FFTs in Graphics and Vision

The Spherical Laplacian

Assignment 1 (Smoothing)

```
SquareGrid<> f;
FourierKey2D<> fK;
double scl = 0.0;
f.resize( res );

for( int y=-res/2 ; y<res/2; y++ ) for( int x=-res/2 ; x<res/2 ; x++ )
        f( x , y ) = exp( - ( x*x + y*y ) /  ( 2. * Smooth.value * Smooth.value ) ) , scl += f( x , y );
scl = res * res / ( scl * 4 * PI * PI );
for( int y=0 ; y<res ; y++ ) for( int x=0 ; x<res ; x++ ) f( x , y ) *= scl;

xForm.ForwardFourier( f , fK );

for( int i=0 ; i< fK.resolution() ; i++ ) for( int j=0 ; j< fK.size() ; j++ )
        gK ( i , j ) *= fK ( i , j ) , bK ( i , j ) *= fK ( i , j ) , rK ( i , j ) *= fK ( i , j ) ;

xForm.InverseFourier( rK , r ) , xForm.InverseFourier( gK , g ) , xForm.InverseFourier( bK , b );
```

Assignment 1 (Continuous Laplacian)

```
for( int i=0 ; i<rK.resolution() ; i++ )
{
        int _i = i<rK.resolution()/2 ? i : rK.resolution()-i;
        for( int j=0 ; j<rK.size() ; j++ )
        {
                double scl = - 4. * PI * PI * ( _i*_i + j*j ) / ( res * res );
                rK (i,j) *= scl , gK (i,j) *= scl , bK (i,j) *= scl;
        }
}

xForm.InverseFourier( rK , r ) , xForm.InverseFourier( gK , g ) , xForm.InverseFourier( bK , b );
```

Ouline

- Stokes' Theorem
- Tangent Spaces
- Gradients
- The Spherical Laplacian
- Applications

Stokes' Theorem

Stokes' Theorem equates the integral of the divergence of a vector field over a region to the integral of the vector field over the boundary:

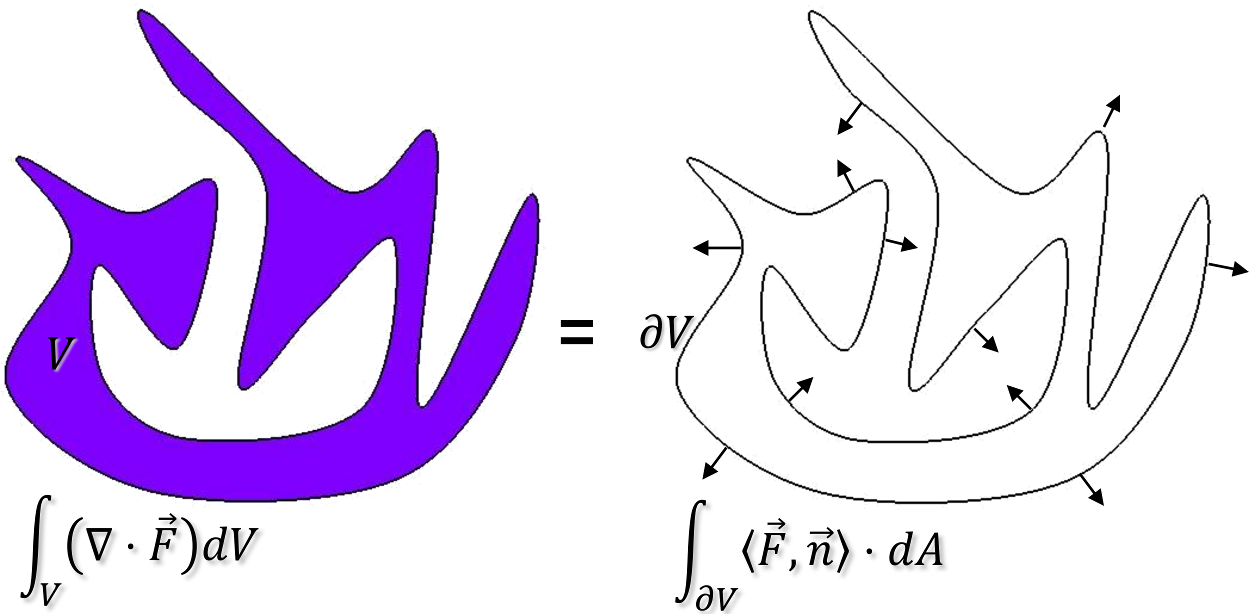
$$\int_V \left(\nabla \cdot \vec{F}\right) dV = \int_{\partial V} \langle \vec{F}, \vec{n} \rangle \cdot dA$$

where $\vec{n}$ is the normal at the boundary.

Stokes' Theorem

Stokes' Theorem equates the integral of the divergence of a vector field over a region to the integral of the vector field over the boundary:

$$\int_V \left(\nabla \cdot \vec{F}\right) dV = \int_{\partial V} \langle \vec{F}, \vec{n} \rangle \cdot dA$$



Ouline

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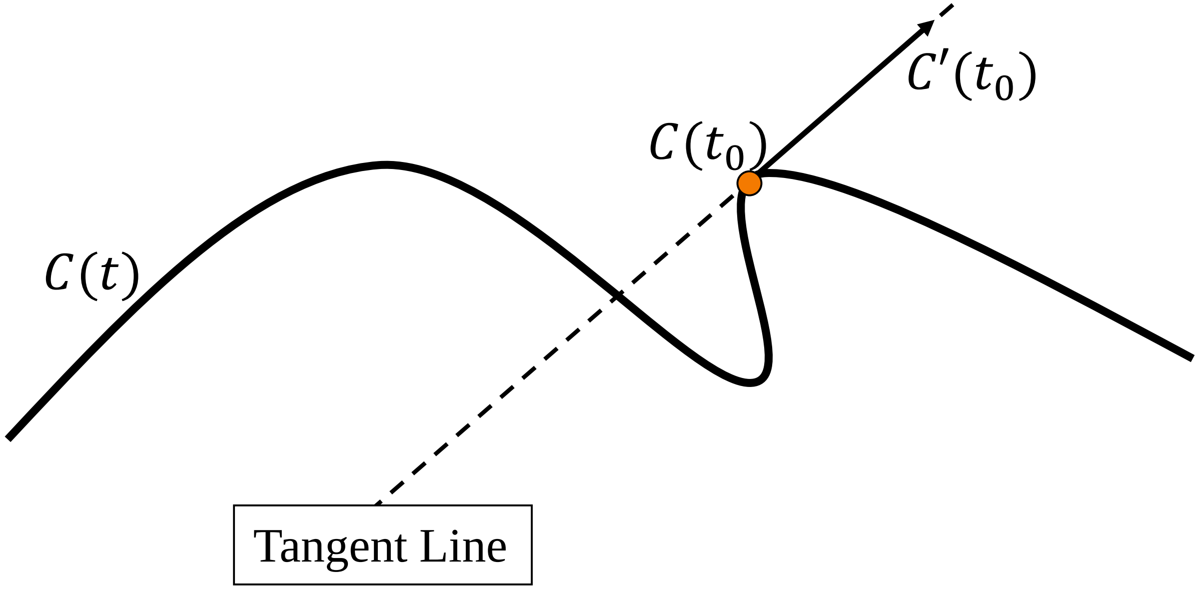
Tangent Spaces

Given a curve $C(t) = \big(x(t), y(t)\big)$, the <u>tangent line</u> to the curve at a point $p_0 = C(t_0)$ is the line passing through p_0 with direction $C'(t_0) = \big(x'(t_0), y'(t_0)\big)$.

This is the line that most closely approximates the curve $C(t)$ at the point p_0.

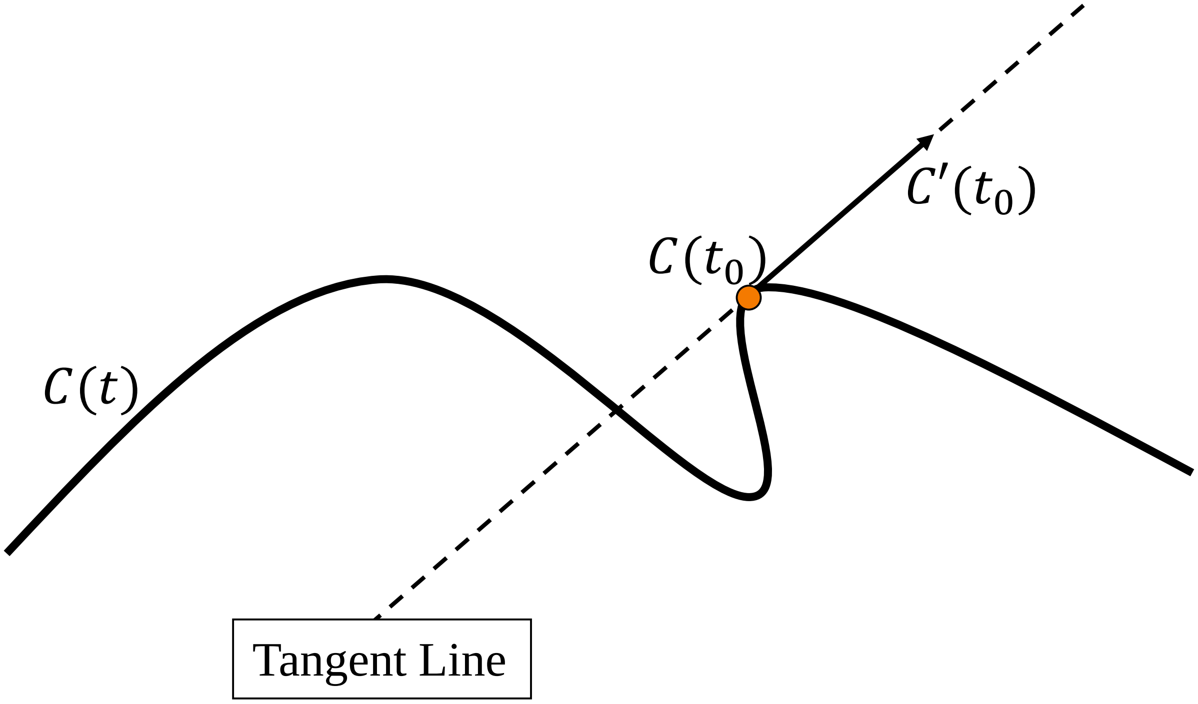


Tangent Spaces

Often, we want a unit vector.

