FFTs in Graphics and Vision

Homogenous Polynomials
and
Irreducible Representations

Outline

The 2π Term in Assignment 1

Homogenous Polynomials

Representations of Functions on the Unit-Circle

- Sub-Representations for the Circle
- Sub-Representations for the Sphere

The 2π Term in Assignment 1

Given an n-dimensional array of values, we would like to treat the values as the regular samples of some continuous, periodic, function:

$$f[\cdot] \leftarrow f(x)$$

What is the domain of $f(x)$?

The 2π Term in Assignment 1

What is the domain of $f(x)$?

Two possible approaches:

- Dimension Dependent $[0, n)$:
$$f[j] = f(j)$$
- Dimension Independent $[0, \rho)$:
$$f[j] = f\left(\frac{j}{n} \cdot \rho\right)$$

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<u>Dimension Dependent Domain $[0, n)$</u>:

This provides a (nearly) norm-preserving map from the space of n-dimensional arrays to the space of functions:

Vector Square Norm	Function Square Norm
$\|f[\cdot]\|^2 = \sum_{j=0}^{n-1} \lvert f[j]\rvert^2 = \sum_{j=0}^{n-1} \lvert f(j)\rvert^2$	$\|f(\cdot)\|^2 = \int_0^n \lvert f(x)\rvert^2 dx = \lim_{l\to\infty} \sum_{j=0}^{l-1} \left\lvert f\left(\frac{j}{l}\cdot n\right)\right\rvert^2 \cdot \frac{n}{l} \approx \sum_{j=0}^{n-1} \lvert f(j)\rvert^2$

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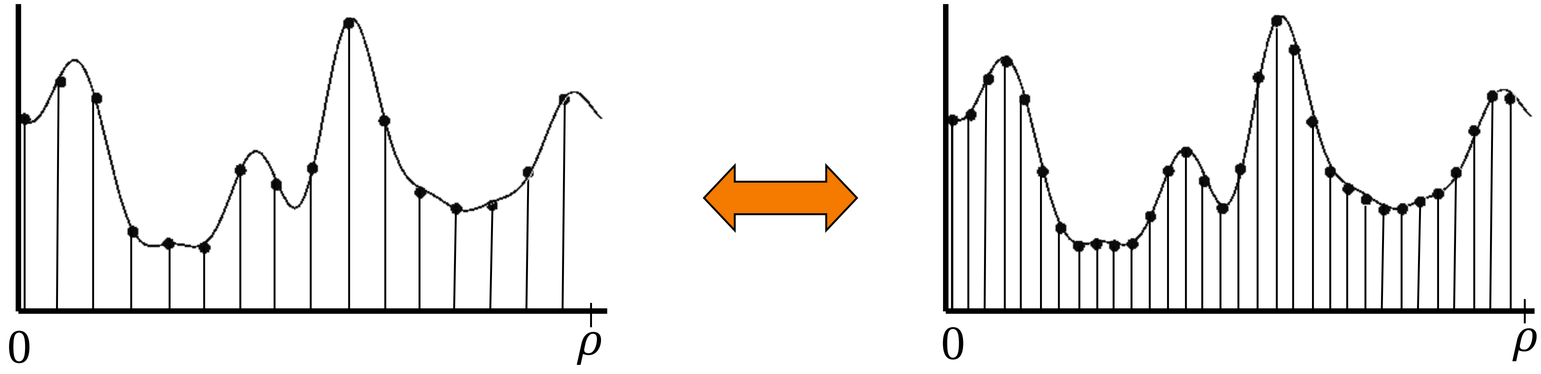
Dimension Independent Domain $[0, \rho)$:

This provides a way for treating two arrays of different dimensions as regular samplings of the same function at different resolutions.



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Dimension Independent Domain $[0, \rho)$:

This does not provide a norm-preserving map from the space of n-dimensional arrays to the space of functions:

Vector Square Norm	Function Square Norm
$\|f[\cdot]\|^2 = \sum_{j=0}^{n-1} \|f[j]\|^2 = \sum_{j=0}^{n-1} \left\| f\left(\frac{j}{n} \cdot \rho\right) \right\|^2$	$\|f(\cdot)\|^2 = \int_0^\rho \|f(x)\|^2 dx = \lim_{l \to \infty} \sum_{j=0}^{l-1} \left\| f\left(\frac{j}{l} \cdot \rho\right) \right\|^2 \cdot \frac{\rho}{l} \approx \sum_{j=0}^{n-1} \left\| f\left(\frac{j}{n} \cdot \rho\right) \right\|^2 \cdot \frac{\rho}{n}$

The 2π Term in Assignment 1

Dimension Independent Domain $[0,\rho)$:

This does not provide a norm-preserving map from the space of n-dimensional arrays to the space of functions:

Vector Square Norm	Function Square Norm
$\|f[\cdot]\|^2$ … $\sum_{j=0}$ … $= \sum_{j=0}^{n-1} \left\| f\left(\frac{j}{n}\cdot\rho\right)\right\|^2$	… $f(x)\|^2 dx$ $= \lim_{l\to\infty} \sum_{j=0}^{l-1} \left\| f\left(\frac{j}{l}\cdot\rho\right)\right\|^2 \cdot \frac{\rho}{l}$ $\approx \sum_{j=0}^{n-1} \left\| f\left(\frac{j}{n}\cdot\rho\right)\right\|^2 \cdot \frac{\rho}{n}$

This mapping scales the square norm by $\frac{\rho}{n}$.

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Dimension Independent Domain $[0,\rho)$:

When we consider periodic functions on a line – i.e. functions on a circle – we set the domain to be equal to the length of a circle: $[0,2\pi)$.

Similarly, for periodic functions on a plane – i.e. functions on the product of two circles (a torus) – we choose the domain to be $[0,2\pi)\times[0,2\pi)$.

The 2π Term in Assignment 1

How does this affect the Fourier coefficients?

The Fourier coefficients of $f[\cdot]$ are the coefficients of $f[\cdot]$ with respect to the Fourier basis:

$$f[\cdot] = \sum_{k=0}^{n-1} \hat{f}[k] \cdot v_k[\cdot]$$

where the $v_k[\cdot]$ correspond to regular samples of the k-th complex exponential at n positions:

$$v_k[\cdot] = \left(e^{i\cdot 2k\pi/n\cdot 0}, \cdots, e^{i\cdot 2k\pi/n\cdot(n-1)}\right)$$

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How does this affect the Fourier coefficients?

