FFTs in Graphics and Vision

Spherical Harmonics
and
Legendre Polynomials
Outline

Math Stuff

Gram-Schmidt Orthogonalization
Completing Homogenous Polynomials

Review

Defining the Harmonics
Gram–Schmidt Orthogonalization

Given an inner product space $V$, and given a basis $\{v_1, \cdots, v_n\}$ we can define an orthonormal basis $\{w_1, \cdots, w_n\}$ for $V$:  

$$\langle w_i, w_j \rangle = \delta_{ij}$$
Gram–Schmidt Orthogonalization

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$$\langle w_i, w_j \rangle = \delta_{ij}$$

**Algorithm:**

Start by making $v_1$ a unit vector:

$$w_1 = \frac{v_1}{\|v_1\|}$$
Gram–Schmidt Orthogonalization

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$$\langle w_i, w_j \rangle = \delta_{ij}$$

Algorithm:

To get the 2\textsuperscript{nd} basis element, subtract from $v_2$ the $w_1$ component and then normalize:

$$w_2 = \frac{v_2 - \langle v_2, w_1 \rangle w_1}{\|v_2 - \langle v_2, w_1 \rangle w_1\|}$$
Gram–Schmidt Orthogonalization

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$$\langle w_i, w_j \rangle = \delta_{ij}$$

Algorithm:

To get the $i$-th basis element, subtract off all the earlier components from $v_i$ and then normalize:

$$w_i = \frac{v_i - \langle v_i, w_{i-1} \rangle w_{i-1} - \cdots - \langle v_i, w_1 \rangle w_1}{\| v_i - \langle v_i, w_{i-1} \rangle w_{i-1} - \cdots - \langle v_i, w_1 \rangle w_1 \|}$$
Gram–Schmidt Orthogonalization

Example:

Consider the space of polynomials of degree $N$ on the interval $[-1,1]$, with the inner-product:

$$\langle f(x), g(x) \rangle = \int_{-1}^{1} f(x) \cdot g(x) \, dx$$

We would like to obtain an orthogonal basis:

$$\{p_0(x), \ldots, p_N(x)\}$$

Start with the basis of monomials:

$$\{1, x, x^2, \ldots, x^N\}$$

and perform Gram-Schmidt orthogonalization.
Gram–Schmidt Orthogonalization

Example:

Consider the space of polynomials of degree $N$ on the interval $[-1,1]$, with the inner-product:

\[ f(x), g(x) = \int_{-1}^{1} f(x) \cdot g(x) \, dx \]

We would like to obtain an orthogonal basis:

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Start with the basis of monomials:

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and perform Gram-Schmidt orthogonalization.

By induction, $p_k(x)$ is a polynomial of degree $k$ since G.S. orthogonalization only subtracts off lower-degree basis functions.
Gram–Schmidt Orthogonalization

Example:

Starting with the constant term, we get:

\[ p_0(x) = \frac{1}{\|1\|} \]

\[ = \frac{1}{\sqrt{\int_{-1}^{1} dx}} \]

\[ = \frac{1}{\sqrt{2}} \]
Gram–Schmidt Orthogonalization

Example:

For the linear term, we get:

\[ p_1(x) = \frac{x - \langle x, p_0(x) \rangle p_0(x)}{\|x - \langle x, p_0(x) \rangle p_0(x)\|} = \frac{x - \left( \int_{-1}^{1} x \cdot \frac{1}{\sqrt{2}} \, dx \right) \frac{1}{\sqrt{2}}}{\|x - \langle x, p_0(x) \rangle p_0(x)\|} = \frac{1}{\sqrt{\int_{-1}^{1} x^2 \, dx}} \sqrt{\int_{-1}^{1} x^2 \, dx} = \sqrt{\frac{3}{2}} x \]
Gram–Schmidt Orthogonalization

Example:

And for the quadratic term:

\[ p_2(x) = \frac{x^2 - \langle x^2, p_1(x) \rangle p_1(x) - \langle x^2, p_0(x) \rangle p_0(x)}{\|x^2 - \langle x^2, p_1(x) \rangle p_1(x) - \langle x^2, p_0(x) \rangle p_0(x)\|} \]

These are the Legendre Polynomials.
Legendre Polynomials

Claim:

The degrees of the monomials comprising the Legendre polynomials are all even/odd:

\[ p_k(x) = a_k x^k + a_{k-1} x^{k-2} + \ldots \]