In this case, we need to normalize:

$$T_C(t) = \frac{C'(t)}{|C'(t)|}$$

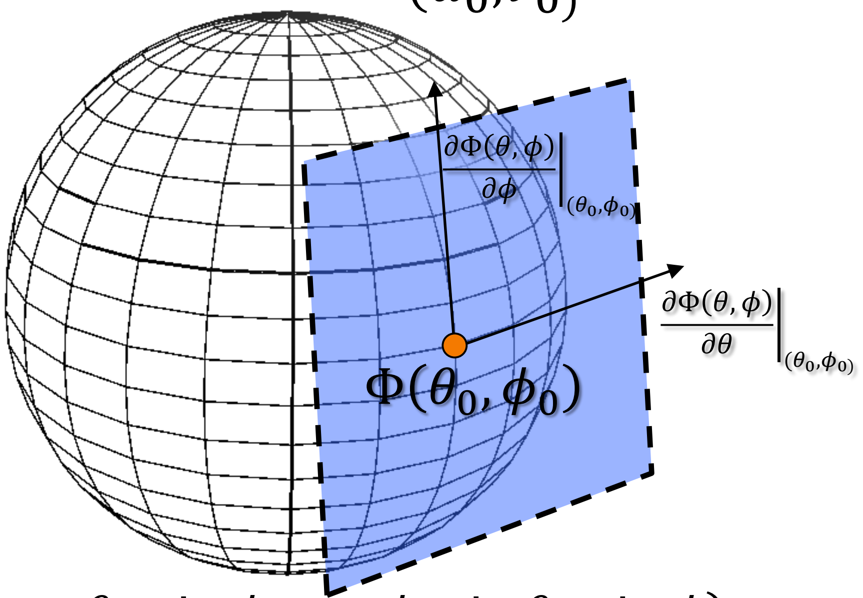


Tangent Spaces

Given a surface $S(u,v)$ the tangent plane to the curve at a point $p_0 = S(u_0,v_0)$ is the plane passing through p_0, parallel to the plane spanned by:

$$\left.\frac{\partial S(u,v)}{\partial u}\right|_{(u_0,v_0)} \quad \text{and} \quad \left.\frac{\partial S(u,v)}{\partial v}\right|_{(u_0,v_0)}$$

This is the plane that most closely approximates $S(u,v)$ at the point p_0.



$$\Phi(\theta,\phi) = (\cos\theta \cdot \sin\phi, \cos\phi, \sin\theta \cdot \sin\phi)$$

Tangent Spaces

In the case of the sphere, points are parameterized by the equation:

$$\Phi(\theta,\phi) = (\cos\theta \cdot \sin\phi\,, \cos\phi\,, \sin\theta \cdot \sin\phi)$$

and two tangent directions are:

$$\frac{\partial\Phi}{\partial\theta} = (-\sin\theta \cdot \sin\phi\,, 0, \cos\theta \cdot \sin\phi)$$
$$\frac{\partial\Phi}{\partial\phi} = (\cos\theta \cdot \cos\phi\,, -\sin\phi\,, \sin\theta \cdot \cos\phi)$$

Tangent Spaces

$$\frac{\partial\Phi}{\partial\theta} = (-\sin\theta\cdot\sin\phi\,, 0, \cos\theta\cdot\sin\phi)$$
$$\frac{\partial\Phi}{\partial\phi} = (\cos\theta\cdot\cos\phi\,, -\sin\phi\,, \sin\theta\cdot\cos\phi)$$

Taking the dot-product of the tangent vectors gives:

$$\left\langle\frac{\partial\Phi}{\partial\theta}, \frac{\partial\Phi}{\partial\theta}\right\rangle = \sin^2\theta\cdot\sin^2\phi + \cos^2\theta\cdot\sin^2\phi$$
$$= \sin^2\phi\cdot(\sin^2\theta + \cos^2\theta)$$
$$= \sin^2\theta$$

Tangent Spaces

$$\frac{\partial\Phi}{\partial\theta} = (-\sin\theta\cdot\sin\phi\,, 0, \cos\theta\cdot\sin\phi)$$
$$\frac{\partial\Phi}{\partial\phi} = (\cos\theta\cdot\cos\phi\,, -\sin\phi\,, \sin\theta\cdot\cos\phi)$$

Taking the dot-product of the tangent vectors gives:

$$\left\langle\frac{\partial\Phi}{\partial\theta},\frac{\partial\Phi}{\partial\theta}\right\rangle = \sin^2\theta$$

$$\begin{aligned}\left\langle\frac{\partial\Phi}{\partial\phi},\frac{\partial\Phi}{\partial\phi}\right\rangle &= \cos^2\theta\cdot\cos^2\phi + \sin^2\phi + \sin^2\theta\cdot\cos^2\phi \\ &= (\cos^2\theta + \sin^2\theta)\cdot\cos^2\phi + \sin^2\phi \\ &= \cos^2\phi + \sin^2\phi \\ &= 1\end{aligned}$$

Tangent Spaces

$$\frac{\partial\Phi}{\partial\theta} = (-\sin\theta\cdot\sin\phi, 0, \cos\theta\cdot\sin\phi)$$
$$\frac{\partial\Phi}{\partial\phi} = (\cos\theta\cdot\cos\phi, -\sin\phi, \sin\theta\cdot\cos\phi)$$

Taking the dot-product of the tangent vectors gives:

$$\left\langle\frac{\partial\Phi}{\partial\theta},\frac{\partial\Phi}{\partial\theta}\right\rangle = \sin^2\theta$$
$$\left\langle\frac{\partial\Phi}{\partial\phi},\frac{\partial\Phi}{\partial\phi}\right\rangle = 1$$
$$\left\langle\frac{\partial\Phi}{\partial\theta},\frac{\partial\Phi}{\partial\phi}\right\rangle = -\sin\theta\cdot\cos\theta\cdot\sin\phi\cdot\cos\phi + \sin\theta\cdot\cos\theta\cdot\sin\phi\cdot\cos\phi$$
$$= 0$$

Tangent Spaces

$$\left\langle \frac{\partial\Phi}{\partial\theta}, \frac{\partial\Phi}{\partial\theta} \right\rangle = \sin^2\theta$$

$$\left\langle \frac{\partial\Phi}{\partial\phi}, \frac{\partial\Phi}{\partial\phi} \right\rangle = 1$$

$$\left\langle \frac{\partial\Phi}{\partial\theta}, \frac{\partial\Phi}{\partial\phi} \right\rangle = 0$$

So, the vectors:

$$\Phi_\theta(\theta,\phi) = \frac{1}{\sin\theta}\frac{\partial\Phi}{\partial\theta} \quad \text{and} \quad \Phi_\phi(\theta,\phi) = \frac{\partial\Phi}{\partial\phi}$$

form an orthonormal basis for the tangent plane to the sphere at the point $\Phi(\theta,\phi)$.

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Function Gradients

The gradient of a function is a vector which tells us how the function changes as we move in different directions.

Function Gradients

The gradient of a function is a vector which tells us how the function changes as we move in different directions.

Given a function f and given a direction v:

$$\left.\frac{d}{dt}\right|_{t=0} f(p+tv) = \langle \nabla f(p), v \rangle$$

Function Gradients

To compute the gradient, we can choose two orthogonal unit vectors u and v, and set:

$$\nabla f(p) = \left.\frac{d}{dt}\right|_{t=0} f(p+tu)\cdot u + \left.\frac{d}{dt}\right|_{t=0} f(p+tv)\cdot v$$

Curve Gradients

Given a curve $C(t)$, and given a function $f(t)$ the gradient of the function is a vector field on the curve telling us how the function changes as we move along the curve.

Curve Gradients

Example:

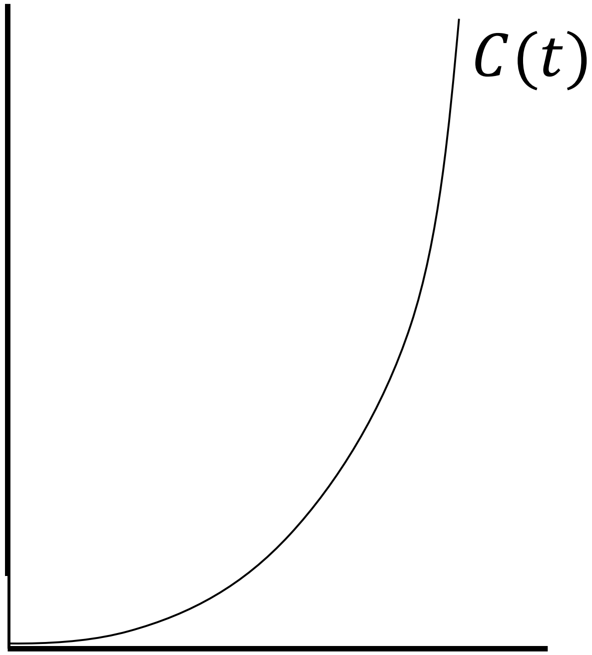
Let C be the curve defined by:

$$C(t) = (t, t^2)$$

and let $f(t)$ be the function on the curve defined by:

$$f(t) = t$$

What is the gradient $\nabla_C f$ of f along the curve?



