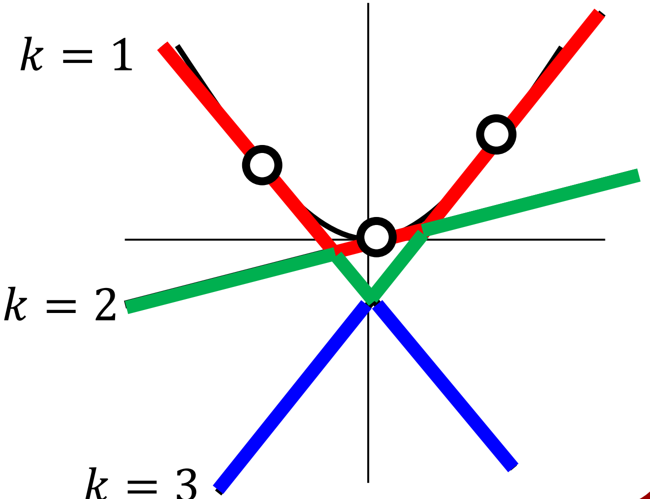


[Edelsbrunner 1987]

Theorem:

The points of intersection of the k-th and $(k+1)$-th levels in the arrangement project to the k-th order Voronoi diagram.



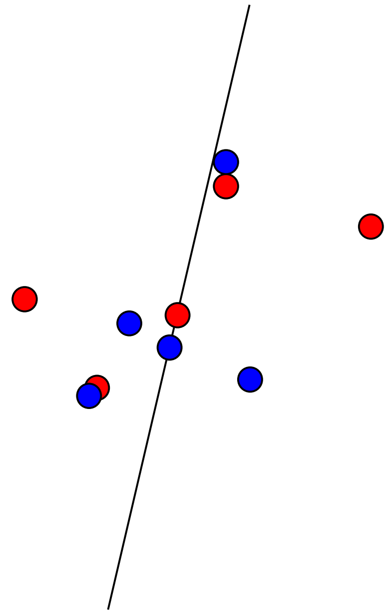
Outline

- Duality
- Generalizing Voronoi Diagrams
- Ham-Sandwich Cuts
  - Red-Blue Matching

Ham-Sandwich Cuts

Claim:

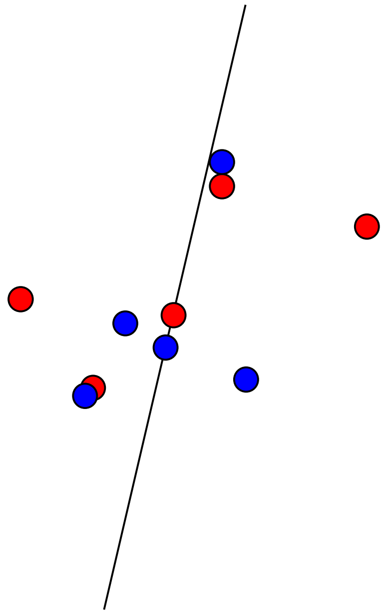
Given two sets of points, P_1 and P_2, in the plane, there is a line that simultaneously bisects both sets.



Ham-Sandwich Cuts

Proof:

Assume general position and, with some loss of generality, that the two point-sets each have an odd number of points.



