Arrangements

O’Rourke, Chapter 6
Outline

• Voronoi Diagrams
• Arrangements
Voronoi Diagrams

Recall:

We can compute the Delaunay Triangulation by raising the points to a paraboloid and computing the projection of the lower hull.
Voronoi Diagrams

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**Recall:**

Given a point \( P(p) = (p, \|p\|^2) \) on the paraboloid, the tangent plane is given by:

\[
z_p(r) = 2 \langle p, r \rangle - \|p\|^2
\]

For any point \( q \) the (vertical) distance between the points on the parabola and the tangent plane are:

\[
P(q) - z_p(q) = \|q\|^2 - (2 \langle q, p \rangle - \|p\|^2)
\]

\[
= \|p - q\|^2
\]
Voronoi Diagrams

⇒ Given points $p$ and $q$, wherever the tangent plane at $p$ is higher than the tangent plane at $q$, we are closer to $p$ than to $q$.

\[ z_p(r) \geq z_q(r) \]

\[ \Leftrightarrow \]

\[ P(r) - z_p(r) \leq P(r) - z_q(r) \]

\[ \Leftrightarrow \]

\[ \|p - r\|^2 \leq \|q - r\|^2 \]
Voronoi Diagrams

⇒ Given points $p$ and $q$, wherever the tangent plane at $p$ is higher than the tangent plane at $q$, we are closer to $p$ than to $q$.

⇒ We can visualize the Voronoi diagram by drawing the tangent planes at the sites and looking down the $z$-axis.
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Arrangements

Definition:

An *arrangement of lines* is a set of lines in the plane, inducing a partition of the domain into (convex) faces, edges, and vertices.

An arrangement is *simple* if all pairs of lines intersect, and no three lines intersect at the same point.
Claim:
A simple arrangement of $n$ lines has

- $\binom{n}{2}$ vertices,
- $n^2$ edges, and
- $\binom{n}{2} + n + 1$ faces.
Proof (Vertices):

Since each pair of lines intersects exactly once, the total number of vertices is the number of distinct line pairs, \( \binom{n}{2} \).
Combinatorics

Proof (Edges):

Since each line is intersected by $n - 1$ other lines, partitioning the lines into $n$ edges, the total number of edges is $n^2$. 
Proof (Faces):

Using stereographic mapping, arrangements of lines in the plane can be thought of as polygonizations of the sphere.
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Note:
The stereographic mapping of the lines intersect at the North Pole.
Combinatorics

Proof (Faces):

By Euler’s theorem the number of faces is:

\[ F = 2 - (V + 1) + E \]
\[ = 2 - \binom{n}{2} - 1 + n^2 \]
\[ = \binom{n}{2} + n + 1 \]
Zone Theorem

Definition:

Given an arrangement $A$ and a line $L$ (s.t. $A \cup \{L\}$ is simple) the *zone* of $L$ in $A$, $Z(L)$, is the set of faces of $A$ intersected by $L$. 

$A (n = 4)$
Zone Theorem

Notation:
The number of edges in $Z(L)$ is denoted $z(L)$.
The max size of $z(L)$ over all lines is denoted $z_n$.
Zone Theorem

Note:

Assuming that no line in the arrangement is horizontal, we can mark each edge as left/right.

\[
L
\]

\[
z(L) = 15 \quad A (n = 4)
\]
Zone Theorem

Claim:

For an arrangement of $n$ lines, $2n \leq z_n \leq 6n$.

Specifically, the number of left/right edges crossed by a line $L$ is bounded by $3n$. 

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Zone Theorem

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For an arrangement of $n$ lines, $2n \leq z \leq 6n$.

Specifically, the number of left/right edges crossed by a line $L$ is bounded by $3n$.

Note:
The expected complexity is $\sqrt{|E|} = O(n)$. This means that we can’t do worse than average.
Zone Theorem

Proof:

Without loss of generality, assume that the line $L$ maximizing the number of edges is horizontal.

With a slight loss of generality, assume that none of the lines are vertical.

Proceed by induction.
Zone Theorem

**Proof (base case):**

Trivially true when $n = 0$. 
Zone Theorem

Proof (inductive case):

Remove the right-most line.

By induction, the number of left edges crossed is less than or equal to $3(n - 1)$.

Need to show that adding the line back contributes at most 3 new $L$ edges.
Zone Theorem

Claim:

Adding the right-most line introduces exactly one new left edge.
Zone Theorem

Proof of Claim:

It introduces one because this will be a left edge of the right-most face.
Zone Theorem

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Exactly one because a right-most line cannot contribute more than one edge.
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Proof of Claim:

It introduces one because this will be a left edge of the right-most face.

Exactly one because a right-most line cannot contribute more than one edge.

If it is split by a line from the left, only one of the two segments will be in the zone, (the one containing $L$.)
Zone Theorem

Proof of Claim:

It introduces one because this will be a left edge of the right-most face.

Exactly one because a right-most line cannot contribute more than one edge.

If it is split by a line from the right, then it wasn’t right-most.
Zone Theorem

Claim:
Adding the right-most line splits at most two existing left edges.
Zone Theorem

Proof of Claim:

As above, if the right-most line splits a left edge in two, the edge has to be on the right-most face.

Since faces are convex, the line can split at most two such edges.
Zone Theorem

**Corollary:**

We can construct a (simple) arrangement of $n$ lines in $O(n^2)$ time.
Zone Theorem

Proof:

Iteratively add lines.

• Find an intersection with any existing line.
• Cycle around faces to the left
• Cycle around faces to the right
Zone Theorem

Proof:

Since the number of edges traversed at each iteration is $O(n)$, the total complexity is $O(n^2)$. 
Zone Theorem

Generalizations:

In $d$-dimensional space:

• The number of faces of any dimension of an arrangement is $O(n^d)$.

• The number of faces in the zone of a hyper-plane is bounded by $O(n^{d-1})$.

• The arrangement can be computed in $O(n^d)$ time.