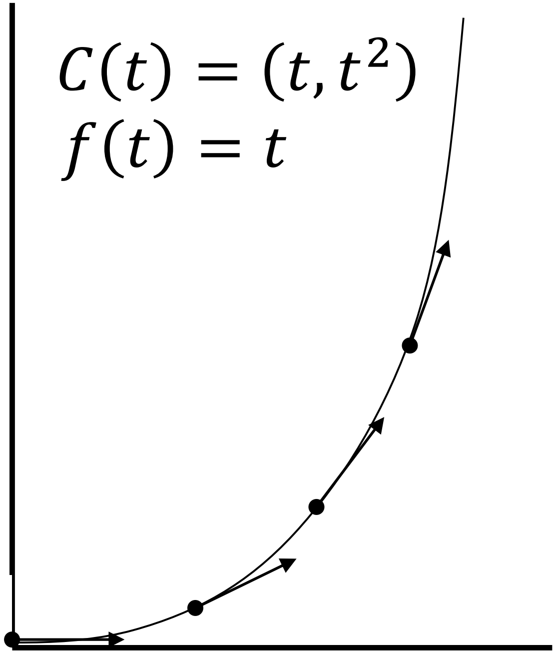
Curve Gradients

Example:

Note that:

$$\nabla_C f \neq 1$$

This would imply that at any point on the curve moving a unit distance forward would change the value by a constant amount.



Curve Gradients

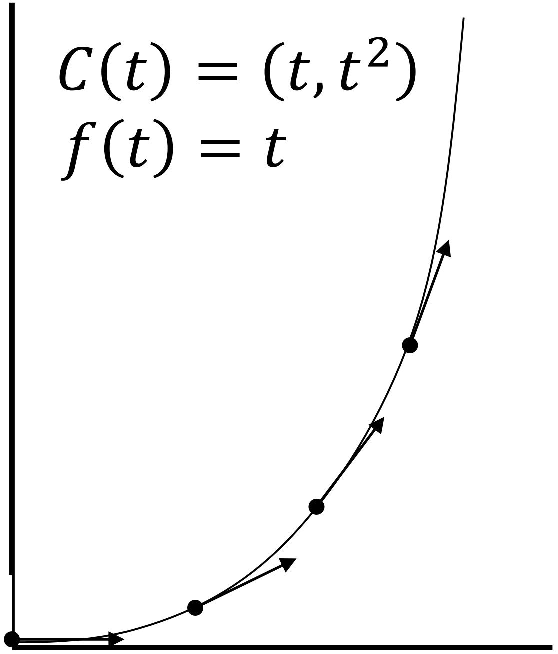
Example:

Note that:

$$\nabla_C f \neq 1$$

As we move from $t = 1$ to $t = 2$, the function changes by a value of 1.

Similarly, as we move from $t = 10$ to $t = 11$, the function changes by a value of 1.



But in the first case, we have moved a distance of:

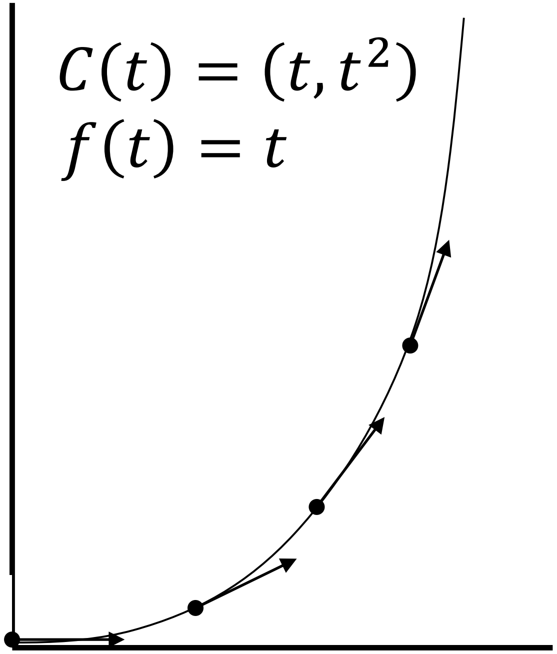
$$d_1 \approx \|C(2) - C(1)\| = \sqrt{1^2 + 3^2}$$

Curve Gradients

Example:

Note that:

$$\nabla_C f \neq 1$$



As we move from $t = 1$ to $t = 2$, the function changes by a value of 1.

Similarly, as we move from $t = 10$ to $t = 11$, the function changes by a value of 1.

And in the first case, we have moved a distance of:

$$d_2 \approx \|C(10) - C(11)\| = \sqrt{1^2 + 21^2}$$

Curve Gradients

Example:

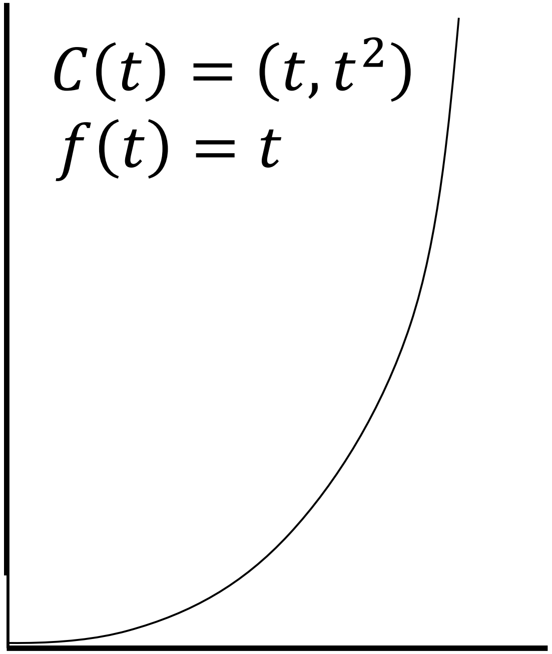
We need to measure the ratio of the change in f over the distance traveled:

$$\nabla_C f(t) \approx \frac{f(t+\varepsilon) - f(t)}{|C(t+\varepsilon) - C(t)|}$$

$$\Downarrow$$

$$\nabla_C f(t) = \frac{f'(t)}{|C'(t)|}$$

$$= \frac{1}{\sqrt{1+2t}}$$



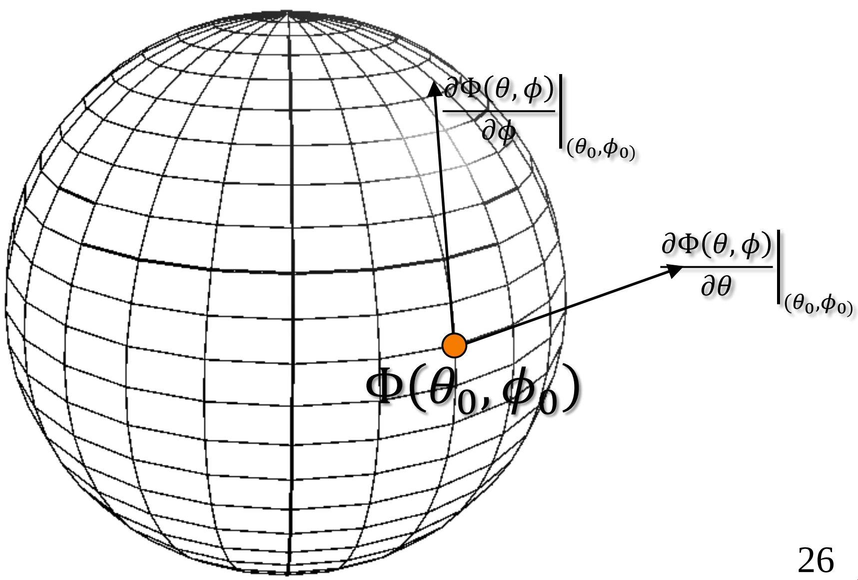
Spherical Gradients

Given a function on the sphere, $f(\theta,\phi)$, we would like to compute the gradient:

$$\nabla f(\theta,\phi)$$

We need to pick two directions in parameter space whose images on the surface of the sphere are orthogonal.

The directions θ and ϕ are two such directions:

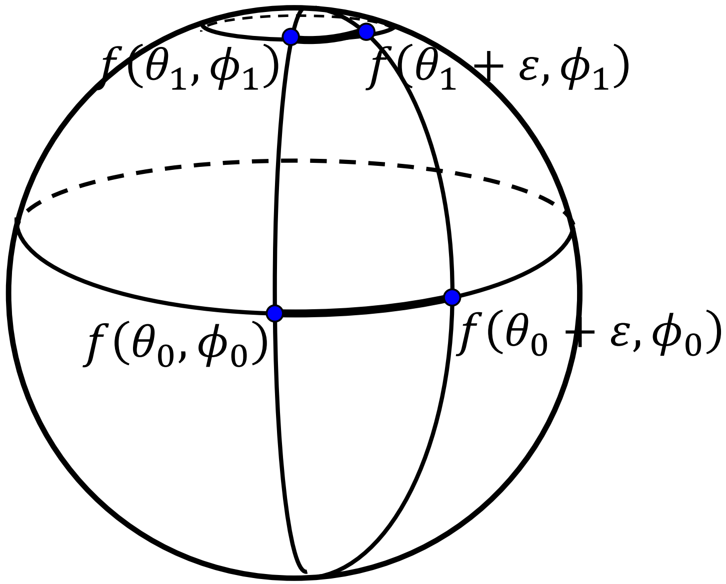


Spherical Gradients

We could try taking the partial derivatives in the θ and ϕ directions:

$$\nabla f(\theta,\phi) = \left(\frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial \phi}\right)$$

But this introduces bias!