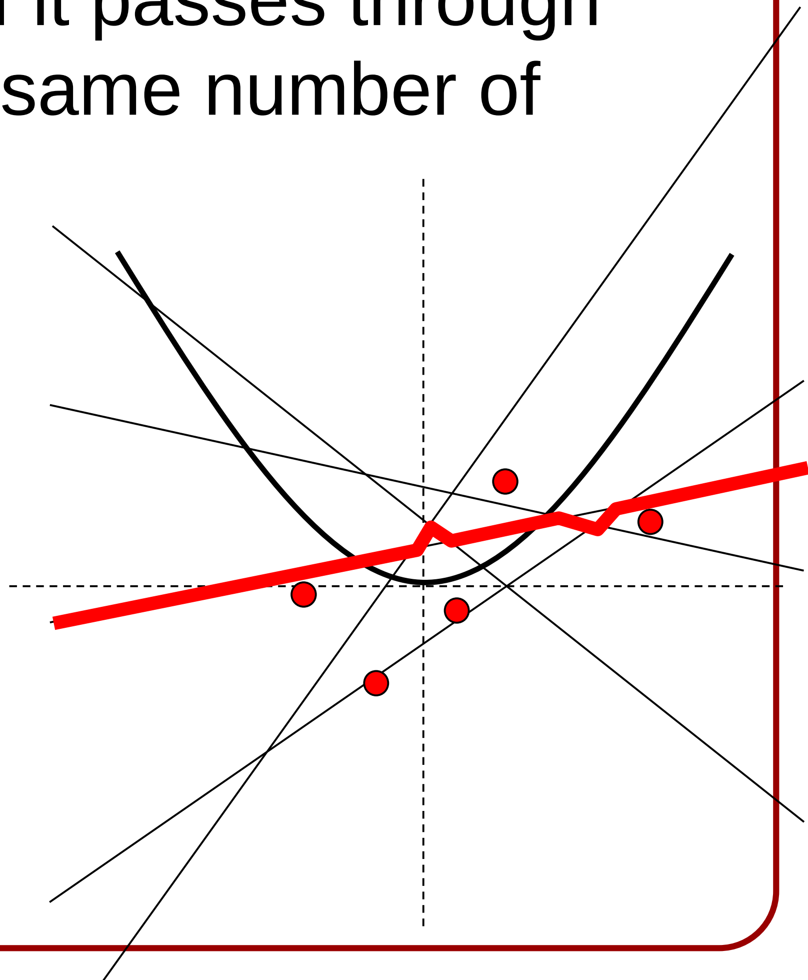
Ham-Sandwich Cuts

<u>Note</u>:

A line splits the points in two if it passes through one of the points and has the same number of points above and below.

⇔ The dual point is on a dual line and has the same number of dual lines above and below.

⇔ The dual point is on the median level of the dual arrangement.