We know that the $v_k[\cdot]$ are perpendicular to each other, and we would like them to have unit-norm so that they form an orthonormal basis:

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Dimension Dependent $[0, n)$	Dimension Independent $[0,2\pi)$
$$\begin{aligned}\|v_k(\cdot)\|^2 &= \int_0^n \left\|e^{i\cdot 2k\pi/n\cdot x}\right\| dx \\ &= \int_0^n 1\, dx \\ &= n\end{aligned}$$	

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$\|v_k(\cdot)\|^2 = \int_0^n \left\|e^{i\cdot 2k\pi/n\cdot x}\right\|dx$ $= n$	

$$v_k[\cdot] \to \frac{v_k[\cdot]}{\sqrt{n}}$$

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Dimension Dependent $[0,n)$	Dimension Independent $[0,2\pi)$
$\|v_k(\cdot)\|^2 = \int_0^n \left\|e^{i\cdot 2k\pi/n\cdot x}\right\| dx$ $= n$ $v_k[\cdot] \to \frac{v_k[\cdot]}{\sqrt{n}}$	$\|v_k(\cdot)\|^2 = \int_0^{2\pi} \left\|e^{i\cdot k\cdot x}\right\| dx$ $= \int_0^{2\pi} 1\, dx$ $= 2\pi$

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How does this affect the Fourier coefficients?

We know that the $v_k[\cdot]$ are perpendicular to each other, and we would like them to have unit-norm so that they form an orthonormal basis:

Dimension Dependent $[0,n)$	Dimension Independent $[0,2\pi)$
$\|v_k(\cdot)\|^2 = \int_0^n \left\|e^{i\cdot 2k\pi/n\cdot x}\right\| dx = n$	$\|v_k(\cdot)\|^2 = \int_0^{2\pi} \left\|e^{i\cdot k\cdot x}\right\| dx = 2\pi$
$v_k[\cdot] \to \frac{v_k[\cdot]}{\sqrt{n}}$	$v_k[\cdot] \to \frac{v_k[\cdot]}{\sqrt{2\pi}}$

The 2π Term in Assignment 1

How does this affect the Fourier coefficients?

So, if we compute the Fourier coefficients of $f[\cdot]$ assuming that the domain is $[0,n)$, to get the Fourier coefficients of $f[\cdot]$ on the domain $[0,2\pi)$, we need to scale:

- $[0,n) \to [0,2\pi)$:

$$\hat{f}[k] \to \sqrt{\frac{n}{2\pi}} \cdot \hat{f}[k]$$

- $[0,2\pi) \to [0,n)$:

$$\hat{f}[k] \to \sqrt{\frac{2\pi}{n}} \cdot \hat{f}[k]$$

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How does this affect 2D Gaussian smoothing?

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How does this affect 2D Gaussian smoothing?

To perform Gaussian smoothing of $f[\cdot][\cdot]$, we want a filter $g[\cdot][\cdot]$ whose entries "sum to one".

Dimension Dependent $[0,n)\times[0,n)$	Dimension Independent $[0,2\pi)\times[0,2\pi)$
$$\begin{aligned} 1 &= \int_0^n \int_0^n g(x,y)\,dy\,dx \\ &= \lim_{l\to\infty} \sum_{j,k=0}^{l-1} g\left(\frac{jn}{l},\frac{kn}{l}\right)\cdot\left(\frac{n}{l}\right)^2 \\ &\approx \sum_{j,k=0}^{l-1} g(j,k) \\ &= \sum_{j,k=0}^{l-1} g[j][k] \end{aligned}$$	

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$1 = \int_0^n \int_0^n g(x,y)\,dy\,dx$	
$= \lim \sum^{l-1} g\left(\frac{jn}{l},\frac{kn}{l}\right)\cdot\left(\frac{n}{l}\right)^2$	
$\sum_{j,k=0} g(j,k)$	
$= \sum_{j,k=0}^{l-1} g[j][k]$	

The Gaussian is normalized if the sum of the entries equals 1.

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How does this affect 2D Gaussian smoothing?

To perform Gaussian smoothing of $f[\cdot][\cdot]$, we want a filter $g[\cdot][\cdot]$ whose entries "sum to one".

Dimension Dependent $[0,n)\times[0,n)$

$$1 = \int_0^n\int_0^n g(x,y)dy\,dx$$

$$= \lim \sum_{j,k=0}^{l-1} g\left(\frac{jn}{l},\frac{kn}{l}\right)\cdot\left(\frac{n}{l}\right)^2$$

$$\approx \sum_{j,k=0}^{l-1} g(j,k)$$

$$= \sum_{j,k=0}^{l-1} g[j][k]$$

The Gaussian is normalized if the sum of the entries equals 1.

Dimension Independent $[0,2\pi)\times[0,2\pi)$

$$1 = \int_0^{2\pi}\int_0^{2\pi} g(x,y)dy\,dx$$

$$= \lim_{l\to\infty} \sum_{j,k=0}^{l-1} g\left(\frac{j2\pi}{l},\frac{k2\pi}{l}\right)\cdot\left(\frac{2\pi}{l}\right)^2$$

$$\approx \sum_{j,k=0}^{l-1} g\left(\frac{j2\pi}{n},\frac{k2\pi}{n}\right)\cdot\left(\frac{2\pi}{n}\right)^2$$

$$= \sum_{j,k=0}^{l-1} g[j][k]\cdot\left(\frac{2\pi}{n}\right)^2$$

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To perform Gaussian smoothing of $f[\cdot][\cdot]$, we want a filter $g[\cdot][\cdot]$ whose entries "sum to one".