Proof by Induction:
Legendre Polynomials

Claim:

The degrees of the monomials comprising the Legendre polynomials are all even/odd:

\[ p_k(x) = a_k x^k + a_{k-1} x^{k-2} + \ldots \]

Proof by Induction \((k = 0)\):

\[ p_0(x) = \frac{1}{\sqrt{2}} \]
Legendre Polynomials

Claim:

The degrees of the monomials comprising the Legendre polynomials are all even/odd:

\[ p_k(x) = a_k x^k + a_{k-1} x^{k-2} + \ldots \]

Proof by Induction (assume true for \( k = n \)):

\[ p_{n+1}(x) = \frac{x^{n+1} - \langle x^{n+1}, p_n(x) \rangle p_n(x) - \ldots - \langle x^{n+1}, p_0(x) \rangle p_0(x)}{\|x^{n+1} - \langle x^{n+1}, p_n(x) \rangle p_n(x) - \ldots - \langle x^{n+1}, p_0(x) \rangle p_0(x)\|} \]

Recall that:

\[ \langle x^{n+1}, p_m(x) \rangle = \int_{-1}^{1} x^{n+1} \cdot p_m(x) \, dx \]
Legendre Polynomials

Claim:

The degrees of the monomials comprising the Legendre polynomials are all even/odd:

\[ p_k(x) = a_k x^k + a_{k-1} x^{k-2} + \ldots \]

Proof by Induction (assume true for \( k = n \)):

\[ \langle x^{n+1}, p_m(x) \rangle = \int_{-1}^{1} x^{n+1} \cdot p_m(x) \, dx \]

Since \( m \leq n \) we can assume that the monomials comprising \( p_m(x) \) are all even/odd if \( m \) is.
Legendre Polynomials

Claim:

The degrees of the monomials comprising the Legendre polynomials are all even/odd:

\[ p_k(x) = a_k x^k + a_{k-1} x^{k-2} + \ldots \]

Proof by Induction (assume true for \( k = n \)):

So if \( n \) and \( m \) are both even/odd, then the polynomial \( x^{n+1} \cdot p_m(x) \) is comprised of strictly odd-powered monomials:

\[ \langle x^{n+1}, p_m(x) \rangle = \int_{-1}^{1} x^{n+1} \cdot p_m(x) \, dx = 0 \]
Legendre Polynomials

Claim:
The degrees of the monomials comprising the Legendre polynomials are all even/odd:
\[ p_k(x) = a_k x^k + a_{k-1} x^{k-2} + \cdots \]

Proof by Induction (assume true for \( k = n \)):
\[ p_{n+1}(x) = \frac{x^{n+1} \langle x^{n+1}, p_n(x) \rangle p_n(x) - \langle x^{n+1}, p_{n-1}(x) \rangle p_{n-1}(x) - \cdots}{\|x^{n+1} \langle x^{n+1}, p_n(x) \rangle p_n(x) - \langle x^{n+1}, p_{n-1}(x) \rangle p_{n-1}(x) - \cdots \|} \]

So \( p_{n+1}(x) \) is obtained by starting with the monomial \( x^{n+1} \) and subtracting off monomials with the same parity.
Gram–Schmidt Orthogonalization

Example:

Consider the space of polynomials of degree $N$ on the interval $[-1,1]$, with an inner-product giving more weight to points near the origin:

$$\langle f(x), g(x) \rangle_m = \int_{-1}^{1} (1 - x^2)^m \cdot f(x) \cdot g(x) \, dx$$

We would like to obtain an orthonormal basis:

$$\{p_0^m(x), \cdots, p_N^m(x)\}$$
Gram–Schmidt Orthogonalization

Example:

\[ \langle f(x), g(x) \rangle_m = \int_{-1}^{1} (1 - x^2)^m \cdot f(x) \cdot g(x) \, dx \]

We proceed as before with the new inner-product.

Since the weighting function is even, if \( f \) is an even function and \( g \) is an odd function (or vice-versa), the inner product must be zero:

\[ \langle f(x), g(x) \rangle_m = 0 \]

Thus, as before, the degree of the monomials comprising \( p_l^m(x) \) must all have the same parity.
Completing Homogenous Polynomials

Given a polynomial \( p(x, y, z) \) of degree \( d \), consisting of monomials of powers \( d, d - 2, \ldots: \)

\[
p(x, y, z) = \sum_{k=0}^{[d/2]} \left( \sum_{l+m+n=d-2k} a_{lmn} \cdot x^l \cdot y^m \cdot z^n \right)
\]

This is not a homogenous polynomial.