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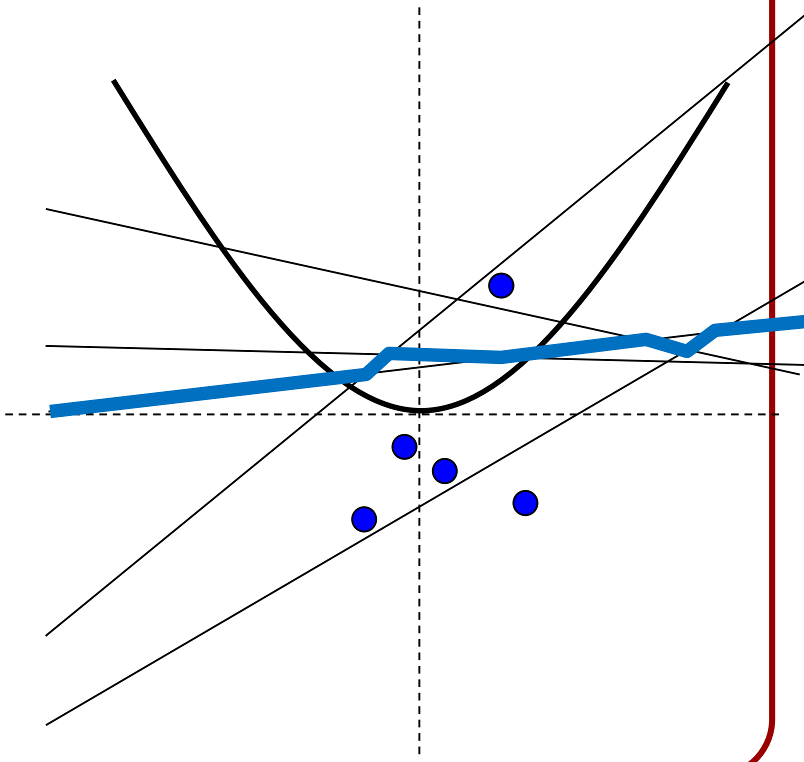
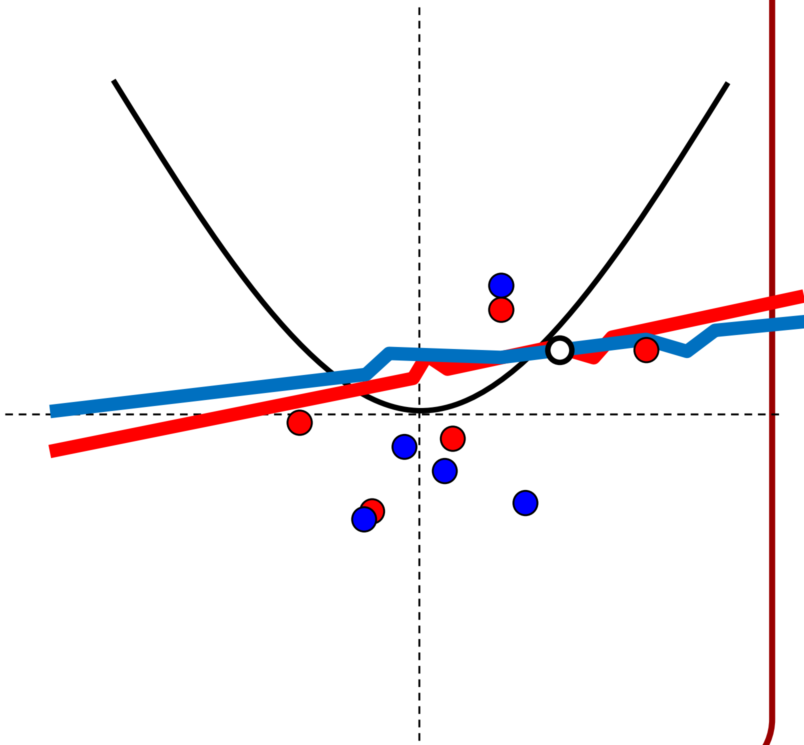


Ham-Sandwich Cuts

Note:

A line splits the points in two if it passes through one of the points and has the same number of points above and below.

⇒ To find the cut, we need to find the intersection of the median levels of the two arrangements.