Shifting by a constant ε will move us different distances depending on where we are on the sphere.

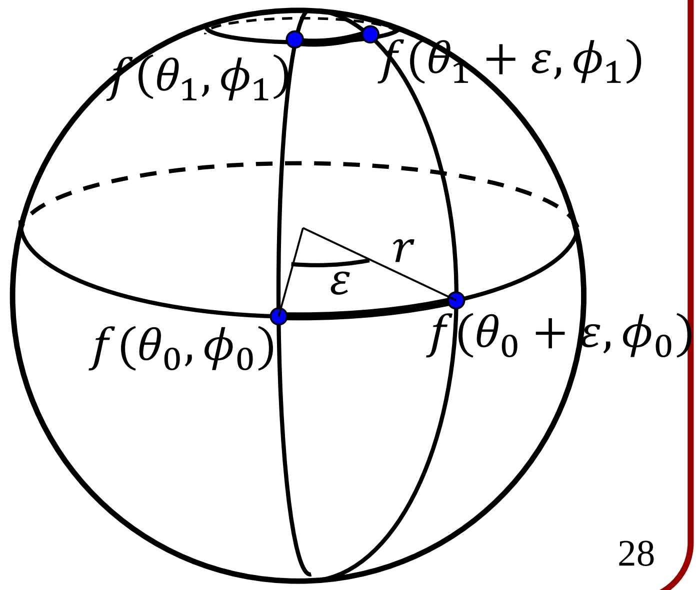
Spherical Gradients

How does the scale change as we change θ or ϕ by a value of ε?

At the point $p = \Phi(\theta, \phi)$, changing the value of θ by ε, moves us a distance of εr along the circle about the y-axis, where r is the radius of the circle.

On the sphere, the radius is defined by:

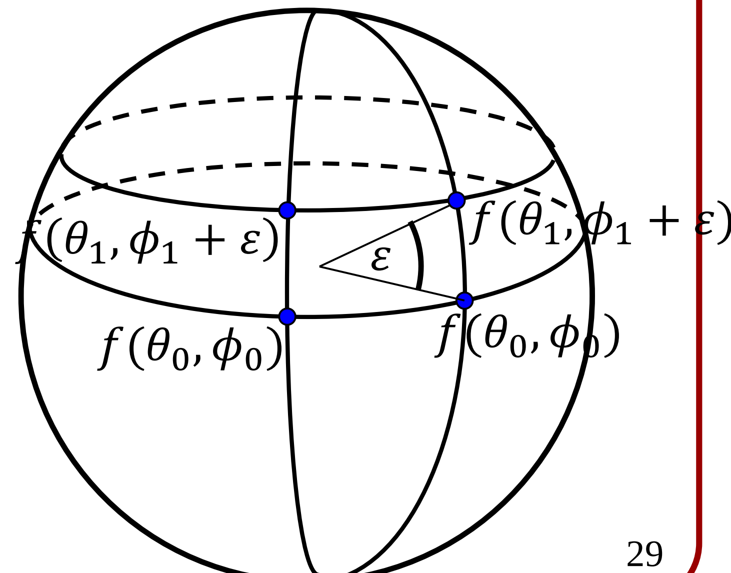
$$r(\phi) = \sin \phi$$



Spherical Gradients

How does the scale change as we change θ or ϕ by a value of ε?

At the point $p = \Phi(\theta, \phi)$, changing the value of ϕ by ε, moves us a distance of ε along a great circle regardless of where on the sphere we are:



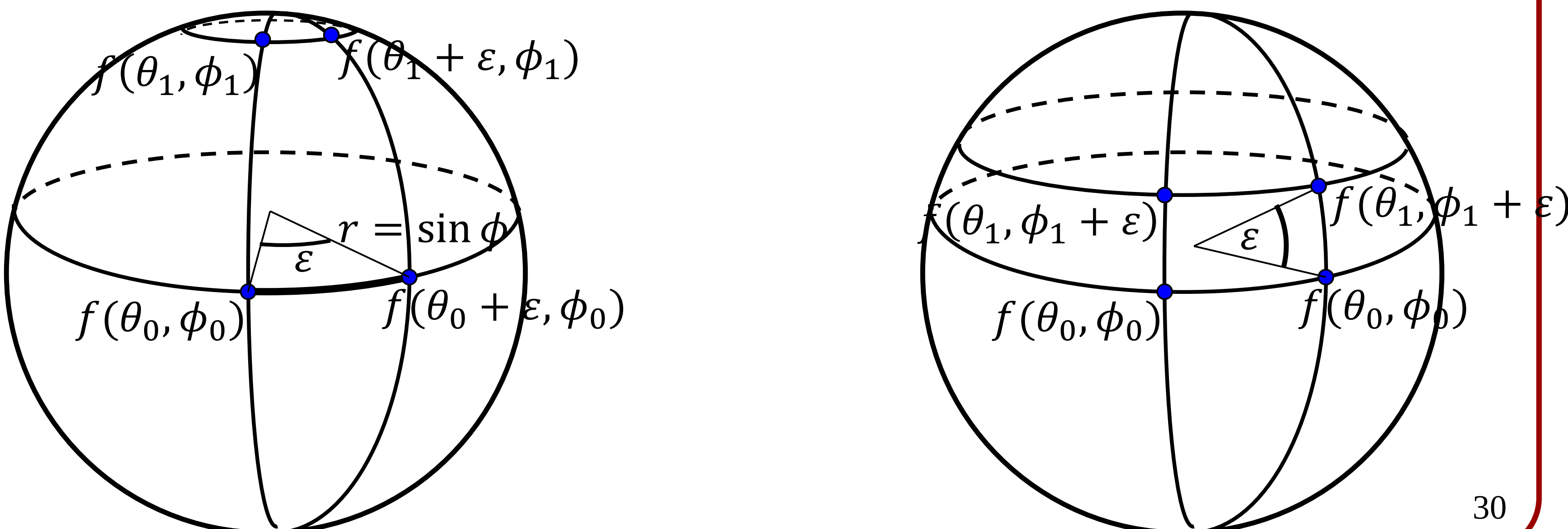
Spherical Gradients

Taking the scaling into account, we get:

$$\nabla f(\theta,\phi) \approx \left(\frac{f(\theta+\varepsilon,\phi) - f(\theta,\phi)}{\varepsilon \sin\phi}, \frac{f(\theta,\phi+\varepsilon) - f(\theta,\phi)}{\varepsilon} \right)$$

$$\Downarrow$$

$$\nabla f(\theta,\phi) = \left(\frac{1}{\sin\phi}\frac{\partial f}{\partial\theta}, \frac{\partial f}{\partial\phi} \right)$$



Ouline

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The Spherical Laplacian

Recall:

The Laplacian operator is self-adjoint (symmetric)

$\Rightarrow$ There is an orthogonal basis of eigenvectors.

The Laplacian operator commutes with rotations

$\Rightarrow$ If F_λ are the eigenfunctions of the Laplacian with eigenvalue λ, rotations fix F_λ.

$\Rightarrow$ The irreducible representations are subspaces of the F_λ.

The Spherical Laplacian

All this implies that for a fixed degree l, the spherical harmonics of degree l:

$$Y_l^m(\theta,\phi) = e^{im\theta} \cdot P_l^m(\cos\phi)$$

(with $|m| \leq l$) must be eigenvectors of the Laplacian with the same eigenvalue.

1. What is the Laplacian?

2. What are the eigenvalues?

The Spherical Laplacian

How do we compute the Laplacian of a spherical function $f(\theta, \phi)$?

Recall:

The Laplacian of a function is the divergence of its gradient:

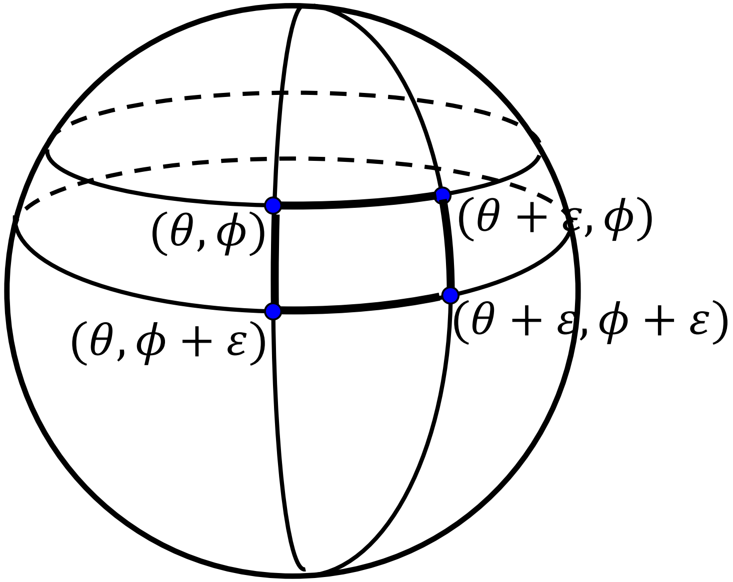
$$\Delta f = \nabla \cdot (\nabla f)$$

The Spherical Laplacian

By Stokes' Theorem, we can compute the integral of the Laplacian (the divergence of the gradient) over a region by computing the surface integral of the gradient field over the boundary.