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$1=\int_0^n\int_0^n g(x,y)\,dy\,dx$	$1=\int_0^{2\pi}\int_0^{2\pi} g(x,y)\,dy\,dx$
$=\lim \sum_{j,k=0}^{l-1} g\left(\frac{jn}{l},\frac{kn}{l}\right)\cdot\left(\frac{n}{l}\right)^2$	$=\lim \sum_{j,k=0}^{l-1} g\left(\frac{j2\pi}{l},\frac{k2\pi}{l}\right)\cdot\left(\frac{2\pi}{l}\right)^2$
$=\sum_{j,k=0} g(j,k)$	$\sum_{j,k=0}$
$=\sum_{j,k=0}^{l-1} g[j][k]$	$=\sum_{j,k=0}^{l-1} g[j][k]\cdot\left(\frac{2\pi}{n}\right)^2$

The Gaussian is normalized if the sum of the entries equals 1.

The Gaussian is normalized if the sum of the entries equals $\left(\frac{n}{2\pi}\right)^2$.

Outline

The 2π Term in Assignment 1

Homogenous Polynomials

Representations of Functions on the Unit-Circle

- Sub-Representations for the Circle
- Sub-Representations for the Sphere

Monomials

Definition:

A monomial in variables $\{x_1, \cdots, x_n\}$ is a product of non-negative integer powers of the variables.

The degree of a monomial is the sum of the powers.

Monomials

Examples:

- Degree 0: 1
- Degree 1: x, y, z
- Degree 2: x^2, y^2, z^2, xy, xz, yz
- Degree 3: x^3, x^2y, x^2z, xy^2, xz^2, xyz, y^3, y^2z, yz^2, z^3

Polynomials

Definition:

A polynomial of degree d in variables $\{x_1, \cdots, x_n\}$ is a linear sum of monomials in $\{x_1, \cdots, x_n\}$, each of whose degree is no greater than d.

Notation:

Denote by $P^d(x_1, \cdots, x_n)$ the set of polynomials in $\{x_1, \cdots, x_n\}$ of degree d.

Polynomials

Examples:

- $d = 0$:
  - $P^0(x) = P^0(x,y) = P^0(x,y,z) = a$
- $d = 1$:
  - $P^1(x) = ax + c$
  - $P^1(x,y) = ax + by + c$
  - $P^1(x,y,z) = ax + by + cz + d$
- $d = 2$:
  - $P^2(x) = ax^2 + bx + c$
  - $P^2(x,y) = ax^2 + by^2 + cxy + dx + ey + f$
  - $P^2(x,y,z) = ax^2 + by^2 + cz^2 + dxy + exz + gyz + hx + iy + jz + k$
- …

Polynomials

Properties:

- The linear sum of polynomials p and q of degree d is a polynomial of degree d:
$$a \cdot p(x_1, \cdots, x_n) + b \cdot p(x_1, \cdots, x_n) \in P^d(x_1, \cdots, x_n)$$
- The product of polynomials p and q of degrees d_1 and d_2 is a polynomial of degree $d_1 + d_2$:
$$p(x_1, \cdots, x_n) \cdot q(x_1, \cdots, x_n) \in P^{d_1+d_2}(x_1, \cdots, x_n)$$
- The k-th power of a polynomial p of degree d is a polynomial of degree $d \cdot k$:
$$p^k(x_1, \cdots, x_n) \in P^{d \cdot k}(x_1, \cdots, x_n)$$

Homogenous Polynomials

Definition:

A degree d polynomial is said to be homogenous if the individual monomials all have degree d.

Notation:

Denote by $HP^d(x_1, \ldots, x_n)$ the set of homogenous polynomials in $\{x_1, \ldots, x_n\}$ of degree d.

Homogenous Polynomials

Examples:

- $d = 0$:
  - $HP^0(x) = HP^0(x,y) = HP^0(x,y,z) = a$
- $d = 1$:
  - $HP^1(x) = ax$
  - $HP^1(x,y) = ax + by$
  - $HP^1(x,y,z) = ax + by + cz$
- $d = 2$:
  - $HP^2(x) = ax^2$
  - $HP^2(x,y) = ax^2 + by^2 + cxy$
  - $HP^2(x,y,z) = ax^2 + by^2 + cz^2 + dxy + exz + gyz$
- …

Homogenous Polynomials

Properties:

- The linear sum of homogenous polynomials p and q of degree d is a homogenous polynomial of degree d:
$$a \cdot p(x_1, \cdots, x_n) + b \cdot p(x_1, \cdots, x_n) \in HP^d(x_1, \cdots, x_n)$$
- The product of homogenous polynomials p and q of degrees d_1 and d_2 is a homogenous polynomial of degree $d_1 + d_2$:
$$p(x_1, \cdots, x_n) \cdot q(x_1, \cdots, x_n) \in HP^{d_1+d_2}(x_1, \cdots, x_n)$$
- The k-th power of a homogenous polynomial p of degree d is a homogenous polynomial of degree $d \cdot k$:
$$p^k(x_1, \cdots, x_n) \in HP^{d \cdot k}(x_1, \cdots, x_n)$$

Homogenous Polynomials

Note 1:

Any degree d polynomial in $\{x_1, \ldots, x_n\}$ can be uniquely expressed as the sum of homogenous polynomials in $\{x_1, \ldots, x_n\}$ of degrees 0 through d:

$$P^d(x_1, \cdots, x_n) = HP^0(x_1, \cdots, x_n) \oplus \cdots \oplus HP^d(x_1, \cdots, x_n)$$

Homogenous Polynomials

Note 1:

$$P^d(x_1,\cdots,x_n) = HP^0(x_1,\cdots,x_n) \oplus \cdots \oplus HP^d(x_1,\cdots,x_n)$$

Example:

- $\underbrace{p(x,y)}_{\in P^2(x,y)} = \underbrace{2x^2 + 3y^2 - xy}_{\in HP^2(x,y)} + \underbrace{5x - 7y}_{\in HP^1(x,y)} + \underbrace{2}_{\in HP^0(x,y)}$

Homogenous Polynomials

Note 2:

Any homogenous polynomial in $\{x_1, \ldots, x_n\}$ of degree d can be uniquely expressed as:

- x_1 times a degree $d-1$ homogenous polynomial in $\{x_1, \ldots, x_n\}$, plus
- a degree d homogenous polynomial in $\{x_2, \ldots, x_n\}$.

$$HP^d(x_1, \cdots, x_n) = x_1 \cdot HP^{d-1}(x_1, \cdots x_n) \oplus HP^d(x_2, \cdots, x_n)$$

Homogenous Polynomials

Note 2:

$$HP^d(x_1,\cdots,x_n) = x_1 \cdot HP^{d-1}(x_1,\cdots x_n) \oplus HP^d(x_2,\cdots,x_n)$$

Example:

- $\underbrace{p(x,y)}_{\in HP^2(x,y)} = 2x^2 + 3y^2 - xy$
  $= x \cdot \underbrace{(2x-y)}_{\in HP^1(x,y)} + \underbrace{3y^2}_{\in HP^2(y)}$

Dimensions

What is the dimension of $P^d(x_1, \ldots, x_n)$?