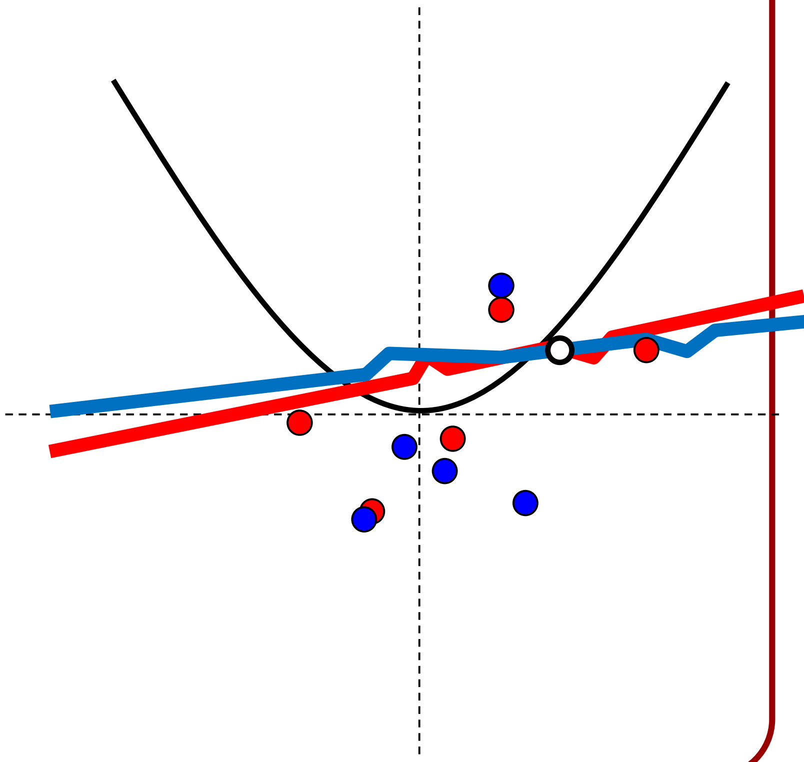
Ham-Sandwich Cuts

Claim:

The median levels of two arrangements must intersect (an odd number of times).

Sub-Claim:

The two infinite edges of the median level are defined by the same line.



Ham-Sandwich Cuts

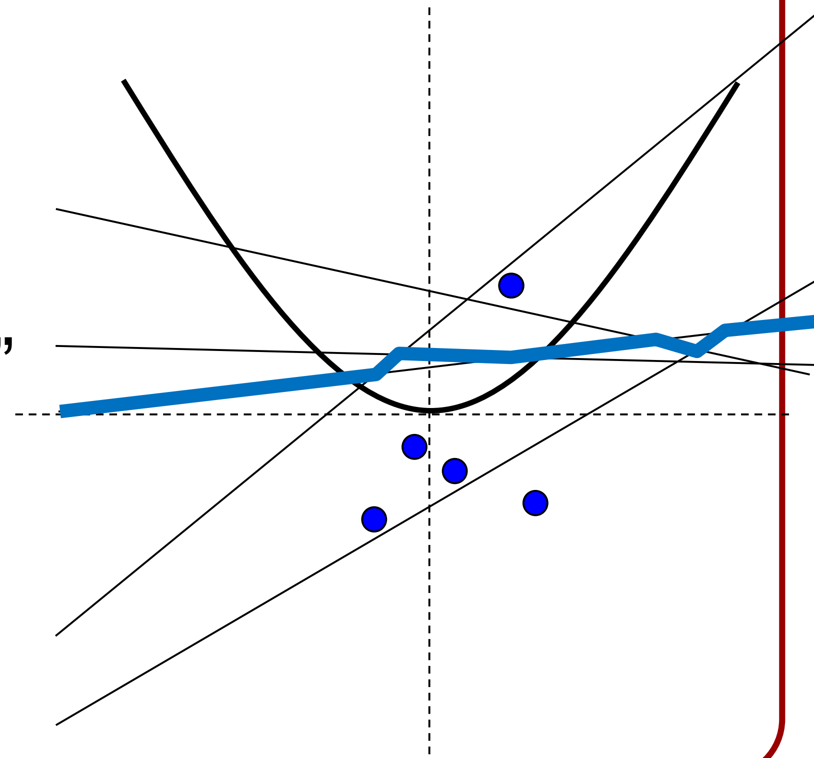
Proof (Sub-Claim):

Let L be the line giving the left median edge.

$\Rightarrow$ As $x \to -\infty$ half the lines are above/below.

$\Rightarrow$ Assuming general position, at $x = \infty$ the "above" lines are "below" and the "below" lines are "above".