The Spherical Laplacian

Consider the "square" on the sphere with vertices (θ,ϕ), $(\theta+\varepsilon,\phi)$, $(\theta+\varepsilon,\phi+\varepsilon)$, and $(\theta,\phi+\varepsilon)$.

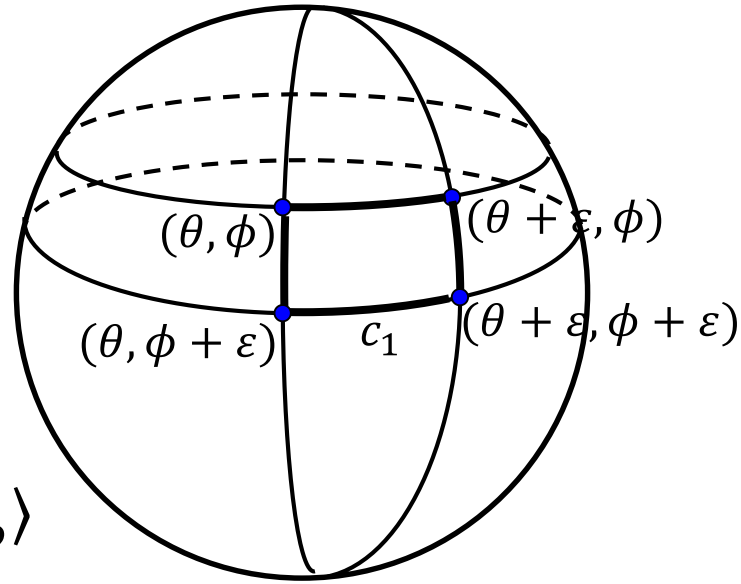


The integral of the Laplacian is approximately:

$$\int_R \Delta f\, dR \approx \text{Area}(R)\cdot\Delta f(\theta,\phi)$$
$$= \varepsilon^2\cdot\sin\phi\cdot\Delta f(\theta,\phi)$$

The Spherical Laplacian

Consider the "square" on the sphere with vertices (θ,ϕ), $(\theta+\varepsilon,\phi)$, $(\theta+\varepsilon,\phi+\varepsilon)$, and $(\theta,\phi+\varepsilon)$.



On the curve c_1 the boundary integral of the Laplacian is approximately:

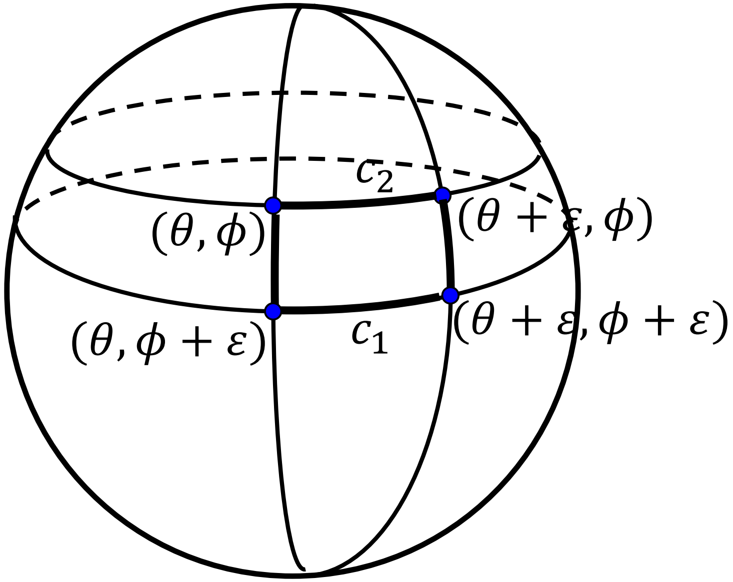
$$\int_{c_1}\langle\nabla f,\vec{n}\rangle\cdot dA \approx \text{Length}(c_1)\cdot\langle\nabla f,\Phi_\phi\rangle$$

$$= \varepsilon\cdot\sin(\phi+\varepsilon)\cdot\left\langle\left(\frac{1}{\sin(\phi+\varepsilon)}\frac{\partial f}{\partial\theta},\frac{\partial f}{\partial f}\right),(0,1)\right\rangle$$

$$= \varepsilon\cdot\sin(\phi+\varepsilon)\cdot\frac{\partial f}{\partial\phi}(\theta,\phi+\varepsilon)$$

The Spherical Laplacian

Consider the “square” on the sphere with vertices (θ, ϕ), $(\theta+\varepsilon, \phi)$, $(\theta+\varepsilon, \phi+\varepsilon)$, and $(\theta, \phi+\varepsilon)$.

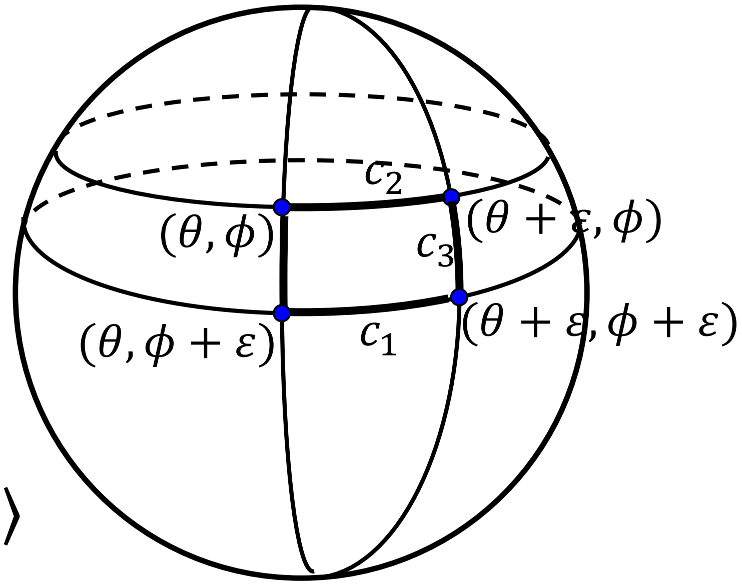


On the curve c_2 the boundary integral of the Laplacian is approximately:

$$\int_{c_2} \langle \nabla f, \vec{n} \rangle \cdot dA \approx -\varepsilon \cdot \sin\phi \cdot \frac{\partial f}{\partial \phi}(\theta, \phi)$$

The Spherical Laplacian

Consider the "square" on the sphere with vertices (θ,ϕ), $(\theta+\varepsilon,\phi)$, $(\theta+\varepsilon,\phi+\varepsilon)$, and $(\theta,\phi+\varepsilon)$.



On the curve c_3 the boundary integral of the Laplacian is approximately:

$$\int_{c_3} \langle \nabla f, \vec{n} \rangle \cdot dA \approx \text{Length}(c_3) \cdot \langle \nabla f, \Phi_\theta \rangle$$

$$= \varepsilon \left\langle \left(\frac{1}{\sin\phi} \frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial \phi} \right), (1,0) \right\rangle$$

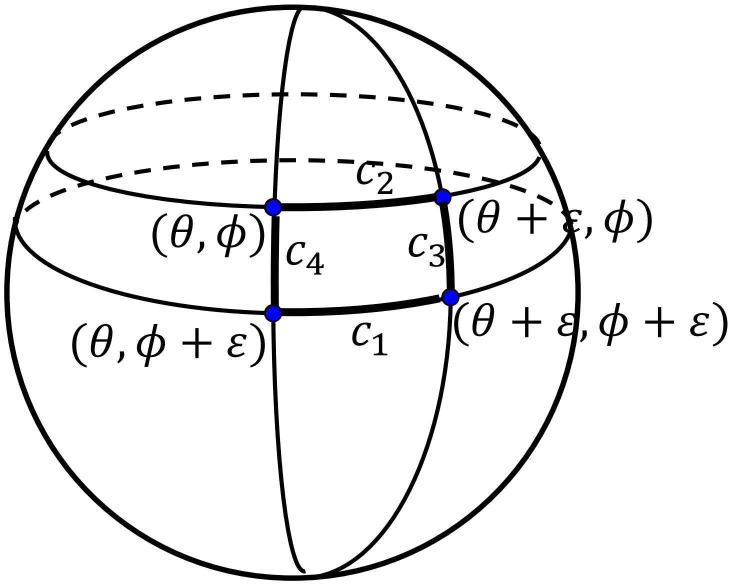
$$= \varepsilon \frac{1}{\sin\phi} \frac{\partial f}{\partial \theta} (\theta+\varepsilon, \phi)$$

The Spherical Laplacian

Consider the "square" on the sphere with vertices (θ,ϕ), $(\theta+\varepsilon,\phi)$, $(\theta+\varepsilon,\phi+\varepsilon)$, and $(\theta,\phi+\varepsilon)$.