However, if we restrict it to the sphere, we can think of it as homogenous:

\[
p(x, y, z) = \sum_{k=0}^{[d/2]} (x^2 + y^2 + z^2)^k \left( \sum_{l+m+n=d-2k} a_{lmn} \cdot x^l \cdot y^m \cdot z^n \right)
\]
Completing Homogenous Polynomials

Example:

\[ p(x, y, z) = x^2 y + y + z \]

is not a homogenous polynomial.

But on the unit-sphere, the polynomial:

\[ q(x, y, z) = x^2 y + (y + z)(x^2 + y^2 + z^2) \]

has identical values and is homogenous of degree 3.
Outline

Math Stuff

Review

Spherical Harmonics

Defining the Harmonics
Spherical Harmonics

For each non-negative integer \( l \), there are \( 2l + 1 \) spherical harmonics of degree \( l \) satisfying:

1. Each spherical harmonic of degree \( l \) can be expressed as the restriction of a homogenous polynomial of degree \( l \) to the unit-sphere.

2. The different spherical harmonics are orthogonal to each other.
Spherical Harmonics

We saw that by considering just rotations about the \( y \)-axis, we could factor the spherical harmonics as:

\[
Y_l^m(\theta, \phi) = e^{im\theta} \cdot P_l^m(\phi) = (\cos \theta + i \sin \theta)^m \cdot P_l^m(\phi)
\]

where \(|m| \leq l\).

Writing \( x = \cos \phi \), the functions \( P_l^m(\phi) \) are given by the associated Legendre polynomials:

\[
P_l^k(\phi) = \frac{(-1)^k}{2^l l!} (1 - x^2)^{k/2} \frac{d^{l+k}}{dx^{l+k}} (x^2 - 1)^l
\]
Outline

Math Stuff

Review

Defining the Harmonics
Defining the Harmonics \((m \geq 0)\)

To define the spherical harmonics, we would like to express the function:

\[ Y_l^m(\theta, \phi) = (\cos \theta + i \sin \theta)^m \cdot P_l^m(\phi) \]

as the restriction of a homogenous polynomial of degree \(l\) to the unit sphere.
Defining the Harmonics ($m \geq 0$)

Using the parameterization of the unit-sphere

$$\Phi(\theta, \phi) = (\cos \theta \cdot \sin \phi, \cos \phi, \sin \theta \cdot \sin \phi)$$

we get:

$$Y_l^m(\theta, \phi) = (\cos \theta + i \sin \theta)^m \cdot P_l^m(\phi)$$

$$= \left( \frac{x}{\sin \phi} + i \frac{z}{\sin \phi} \right)^m \cdot P_l^m(\phi)$$

$$= (x + iz)^m \cdot \frac{P_l^m(\phi)}{\sin^m \phi}$$

$$= (x + iz)^m \cdot \frac{P_l^m(\cos^{-1} y)}{\left( \sqrt{1 - y^2} \right)^m}$$
Defining the Harmonics \((m \geq 0)\)

\[ Y_l^m(\theta, \phi) = (x + iz)^m \cdot \frac{P_l^m(\cos^{-1} y)}{(1 - y^2)^{m/2}} \]

This is a homogenous polynomial of degree \(l\).

This is a homogenous polynomial of degree \(m\).

So we want:

1. This to complete to a homogenous polynomial of degree \(l - m\).

2. The different \(Y_l^m\) to be orthogonal
Defining the Harmonics \((m \geq 0)\)

\[
Y_l^m(\theta, \phi) = (x + iz)^m \cdot \frac{P_l^m(\cos^{-1} y)}{(1 - y^2)^{m/2}}
\]

Homogeneous Completion:

To satisfy the homogeneity constraint, we need:

\[
\frac{P_l^m(\cos^{-1} y)}{(1 - y^2)^{m/2}} = a_{l-m}y^{l-m} + a_{l-m-2}y^{l-m-2} + \ldots
\]