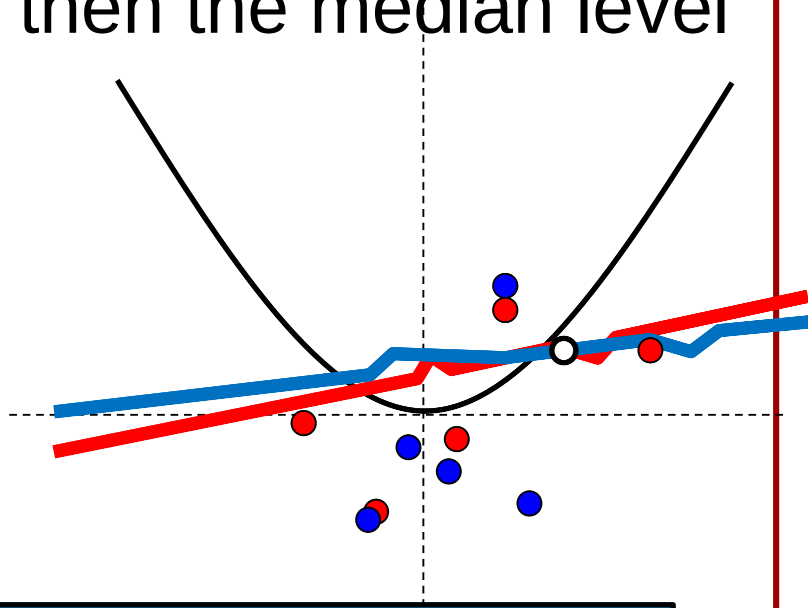
$\Rightarrow$ L also defines the right median edge.



Ham-Sandwich Cuts

<u>Proof (Claim)</u>:

Since the left/right-most edges lie on the same line, if the median level of P_1 is above (resp. below) the median level of P_2 as $x \to -\infty$ then the median level of P_1 is below (resp. above) the median level of P_2 as $x \to \infty$.



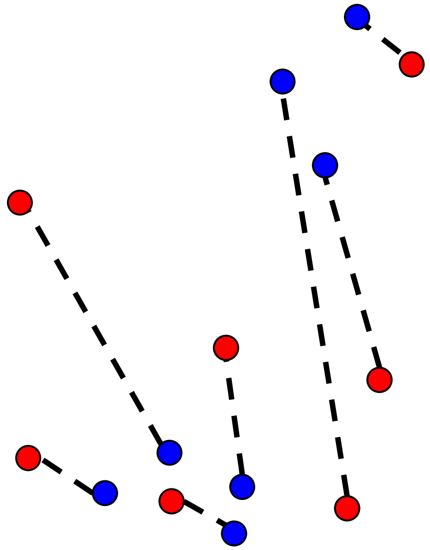
⇒ The median levels cross (an odd number of times).

<u>[Lo, Maoutsek, and Steiger, 1994]</u>:
The intersection can be found in $O(|P_1| + |P_2|)$ time.

Red-Blue Matching

Claim:

Given n red and n blue points in the plan, we can pair them up using non-intersecting line segments.



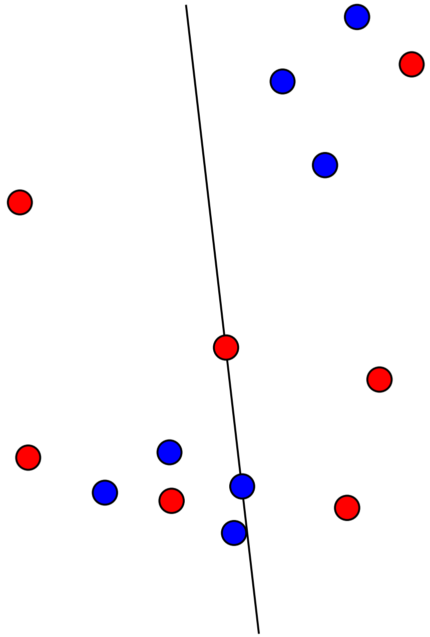
Red-Blue Matching

<u>Claim</u>:

Given n red and n blue points in the plan, we can pair them up using non-intersecting line segments.

<u>Proof (by Algorithm)</u>:

- Compute a ham-sandwich cut
- (Recursively) compute a matching.