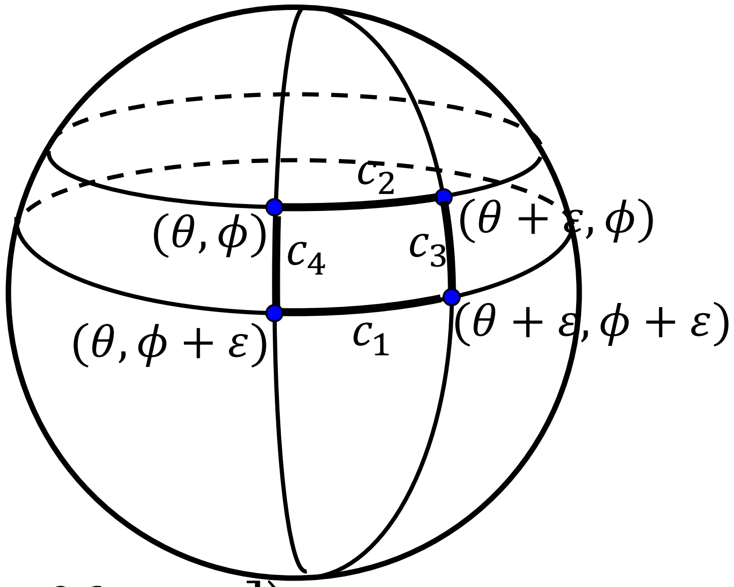
On the curve c_4 the boundary integral of the Laplacian is approximately:

$$\int_{c_4}\langle\nabla f,\vec{n}\rangle\cdot dA \approx -\varepsilon\frac{1}{\sin\phi}\frac{\partial f}{\partial\theta}(\theta,\phi)$$



The Spherical Laplacian

Consider the "square" on the sphere with vertices (θ,ϕ), $(\theta+\varepsilon,\phi)$, $(\theta+\varepsilon,\phi+\varepsilon)$, and $(\theta,\phi+\varepsilon)$.

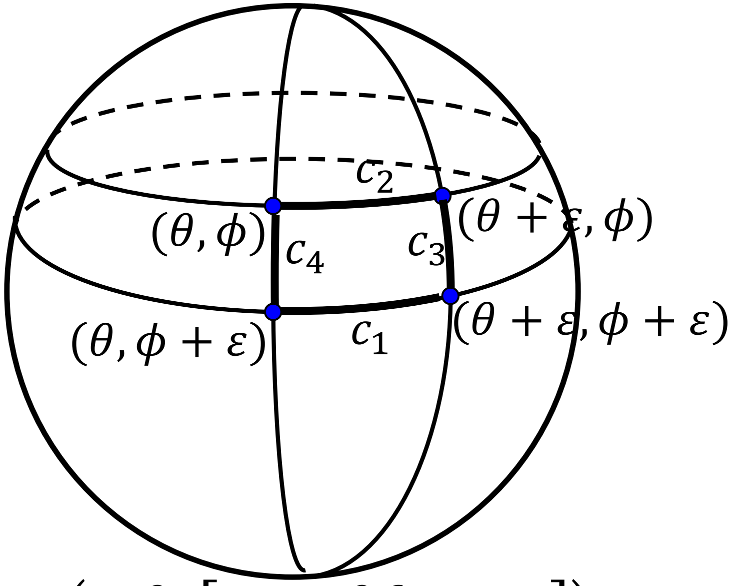


Summing these together, we can approximate the boundary integral by:

$$\int_{\partial R}\langle\nabla f,\vec{n}\rangle\cdot dA \approx \varepsilon\left(\frac{1}{\sin\phi}\left[\frac{\partial f}{\partial\theta}(\theta+\varepsilon,\phi)-\frac{\partial f}{\partial\theta}(\theta,\phi)\right]\right)+$$
$$\varepsilon\left(\sin(\phi+\varepsilon)\frac{\partial f}{\partial\phi}(\theta,\phi+\varepsilon)-\sin\phi\frac{\partial f}{\partial\phi}(\theta,\phi)\right)$$

The Spherical Laplacian

Consider the “square” on the sphere with vertices (θ,ϕ), $(\theta+\varepsilon,\phi)$, $(\theta+\varepsilon,\phi+\varepsilon)$, and $(\theta,\phi+\varepsilon)$.

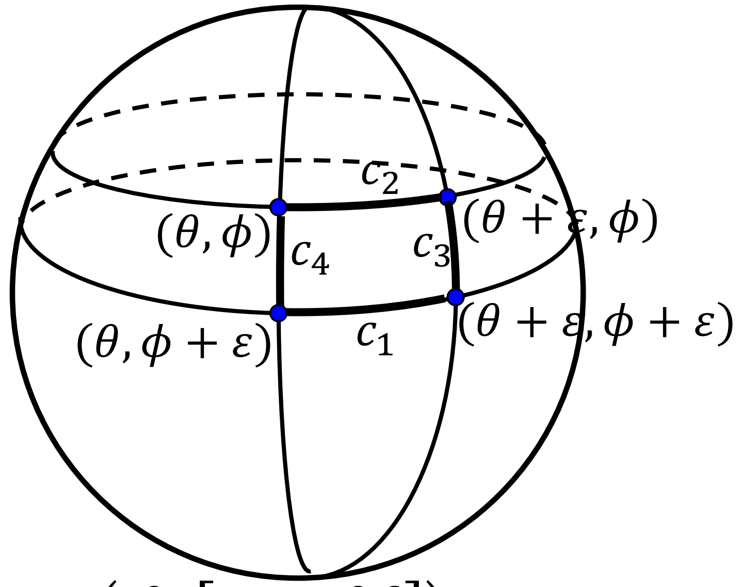


Summing these together, we can approximate the boundary integral by:

$$\int_{\partial R}\langle\nabla f,\vec{n}\rangle\cdot dA \approx \varepsilon\left(\frac{1}{\sin\phi}\varepsilon\frac{\partial}{\partial\theta}\left[\frac{\partial f}{\partial\theta}(\theta,\phi)\right]\right)+\varepsilon\left(\varepsilon\frac{\partial}{\partial\phi}\left[\sin\phi\frac{\partial f}{\partial\phi}(\theta,\phi)\right]\right)$$

The Spherical Laplacian

Consider the "square" on the sphere with vertices (θ,ϕ), $(\theta+\varepsilon,\phi)$, $(\theta+\varepsilon,\phi+\varepsilon)$, and $(\theta,\phi+\varepsilon)$.



Summing these together, we can approximate the boundary integral by:

$$\int_{\partial R}\langle\nabla f,\vec{n}\rangle\cdot dA \approx \varepsilon^2\left(\frac{1}{\sin\phi}\frac{\partial}{\partial\theta}\left[\frac{\partial f}{\partial\theta}\right]\right)+\varepsilon^2\left(\frac{\partial}{\partial\phi}\left[\sin\phi\,\frac{\partial f}{\partial\phi}\right]\right)$$

The Spherical Laplacian

Using the fact that the boundary integral can be approximated by:

$$\int_{\partial R} \langle \nabla f, \vec{n} \rangle \cdot dA \approx \varepsilon^2 \left(\frac{1}{\sin\phi} \frac{\partial}{\partial\theta} \left[\frac{\partial f}{\partial\theta} \right] \right) + \varepsilon^2 \left(\frac{\partial}{\partial\phi} \left[\sin\phi \frac{\partial f}{\partial\phi} \right] \right)$$

and the surface integral can be approximated by:

$$\int_R \Delta f \, dR \approx \varepsilon^2 \cdot \sin\phi \cdot \Delta f(\theta, \phi)$$

We can apply Stokes' Theorem to get:

$$\Delta f = \frac{1}{\sin^2\phi} \frac{\partial^2 f}{\partial\theta^2} + \frac{1}{\sin\phi} \frac{\partial}{\partial\phi} \left[\sin\phi \frac{\partial f}{\partial\phi} \right]$$

The Spherical Laplacian

$$\Delta f = \frac{1}{\sin^2 \phi}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{\sin \phi}\frac{\partial}{\partial \phi}\left[\sin \phi \frac{\partial f}{\partial \phi}\right]$$

To compute the eigenvalues of the Laplacian associated to the spherical harmonics, we need to compute:

$$\Delta Y_l^m(\theta, \phi) = \Delta\left(e^{ik\theta} \cdot P_l^m(\cos \phi)\right)$$

The Spherical Laplacian

$$\Delta f = \frac{1}{\sin^2\phi}\frac{\partial^2 f}{\partial\theta^2} + \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left[\sin\phi\frac{\partial f}{\partial\phi}\right]$$

Taking the derivative with respect to θ is easy:

$$\begin{aligned}\frac{1}{\sin^2\phi}\frac{\partial^2}{\partial\theta^2}Y_l^m(\theta,\phi) &= \frac{1}{\sin^2\phi}\frac{\partial^2}{\partial\theta^2}\left(e^{im\theta}\cdot P_l^m(\cos\phi)\right)\\ &= -\frac{m^2}{\sin^2\phi}e^{im\theta}\cdot P_l^m(\cos\phi)\\ &= -\frac{m^2}{\sin^2\phi}Y_l^m(\theta,\phi)\end{aligned}$$

The Spherical Laplacian

$$\Delta f = \frac{1}{\sin^2\phi}\frac{\partial^2 f}{\partial\theta^2} + \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left[\sin\phi\frac{\partial f}{\partial\phi}\right]$$

Taking the derivative with respect to ϕ is more complicated:

$$\begin{aligned}\frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left[\sin\phi\frac{\partial}{\partial\phi}Y_l^m(\theta,\phi)\right] &= \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left[\sin\phi\frac{\partial}{\partial\phi}\left(e^{im\theta}\cdot P_l^m(\theta,\phi)\right)\right] \\ &= \frac{e^{im\theta}}{\sin\phi}\frac{\partial}{\partial\phi}\left[\sin\phi\frac{\partial P_l^m}{\partial\phi}(\cos\phi)\right]\end{aligned}$$

This requires taking the derivatives of the associated Legendre polynomials.