Or equivalently:

\[
P_l^m(\cos^{-1} y) = q_l^m(y) \cdot (1 - y^2)^{m/2}
\]

for some polynomial:

\[
q_l^m(y) = a_{l-m}y^{l-m} + a_{l-m-2}y^{l-m-2} + \ldots
\]
Defining the Harmonics \((m \geq 0)\)

\[
Y_l^m(\theta, \phi) = e^{im\theta} \cdot P_l^m(\cos^{-1} y)
\]

Orthogonality:

To satisfy the orthogonality constraint, we need:

\[
\langle Y_l^m(\theta, \phi), Y_{l'}^{m'}(\theta, \phi) \rangle = 0 \quad \forall \ l \neq l' \text{ or } m \neq m'
\]
Defining the Harmonics \((m \geq 0)\)

\[
Y_l^m(\theta, \phi) = e^{im\theta} \cdot P_l^m(\cos^{-1} y)
\]

Orthogonality \((m \neq m')\):

Since we have:

\[
Y_l^m(\theta, \phi) = e^{im\theta} \cdot P_l^m(\phi)
\]

we know that:

\[
\langle Y_l^m(\theta, \phi), Y_{l'}^{m'}(\theta, \phi) \rangle = \int_0^\pi \int_0^{2\pi} e^{im\theta} \cdot P_l^m(\phi) \cdot \overline{e^{im'\theta} \cdot P_{l'}^{m'}(\phi)} \, d\theta \sin \phi \, d\phi
\]

\[
= \left( \int_0^\pi P_l^m(\phi) \cdot \overline{P_{l'}^{m'}(\phi)} \sin \phi \, d\phi \right) \cdot \left( \int_0^{2\pi} e^{i(m-m')\theta} \, d\theta \right)
\]
Defining the Harmonics \((m \geq 0)\)

\[
Y^m_l(\theta, \phi) = e^{im\theta} \cdot P^m_l(\cos^{-1} y)
\]

Orthogonality \((m \neq m')\):

\[
\langle Y^m_l(\theta, \phi), Y^{m'}_l(\theta, \phi) \rangle = \left( \int_0^\pi P^m_l(\phi) \cdot \overline{P^{m'}_l(\phi)} \sin \phi \, d\phi \right) \cdot \left( \int_0^{2\pi} e^{i(m-m')\theta} \, d\theta \right)
\]

But this is zero whenever \(m \neq m'\):

\[
\int_0^{2\pi} e^{i(m-m')\theta} \, d\theta = \frac{1}{i(m - m')} \cdot e^{i(m-m')\theta} \bigg|_0^{2\pi} = 0
\]
Defining the Harmonics \((m \geq 0)\)

\[ Y_{l}^{m}(\theta, \phi) = e^{im\theta} \cdot P_{l}^{m}(\cos^{-1} y) \]

Orthogonality \((m = m' \text{ and } l \neq l')\):

We have to choose the function:

\[ P_{l}^{m}(\cos^{-1} y) = q_{l}^{m}(y) \cdot (1 - y^2)^{m/2} \]

so that:

\[
0 = \int_{0}^{\pi} \int_{0}^{2\pi} e^{im\theta} \cdot P_{l}^{m}(\phi) \cdot \overline{e^{im\theta} \cdot P_{l'}^{m}(\phi)} \, d\theta \sin \phi \, d\phi
\]

\[
\downarrow
\]

\[ 0 = \int_{0}^{\pi} P_{l}^{m}(\phi) \cdot \overline{P_{l'}^{m}(\phi)} \cdot \sin \phi \, d\phi \]
Defining the Harmonics \((m \geq 0)\)

\[
Y_l^m(\theta, \phi) = e^{im\theta} \cdot P_l^m(\cos^{-1} y)
\]

Orthogonality \((m = m' \text{ and } l \neq l')\):

We have to choose the function:

\[
P_l^m(\cos^{-1} y) = q_l^m(y) \cdot (1 - y^2)^{m/2}
\]

Using the change of variables:

\[
\int_0^\pi P_l^m(\phi) \cdot \overline{P_{l'}^m(\phi)} \cdot \sin \phi \, d\phi = \int_{-1}^1 P_l^m(\cos^{-1} y) \cdot \overline{P_{l'}^m(\cos^{-1} y)} \, dy
\]

\[
= \int_{-1}^1 q_l^m(y) \cdot \overline{q_{l'}^m(y)} \cdot (1 - y^2)^m \, dy
\]
Defining the Harmonics \((m \geq 0)\)

\[ Y_l^m(\theta, \phi) = e^{im\theta} \cdot P_l^m(\cos^{-1} y) \]
\[ P_l^m(\cos^{-1} y) = q_l^m(y) \cdot (1 - y^2)^{m/2} \]