What is the dimension of $HP^d(x_1, \ldots, x_n)$?

Dimensions

What is the dimension of $P^d(x_1, ..., x_n)$?

Since every polynomial of degree d can be uniquely expressed as the sum of homogenous polynomials of degrees 0 through d:

$$\dim\left(P^d(x_1, \cdots, x_n)\right) = \dim\left(HP^0(x_1, \cdots, x_n)\right) + \cdots + \dim\left(HP^d(x_1, \cdots, x_n)\right)$$

Dimensions

What is the dimension of $HP^d(x_1, \ldots, x_n)$?

Dimensions

Three properties give us a recursive definition:

1. A homogenous polynomial of degree d factors as:
   $HP^d(x_1, \cdots, x_n) = x_1 \cdot HP^{d-1}(x_1, \cdots, x_n) \oplus HP^d(x_2, \cdots, x_n)$

2. The space of homogenous polynomials in $\{x_1, \cdots, x_n\}$ of degree 0 is one-dimensional:
   $$HP^0(x_1, \cdots, x_n) = a$$

3. The space of homogenous polynomials in $\{x\}$ of degree d is one-dimensional:
   $$HP^d(x) = ax^d$$

Dimensions

Homogenous Polynomials of Degree Zero:

The dimension of the space of homogenous polynomials of degree 0 in any number of variables is one:

$$\dim[HP^0(x_1, \cdots, x_n)] = 1$$

Dimensions

Homogenous Polynomials in One Variable:

The dimension of the space of homogenous polynomials of degree d in one variable is one, for all degrees d:

$$\dim[HP^d(x)] = 1$$

Dimensions

Homogenous Polynomials in n Variables:

The dimension of the space of homogenous polynomials of degree d in n variables is:

$$\dim[HP^d(x_1,\cdots,x_n)] = \dim[HP^d(x_2,\cdots x_n)] + \dim[HP^{d-1}(x_1,\cdots,x_n)]$$

Dimensions

Homogenous Polynomials in n Variables:

The dimension of the space of homogenous polynomials of degree d in n variables is:

$$\begin{aligned}\dim[HP^d(x_1,\cdots,x_n)] &= \dim[HP^d(x_2,\cdots x_n)]\\ &+ \dim[HP^{d-1}(x_2,\cdots,x_n)]\\ &+ \dim[HP^{d-2}(x_1,\cdots,x_n)]\end{aligned}$$

Dimensions

Homogenous Polynomials in n Variables:

The dimension of the space of homogenous polynomials of degree d in n variables is:

$$\dim[HP^d(x_1,\cdots,x_n)] = \sum_{i=1}^{d} \dim[HP^i(x_2,\cdots x_n)] + \dim[HP^0(x_1,\cdots,x_n)]$$

Dimensions

Homogenous Polynomials in n Variables:

The dimension of the space of homogenous polynomials of degree d in n variables is:

$$\dim[HP^d(x_1,\cdots,x_n)] = \sum_{i=1}^{d} \dim[HP^i(x_2,\cdots x_n)] + 1$$

Dimensions

Homogenous Polynomials in n Variables:

One Variable:

$$\dim[HP^d(x)] = 1$$

Dimensions

Homogenous Polynomials in n Variables:

One Variable: $\dim[HP^d(x)] = 1$

Two Variables:

$$\dim[HP^d(x,y)] = 1 + \sum_{i=1}^{d} \dim\left[HP^i(x)\right]$$
$$= 1 + \sum_{i=1}^{d} 1$$
$$= 1 + d$$

Dimensions

Homogenous Polynomials in n Variables:

One Variable: $\dim[HP^d(x)] = 1$

Two Variables: $\dim[HP^d(x,y)] = 1 + d$

Three Variables:

$$\dim[HP^d(x,y,z)] = 1 + \sum_{i=1}^{d} \dim[HP^i(x,y)]$$

$$= 1 + \sum_{i=1}^{d} (i+1)$$

$$= \frac{(d+2)\cdot(d+1)}{2}$$

Dimensions

Homogenous Polynomials in n Variables:

One Variable: $\dim[HP^d(x)] = 1$

Two Variables: $\dim[HP^d(x,y)] = 1 + d$

Three Variables: $\dim[HP^d(x,y,z)] = \frac{(d+2)\cdot(d+1)}{2}$

Outline

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Homogenous Polynomials

Representations of Functions on the Unit-Circle

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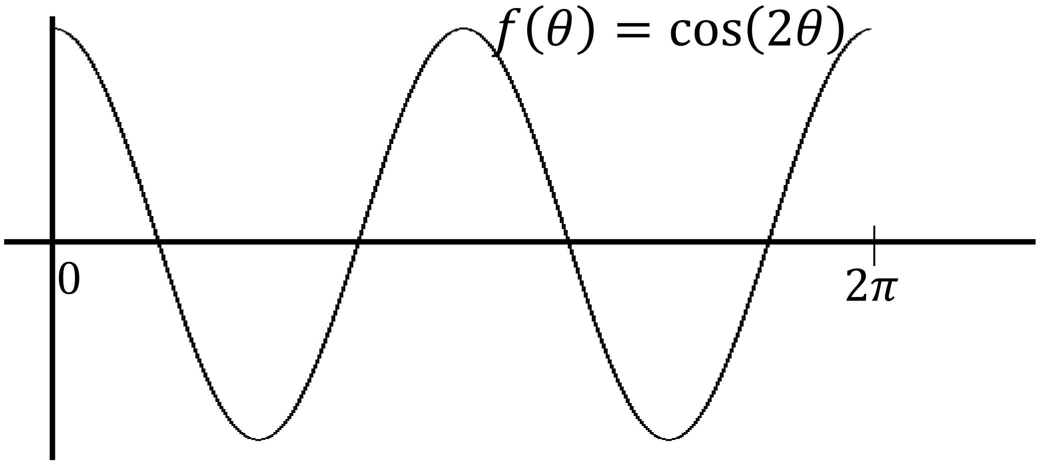
Representing Functions on the Unit-Circle

There are two ways we can represent a function on the unit-circle:

1. By Parameter: Every point on the circle can be represented by an angle in the range $[0,2\pi)$.

$\Rightarrow$ We can represent circular functions as 1D functions on the domain $[0,2\pi)$.