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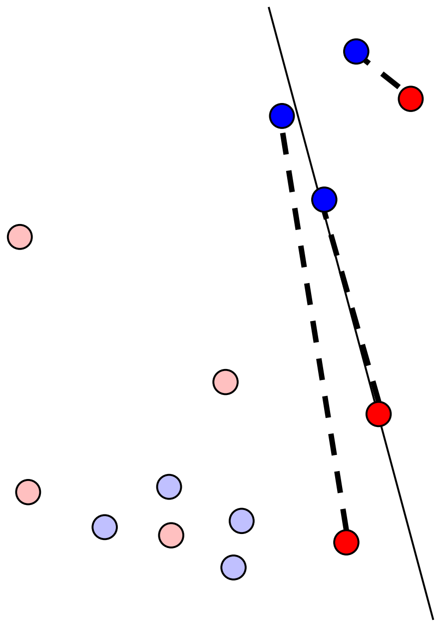

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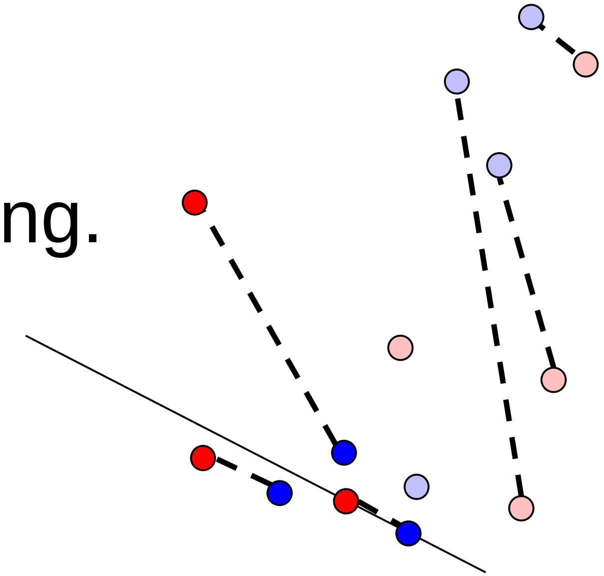

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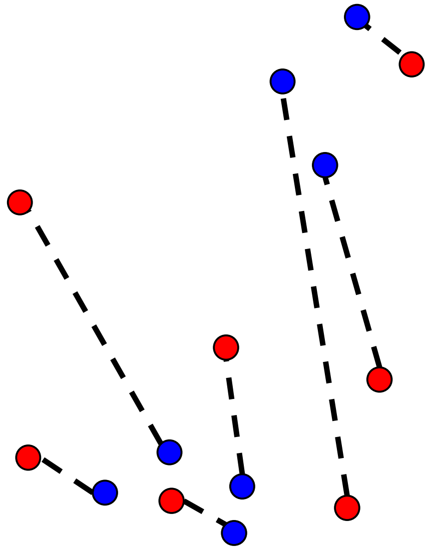

Since the line-segments for each sub-problem are on one side of the cut, the segments from the two sub-problems do not intersect.

Red-Blue Matching

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- Compute a ham-sandwich cut
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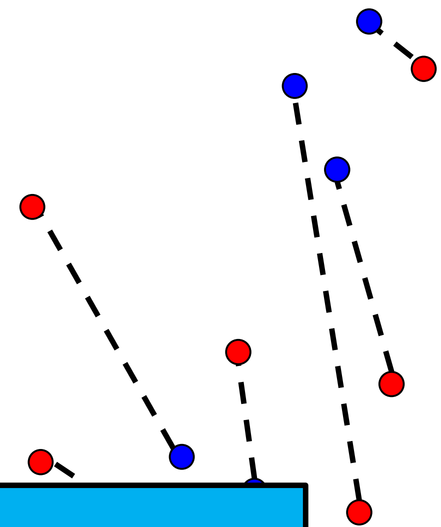

Since the line-segments for each sub-problem are on one side of the cut

sub

[Lo, Maoutsek, and Steiger, 1994]:
The matching can be found in $O(n \log n)$ time.