Thus, the polynomials \(q_l^m(y)\) should:

1. Complete to homogeneous polynomials of degree \(l - m\):
\[ q_l^m(y) = a_{l-m}y^{l-m} + a_{l-m-2}y^{l-m-2} + \ldots \]

2. Satisfy the orthogonality condition:
\[ 0 = \int_{-1}^{1} q_l^m(y) \cdot q_l^m(y) \cdot (1 - y^2)^m \, dy \]
Defining the Harmonics \((m \geq 0)\)

\[
Y_l^m(\theta, \phi) = e^{im\theta} \cdot P_l^m(\cos^{-1} y)
\]

\[
P_l^m(\cos^{-1} y) = q_l^m(y) \cdot (1 - y^2)^{m/2}
\]

This is what we get with G.S. orthogonalization
\[\{1, y, y^2, \cdots\} \rightarrow \{p_0^m(y), p_1^m(y), p_2^m(y), \cdots\}\] relative to the inner-product:

\[
\langle f(y), g(y) \rangle_m = \int_{-1}^{1} f(y) \cdot g(y) \cdot (1 - y^2)^m \, dy
\]

and set:

\[
q_l^m(y) = p_{l-m}^m(y)
\]
Defining the Harmonics \((m \geq 0)\)

\[
Y_l^m(\theta, \phi) = e^{im\theta} \cdot P_l^m(\cos^{-1} y)
\]
\[
P_l^m(\cos^{-1} y) = q_l^m(y) \cdot (1 - y^2)^{m/2}
\]

In sum, we get an expression for the spherical harmonics as:

\[
Y_l^m(\theta, \phi) = e^{im\theta} \cdot p_{l-m}^m(\cos \phi) \cdot \left(\sqrt{1 - \cos^2 \phi}\right)^m
\]
\[
= e^{im\theta} \cdot p_{l-m}^m(\cos \phi) \cdot \sin^m \phi
\]

where \(p_{l-m}^m(y)\) is a (homogeneously completable) polynomial of degree \(l - m\).
Defining the Harmonics \((m \geq 0)\)

\[ Y_l^m(\theta, \phi) = e^{im\theta} \cdot p_{l-m}(\cos \phi) \cdot \sin^m \phi \]

Examples \((l = 0)\):

\[ Y_0^0(\theta, \phi) = \frac{1}{\sqrt{4\pi}} \]
Defining the Harmonics \((m \geq 0)\)

\[ Y_l^m(\theta, \phi) = e^{i m \theta} \cdot p_{l-m}^m(\cos \phi) \cdot \sin^m \phi \]

Examples \((l = 1)\):

\[ Y_1^{-1}(\theta, \phi) = \sqrt{\frac{3}{8\pi}} \sin \phi \cdot e^{-i \theta} \]

\[ Y_1^0(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos \phi \]

\[ Y_1^1(\theta, \phi) = \sqrt{\frac{3}{8\pi}} \sin \phi \cdot e^{i \theta} \]
Defining the Harmonics ($m \geq 0$)

$$Y_l^m(\theta, \phi) = e^{im\theta} \cdot p_{l-m}^m(\cos \phi) \cdot \sin^m \phi$$

Examples ($l = 2$):

$$Y_2^{-2}(\theta, \phi) = \sqrt{\frac{15}{32\pi}} \sin^2 \phi \cdot e^{-i2\theta}$$

$$Y_2^{-1}(\theta, \phi) = \sqrt{\frac{15}{8\pi}} \sin \phi \cdot \cos \phi \cdot e^{-i\theta}$$

$$Y_2^0(\theta, \phi) = \sqrt{\frac{5}{16\pi}} (3 \cos^2 \phi - 1)$$

$$Y_2^1(\theta, \phi) = \sqrt{\frac{15}{8\pi}} \sin \phi \cdot \cos \phi \cdot e^{i\theta}$$

$$Y_2^2(\theta, \phi) = \sqrt{\frac{15}{32\pi}} \sin^2 \phi \cdot e^{i2\theta}$$