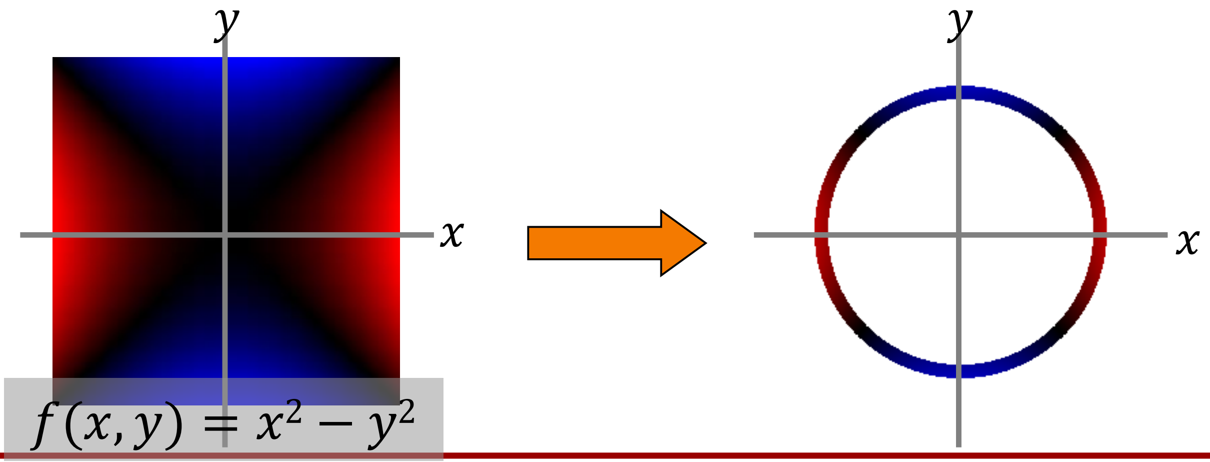
Representing Functions on the Unit-Circle

There are two ways we can represent a function on the unit-circle:

2. By Restriction: We know that the unit-circle "lives" in 2D, i.e. it is the set of points (x,y) satisfying:

$$x^2 + y^2 = 1$$

$\Rightarrow$ We can represent circular functions by looking at the restriction of 2D functions to the unit-circle.



Representing By Restriction

Observation 1:

On a circle, a point with angle θ has x- and y-coordinates given by:

$$x = \cos(\theta) \qquad y = \sin(\theta)$$

This lets us transform a (circular) function represented by the restriction of a 2D function $f(x, y)$ to a function represented by parameter:

$$f(x, y) \rightarrow g(\theta) \equiv f(\cos\theta, \sin\theta)$$

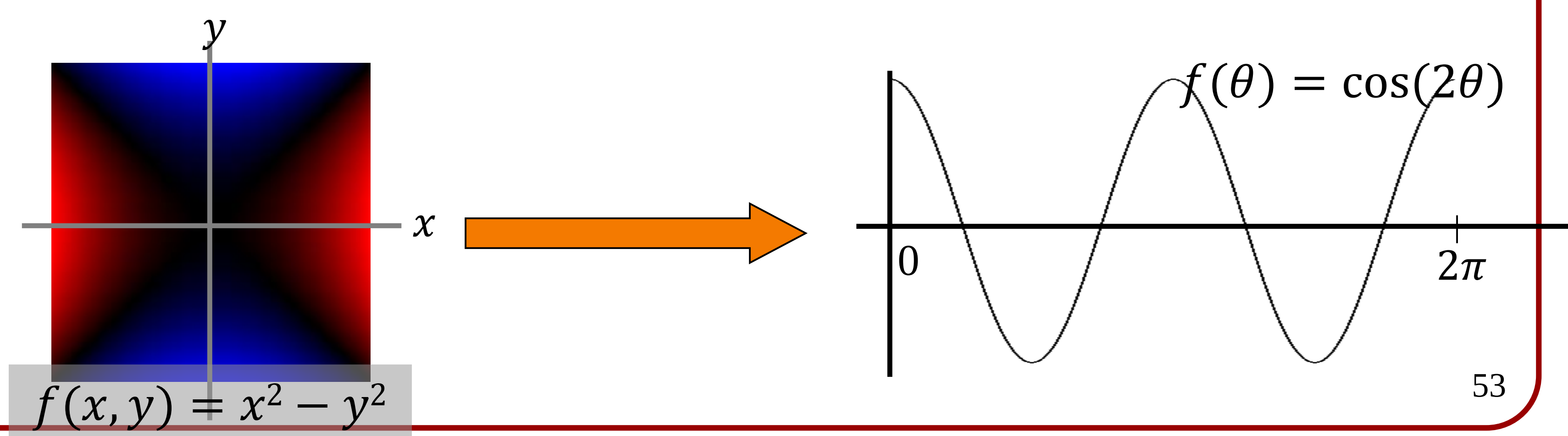
Representing By Restriction

Example: If the circular function is defined as the restriction of the 2D function:

$$f(x,y) = x^2 - y^2$$

Then the representation in terms of angle is:

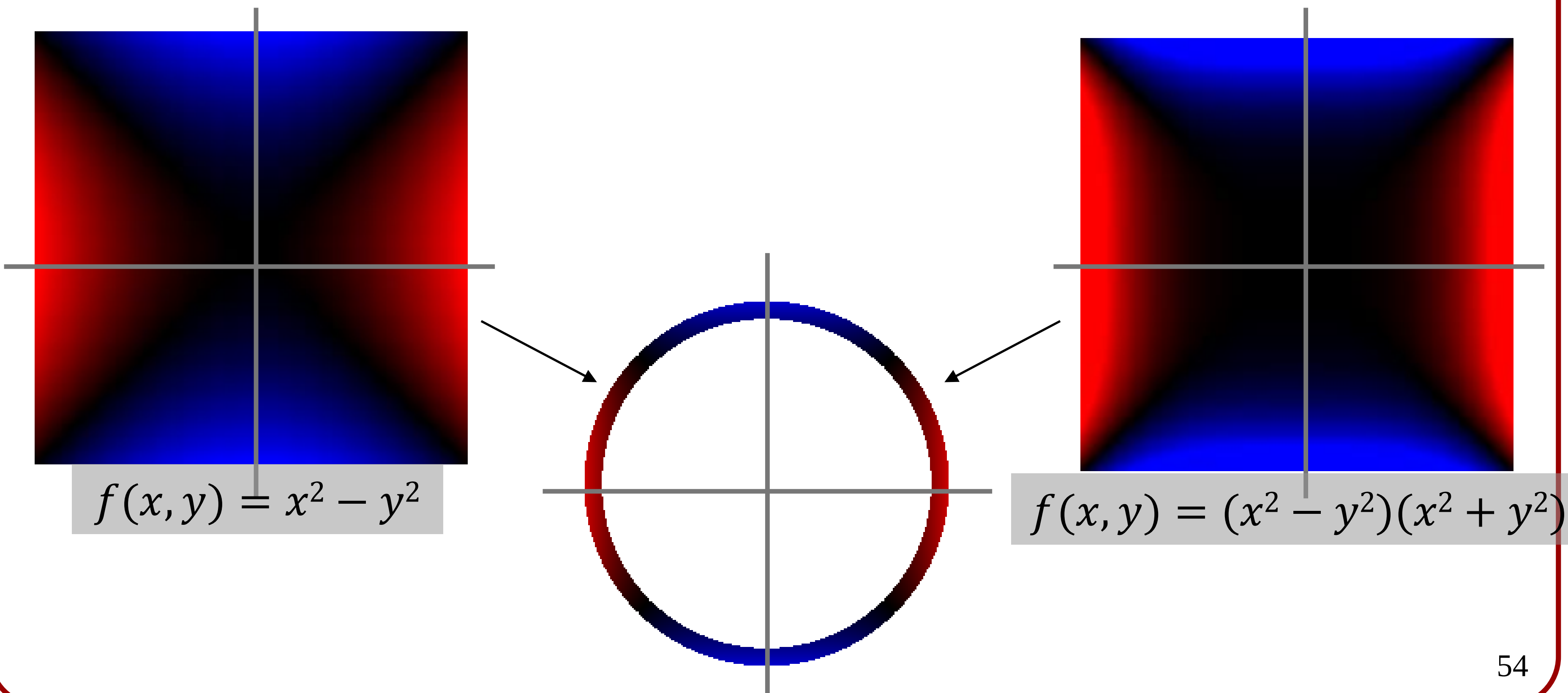
$$\begin{aligned} g(\theta) &= \cos^2\theta - \sin^2\theta \\ &= \cos 2\theta \end{aligned}$$



Representing By Restriction

<u>Observation 2</u>:

Two different functions in 2D, can have the same restriction to the unit-circle.



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Irreducible Representations

Recall:

In essential shape/image analysis tasks:

- Rotation invariant representation
- Image filtering
- Symmetry detection
- (2D) Rotational alignment

we needed to consider the representation of the group of 2D rotations on the space of circular functions.

Irreducible Representations

Recall:

To perform these tasks efficiently and/or effectively, we depended on Schur's Lemma:

> Since the group was commutative, the irreducible representations were all one (complex) dimensional

Irreducible Representations

Challenge:

We know that the irreducible representations exist. How do we find them?

Sub-Representations

How do we find a sub-space of functions that is also a sub-representation?

That is, how do we find a space of functions with the property that a rotation of a function from this space, will give some other function in the space.

Fourier Basis

For the circles, we know that these spaces are tone-dimensional, spanned by:

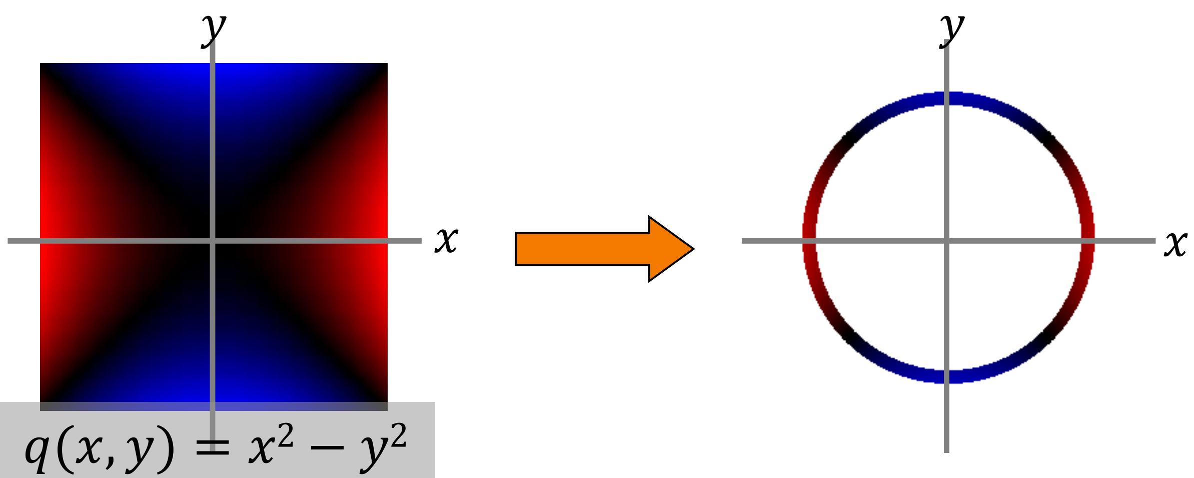
$$f_k(\theta) = e^{ik\theta}$$

But how would we go about finding them if we didn't know?