Associated Legendre Polynomials

Recall:

The associated Legendre polynomials are defined by the (somewhat hairy) formula:

$$P_l^m(x) = \frac{(-1)^m}{2^l l!}(1-x^2)^{\frac{m}{2}} \frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l$$

Associated Legendre Polynomials

One can show, (but we won't) that the associated Legendre polynomials satisfy the identities:

$$\frac{dP_l^m(\cos\phi)}{d\phi} = \frac{l \cdot \cos\phi \cdot P_l^m(\cos\phi) - (l+m) \cdot P_{l-1}^m(\cos\phi)}{\sin\phi}$$

$$\begin{aligned} 0 = \quad & (l-m) \cdot P_l^m(\cos\phi) \\ & - \cos\phi \cdot (2l-1) \cdot P_{l-1}^m(\cos\phi) \\ & + (l+m-1) \cdot P_{l-2}^m(\cos\phi) \end{aligned}$$

The Spherical Laplacian

Plugging these identities into the equation for the Laplacian, we get (see appendix):

$$\frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left[\sin\phi\frac{\partial Y_l^m}{\partial\phi}\right] = (-l^2 - l)\cdot Y_l^m + m^2\cdot\frac{Y_l^m}{\sin^2\phi}$$

$$\Downarrow$$

$$\frac{1}{\sin^2\phi}\frac{\partial^2 Y_l^m}{\partial\theta^2} + \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left[\sin\phi\frac{\partial Y_l^m}{\partial\phi}\right] = -l(l+1)Y_l^m$$

$$\Downarrow$$

$$\Delta Y_l^m(\theta,\phi) = -l(l+1)Y_l^m(\theta,\phi)$$

Ouline

- Stokes' Theorem
- Tangent Spaces
- Gradients
- The Spherical Laplacian
- Applications

Smoothing

In the case of a functions on a plane, we had Newton's Law of Cooling:

"*The rate of heat loss of a body is proportional to the difference in temperatures between the body and its surroundings.*"

Smoothing

In the case of a functions on a plane, we had Newton's Law of Cooling:

"*The rate of heat loss of a body is proportional to the difference in temperatures between the body and its surroundings.*"

This can be expresses as a PDE:

$$\frac{\partial F}{\partial t} = \eta \cdot \Delta F$$

Smoothing

Using the fact that the spherical harmonics are eigenvectors of the Laplacian, we get solutions of the form:

$$F_l^m(\theta,\phi,t) = e^{-\lambda\cdot l(l+1)t}\cdot Y_l^m(\theta,\phi)$$

Since the linear sum of solutions is also a solution, solutions to the PDE will have the form:

$$F(\theta,\phi,t) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_l^m\cdot e^{-\lambda\cdot l(l+1)t}\cdot Y_l^m(\theta,\phi)$$

and we have freedom in choosing the linear coefficients.

Smoothing

To satisfy the initial condition:

$$F(\theta, \phi, 0) = f(\theta, \phi)$$

we need to compute the spherical harmonic decomposition of f:

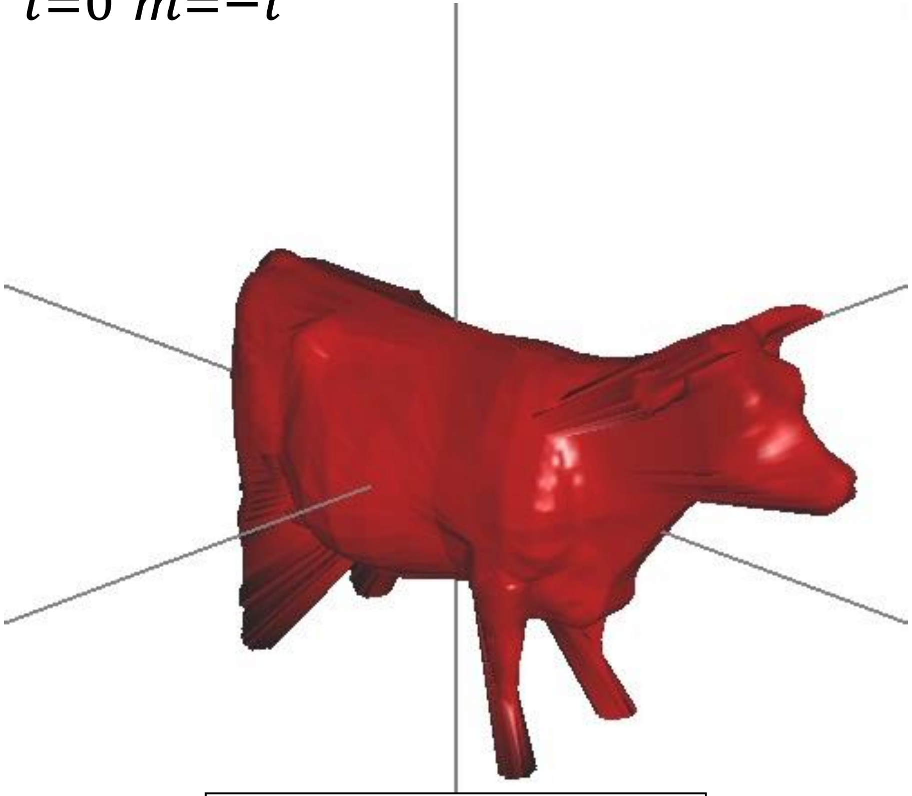
$$f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \hat{f}(l, m) \cdot Y_l^m(\theta, \phi)$$

Then we set the solution to be:

$$F(\theta, \phi, t) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \hat{f}(l, m) \cdot e^{-\lambda \cdot l(l+1)t} \cdot Y_l^m(\theta, \phi)$$

Smoothing

$$F(\theta,\phi,t)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\hat{f}(l,m)\cdot e^{-\lambda\cdot l(l+1)t}\cdot Y_l^m(\theta,\phi)$$



Cooling Cow

The Spherical Wave Equation

We can repeat the same type of argument for the wave equation, where the acceleration is proportional to the difference in values:

$$\frac{\partial^2 F}{\partial t^2} = \eta \cdot \Delta \mathrm{F}$$

The Spherical Wave Equation

Again, using the fact that the spherical harmonics Y_l^m are eigenvectors of the Laplacian with eigenvalues $l(l+1)$ we get solutions of the form:

$$F_l^{m+}(\theta,\phi,t) = e^{\;i\sqrt{\eta\cdot l(l+1)}t} \cdot Y_l^m(\theta,\phi)$$

$$F_l^{m-}(\theta,\phi,t) = e^{-i\sqrt{\eta\cdot l(l+1)}t} \cdot Y_l^m(\theta,\phi)$$

The Spherical Wave Equation

Thus, given the initial conditions:

$$F(\theta,\phi,0) = f(\theta,\phi)$$

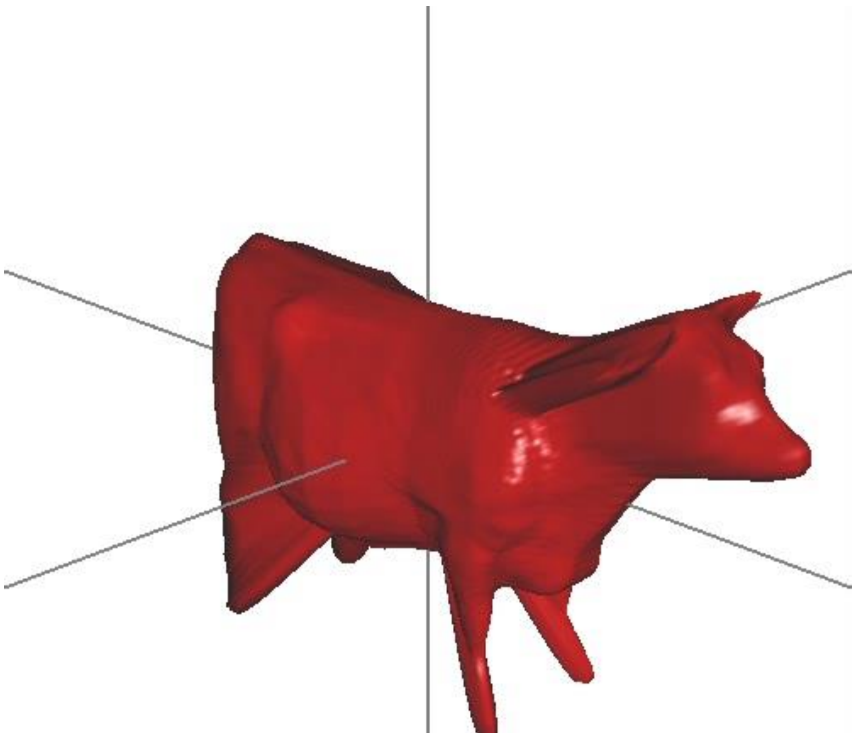
$$\frac{\partial}{\partial t}F(\theta,\phi,0) = 0$$

we get the solution:

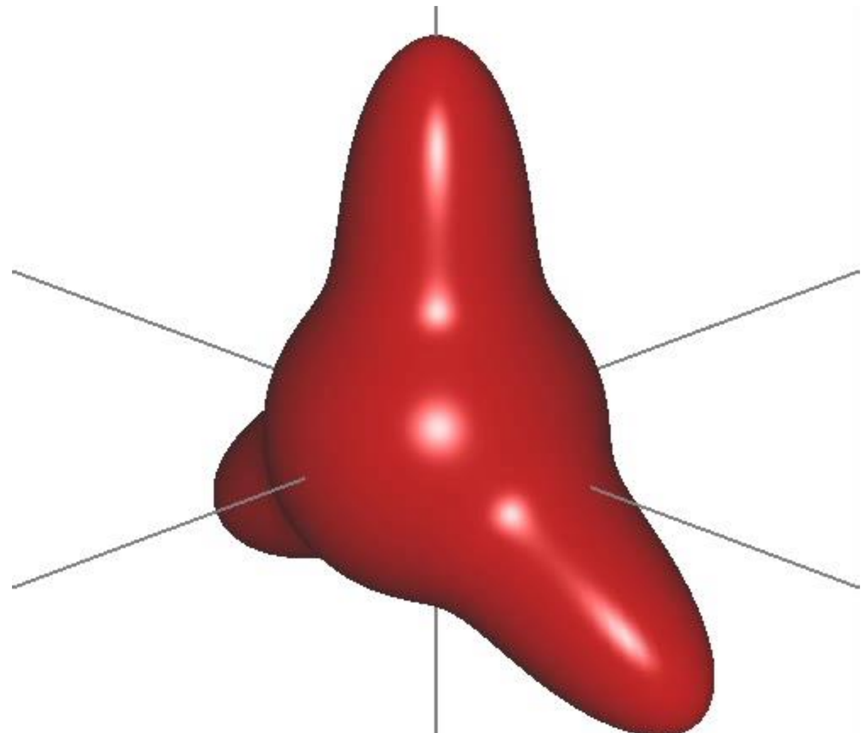
$$F(\theta,\phi,t) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\hat{f}(k,l)\cdot\cos\left(\sqrt{\eta\cdot l(l+1)}t\right)Y_l^m(\theta,\phi)$$

The Spherical Wave Equation

$$F(\theta,\phi,t)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\hat{f}(k,l)\cdot\cos\left(\sqrt{\eta\cdot l(l+1)}t\right)Y_l^m(\theta,\phi)$$



Waving Cow



Waving Gaussians

Appendix

$$\frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left[\sin\phi\frac{\partial}{\partial\phi}Y_l^m(\theta,\phi)\right] = e^{im\theta}\frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left[\sin\phi\frac{\partial P_l^m}{\partial\phi}(\cos\phi)\right]$$

$$= e^{im\theta}\left(\frac{\cos\phi}{\sin\phi}\frac{\partial P_l^m(\cos\phi)}{\partial\phi} + \frac{\partial^2 P_l^m(\cos\phi)}{\partial\phi^2}\right)$$

Appendix

$$\frac{dP_l^m(\cos\phi)}{d\phi} = \frac{l\cdot\cos\phi\cdot P_l^m(\cos\phi) - (l+m)P_{l-1}^m(\cos\phi)}{\sin\phi}$$

$$0 = (l-m)\cdot P_l^m(\cos\phi) - \cos\phi\cdot(2l-1)\cdot P_{l-1}^m(\cos\phi) + (l+m-1)\cdot P_{l-2}^m(\cos\phi)$$

$$\begin{aligned}
\left(\frac{\cos\phi}{\sin\phi}\frac{\partial P_l^m(\cos\phi)}{\partial\phi} + \frac{\partial^2 P_l^m(\cos\phi)}{\partial\phi^2}\right) &= \frac{\cos\phi}{\sin\phi}\frac{l\cdot\cos\phi\cdot P_l^m(\cos\phi) - (l+m)\cdot P_{l-1}^m(\cos\phi)}{\sin\phi} + \frac{\partial}{\partial\phi}\left(\frac{l\cdot\cos\phi\cdot P_l^m(\cos\phi) - (l+m)\cdot P_{l-1}^m(\cos\phi)}{\sin\phi}\right) \\
&= \frac{\cos\phi}{\sin^2\phi}\left(l\cdot\cos\phi\cdot P_l^m(\cos\phi) - (l+m)\cdot P_{l-1}^m(\cos\phi)\right) \\
&- \frac{\cos\phi}{\sin^2\phi}\left(l\cdot\cos\phi\cdot P_l^m(\cos\phi) - (l+m)\cdot P_{l-1}^m(\cos\phi)\right) \\
&+ \left(\frac{-l\cdot\sin\phi\cdot P_l^m(\cos\phi) + l\cdot\cos\phi\frac{\partial P_l^m(\cos\phi)}{\partial\phi} - (l+m)\frac{\partial P_{l-1}^m(\cos\phi)}{\partial\phi}}{\sin\phi}\right) \\
&= \frac{-l\cdot\sin\phi\cdot P_l^m(\cos\phi) + l\cdot\cos\phi\frac{\partial P_l^m(\cos\phi)}{\partial\phi} - (l+m)\frac{\partial P_{l-1}^m(\cos\phi)}{\partial\phi}}{\sin\phi}
\end{aligned}$$

Appendix

$$\frac{dP_l^m(\cos\phi)}{d\phi} = \frac{l \cdot \cos\phi \cdot P_l^m(\cos\phi) - (l+m)P_{l-1}^m(\cos\phi)}{\sin\phi}$$

$$0 = (l-m) \cdot P_l^m(\cos\phi) - \cos\phi \cdot (2l-1) \cdot P_{l-1}^m(\cos\phi) + (l+m-1) \cdot P_{l-2}^m(\cos\phi)$$

$$\frac{-l \cdot \sin\phi \cdot P_l^m(\cos\phi) + l \cdot \cos\phi \frac{\partial P_l^m(\cos\phi)}{\partial\phi} - (l+m)\frac{\partial P_{l-1}^m(\cos\phi)}{\partial\phi}}{\sin\phi}$$

$$= -l \cdot P_l^m(\cos\phi) + l \cdot \frac{\cos\phi}{\sin\phi}\frac{l \cdot \cos\phi \cdot P_l^m(\cos\phi) - (l+m)P_{l-1}^m(\cos\phi)}{\sin\phi} - (l+m)\frac{1}{\sin\phi}\frac{(l-1) \cdot \cos\phi \cdot P_{l-1}^m(\cos\phi) - (l-1+m) \cdot P_{l-2}^m(\cos\phi)}{\sin\phi}$$

$$= \frac{-l \cdot \sin^2\phi \cdot P_l^m(\cos\phi) + l^2 \cdot \cos^2\phi \cdot P_l^m(\cos\phi) - l \cdot (l+m) \cdot \cos\phi \cdot P_{l-1}^m(\cos\phi) - (l+m)(l-1) \cdot \cos\phi \cdot P_{l-1}^m(\cos\phi) + (l+m)(l-1+m) \cdot P_{l-2}^m(\cos\phi)}{\sin^2\phi}$$

$$= \frac{(-l \cdot \sin^2\phi \cdot + l^2 \cdot \cos^2\phi) \cdot P_l^m(\cos\phi) - (l+m)(2l-1) \cdot \cos\phi \cdot P_{l-1}^m(\cos\phi) + (l+m)(l-1+m) \cdot P_{l-2}^m(\cos\phi)}{\sin^2\phi}$$

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$$\frac{(-l \cdot \sin^2\phi \cdot + l^2 \cdot \cos^2\phi) \cdot P_l^m(\cos\phi) - (l+m)(2l-1) \cdot \cos\phi \cdot P_{l-1}^m(\cos\phi) + (l+m)(l-1+m) \cdot P_{l-2}^m(\cos\phi)}{\sin^2\phi}$$

$$= \frac{\left((-l^2 - l) \cdot \sin^2\phi + l^2\right) \cdot P_l^m(\cos\phi) - (l+m)(2l-1) \cdot \cos\phi \cdot P_{l-1}^m(\cos\phi) + (l+m)(l-1+m) \cdot P_{l-2}^m(\cos\phi)}{\sin^2\phi}$$

$$= (-l^2 - l) \cdot P_l^m(\cos\phi) + \frac{l^2 \cdot P_l^m(\cos\phi) - (l+m)(2l-1) \cdot \cos\phi \cdot P_{l-1}^m(\cos\phi) + (l+m)(l-1+m) \cdot P_{l-2}^m(\cos\phi)}{\sin^2\phi}$$

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$$(-l^2-l)\cdot P_l^m(\cos\phi) + \frac{l^2\cdot P_l^m(\cos\phi) - (l+m)(2l-1)\cdot\cos\phi\cdot P_{l-1}^m(\cos\phi) + (l+m)(l-1+m)\cdot P_{l-2}^m(\cos\phi)}{\sin^2\phi}$$

$$= (-l^2-l)\cdot P_l^m(\cos\phi) + \frac{m^2\cdot P_l^m(\cos\phi) + (l-m)(l+m)\cdot P_l^m(\cos\phi) - (l+m)(2l-1)\cdot\cos\phi\cdot P_{l-1}^m(\cos\phi) + (l+m)(l-1+m)\cdot P_{l-2}^m(\cos\phi)}{\sin^2\phi}$$

$$= (-l^2-l)\cdot P_l^m(\cos\phi) + m^2\cdot\frac{P_l^m(\cos\phi)}{\sin^2\phi} + \frac{l+m}{\sin^2\phi}\frac{(l-m)\cdot P_l^m(\cos\phi) - (2l-1)\cdot\cos\phi\cdot P_{l-1}^m(\cos\phi) + (l-1+m)\cdot P_{l-2}^m(\cos\phi)}{\sin^2\phi}$$

$$= (-l^2-l)\cdot P_l^m(\cos\phi) + m^2\cdot\frac{P_l^m(\cos\phi)}{\sin^2\phi}$$