Polynomials

Consider the circular functions that are obtained by restricting degree d polynomials to the circle:

$$q(x,y) = \sum_{j+k \le d} a_{jk} \cdot x^j \cdot y^k$$



Polynomials

Consider the circular functions that are obtained by restricting degree d polynomials to the circle:

$$q(x,y) = \sum_{j+k\leq d} a_{jk} \cdot x^j \cdot y^k$$

How does a rotation act on this function?

$$R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Polynomials

Rotations act on the space of functions by rotating the domain of evaluation:

$$\big(\rho_R(q)\big)(x,y) = q\big(R^{-1}(x,y)\big)$$

Since the inverse of a rotation is its transpose, the rotation R^{-1}, acts on the 2D space by:

$$R^{-1}(x,y) = (ax + cy, bx + dy)$$

Polynomials

This means that the rotation acts on the polynomial by sending:

$$q(x,y) = \sum_{j+k\leq d} a_{jk} \cdot x^j \cdot y^k$$

$$\Updownarrow$$

$$\big(\rho_R(q)\big)(x,y) = \sum_{j+k\leq d} a_{jk} \cdot (ax+cy)^j \cdot (bx+dy)^k$$

Polynomials

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$$q(x,y) = \sum_{j+k\leq d} a_{jk} \cdot x^j \cdot y^k$$

$$\Updownarrow$$

$$\big(\rho_R(q)\big)(x,y) = \sum_{j+k\leq d} a_{jk} \cdot \underbrace{(ax+cy)}_{\text{Degree 1}}{}^j \cdot \underbrace{(bx+dy)}_{\text{Degree 1}}{}^k$$

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⇒ Since $j + k \leq d$, the rotation of $q(x,y)$ by R must also be a polynomial of degree d.

Polynomials

If we start with a polynomial of degree d:

$$q(x, y) \in P^d(x, y)$$

and we apply any rotation R to it, the rotated polynomial will also be a polynomial of degree d:

$$\rho_R(q) \in P^d(x, y)$$

Polynomials

Thus, the space of functions obtained by restricting polynomials of degree d to the unit circle is a sub-representation.

Polynomials

We can repeat the argument for restrictions of homogenous polynomials:

$$q(x,y) = \sum_{j+k=d} a_{jk} \cdot x^j \cdot y^k$$

$$\Updownarrow$$

$$\big(\rho_R(q)\big)(x,y) = \sum_{j+k=d} a_{jk} \cdot (ax+cy)^j \cdot (bx+dy)^k$$

Polynomials

We can repeat the argument for restrictions of homogenous polynomials:

$$q(x,y)=\sum_{j+k=d} a_{jk}\cdot x^j\cdot y^k$$

$$\Updownarrow$$

$$\big(\rho_R(q)\big)(x,y)=\sum_{j+k=d} a_{jk}\cdot \underbrace{(ax+cy)}_{\text{Degree 1}}{}^{j}\cdot \underbrace{(bx+dy)}_{\text{Degree 1}}{}^{k}$$

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$$\big(\rho_R(q)\big)(x,y) = \sum_{j+k=d} a_{jk} \cdot \underbrace{(ax+cy)^j \cdot (bx+dy)^k}_{\text{Degree } j+k}$$

Homogenous Polynomials

Thus, the space of functions obtained by restricting homogenous polynomials of degree d to the unit circle is a sub-representation.

Homogenous Polynomials

How small are these sub-representations?

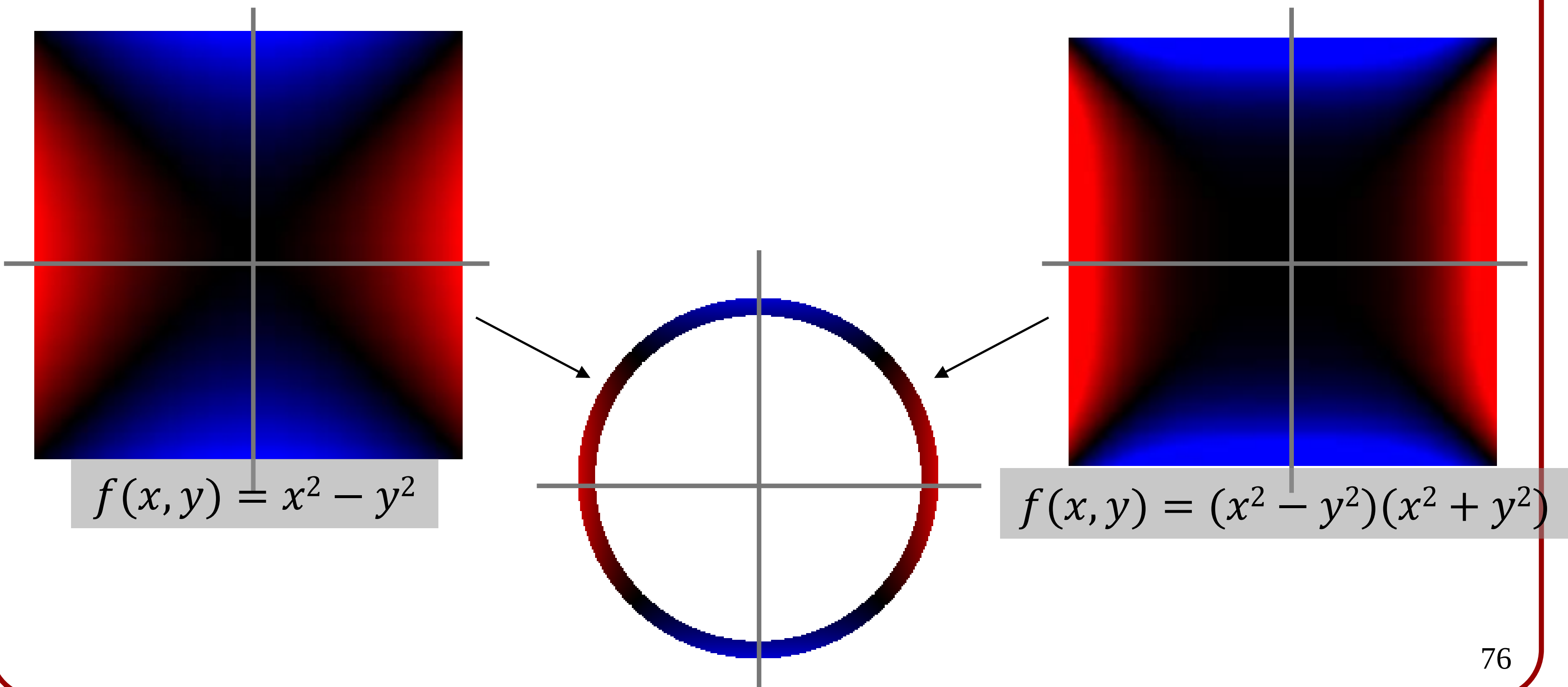
The space of homogenous polynomials of degree d in two variables is $(d+1)$-dimensional.

We know that the irreducible representations all have to be one-dimensional – what's going on?

Homogenous Polynomials

Recall:

Two different functions in 2D, can have the same restriction to the unit-circle.



Homogenous Polynomials

In particular, any point (x, y) on the circle satisfies the condition:

$$x^2 + y^2 + 1$$

So for any $q(x, y) \in HP^d(x, y)$, the homogenous polynomial $q(x, y) \cdot (x^2 + y^2) \in HP^{d+2}(x, y)$ will have the same restriction to the unit circle.

Homogenous Polynomials

When considering the restriction of homogenous polynomials to the circle, degree d polynomials are "contained" in the restriction of the degree $(d+2)$ polynomials.

Since the restrictions of degree d polynomials to the circle form a sub-representation, we want the polynomials of degree $(d+2)$ whose restrictions are orthogonal to those of degree d polynomials.

Homogenous Polynomials

Example:

- $d = 0$:

  $HP^d(x, y)$ is spanned by $\{1\}$ so the restriction is the space of constant functions.

Homogenous Polynomials

Example:

- $d = 1$:

  $HP^d(x, y)$ is spanned by $\{x, y\}$ so the restriction is the space of functions $ax + by$.

  Since we can write out the x and y coordinates in terms of the circular angle θ:

$$x = \cos\theta \qquad y = \sin\theta$$

  this gives the space of circular functions of the form:

$$f(\theta) = a \cdot \cos\theta + b \cdot \sin\theta$$

Homogenous Polynomials

<u>Example</u>:

- $d = 2$:

  $HP^d(x,y)$ is spanned by $\{x^2, xy, y^2\}$ so the restriction is the space of functions of the form $ax^2 + bxy + cy^2$.

  In terms of the circular angle, this gives the space of circular functions of the form:

  $f(\theta) = a \cdot \cos^2\theta + b \cdot \cos\theta \cdot \sin\theta + c \cdot \sin^2\theta$

Homogenous Polynomials

Example:

- $d = 2$:

$$f(\theta) = a \cdot \cos^2\theta + b \cdot \cos\theta \cdot \sin\theta + c \cdot \sin^2\theta$$

Since we know that:

$$\cos^2\theta + \sin^2\theta = 1$$

is a constant function accounted for by the $d = 0$ case, we want the space of homogenous polynomial restrictions that are perpendicular to those accounted for by the $d = 0$ case.

Homogenous Polynomials

<u>Example</u>:

- $d = 2$:

    A function of the form:

    $$f(\theta) = a \cdot \cos^2\theta + b \cdot \cos\theta \cdot \sin\theta + c \cdot \sin^2\theta$$

    is perpendicular to the function:

    $$\cos^2\theta + \sin^2\theta = 1$$

    if and only if:

    $$0 = \langle 1, a \cdot \cos^2\theta + b \cdot \cos\theta \cdot +c \cdot \sin^2\theta \rangle$$

Homogenous Polynomials

Example:

- $d = 2$:

$$0 = \langle 1, a \cdot \cos^2\theta + b \cdot \cos\theta \cdot \sin\theta + c \cdot \sin^2\theta \rangle$$

$$\Updownarrow$$

$$0 = \int_0^{2\pi} (a \cdot \cos^2\theta + b \cdot \cos\theta \cdot \sin\theta + c \cdot \sin^2\theta) d\theta$$

$$\Updownarrow$$

$$0 = a \cdot \pi + c \cdot \pi$$

$$\Updownarrow$$

$$a = -c$$

Homogenous Polynomials

Example:

- $d = 2$:

  Homogenous polynomials of can be expressed as:

  $$f(\theta) = a \cdot \cos^2\theta + b \cdot \cos\theta \cdot \sin\theta + c \cdot \sin^2\theta$$

  and orthogonality implies that:

  $$c = -a$$

  $\Rightarrow$ A basis for the sub-representation is:

  $$\{\cos^2\theta - \sin^2\theta, \cos\theta \cdot \sin\theta\}$$

  $$\Updownarrow$$

  $$\{\cos 2\theta, \sin 2\theta\}$$

Homogenous Polynomials

Example:

- $d \geq 2$:

  As in the $d = 2$ case, we start with the space of homogenous polynomials of degree d.

  Since the space of homogenous polynomials of degree $d - 2$ is contained in this space, we need to "throw out" the degree $d - 2$ polynomials.

  Thus, the final dimension of the sub-representation is:

$$\dim[HP^d(x,y)] - \dim[HP^{d-2}(x,y) = (d+1) - (d-1) \\ = 2$$

Homogenous Polynomials

<u>Example</u>:

- $d \geq 2$:

  As in the $d = 2$ case, that the two functions:

  $$\{\cos d\theta, \sin d\theta\}$$

  are a basis for the sub-representation.

Homogenous Polynomials

Note:

These sub-representations are not irreducible.

By Schur's lemma, the irreducible representations are all one-dimensional and for $d > 0$, we are getting two-dimensional sub-representations.

Homogenous Polynomials

Note:

These sub-representations are not irreducible.

By Schur's lemma, the irreducible representations are all one-dimensional and for $d > 0$, we are getting two-dimensional sub-representations.

To get the irreducible representations, we need to further break apart these sub-representations.

$$\{\cos d\theta\,, \sin d\theta\} = \begin{Bmatrix} \cos d\theta + i\sin d\theta \\ \cos d\theta + i\sin d\theta \end{Bmatrix} = \begin{Bmatrix} e^{i\theta} \\ e^{-i\theta} \end{Bmatrix}$$

Outline

The 2π Term in Assignment 1

Homogenous Polynomials

Representations of Functions on the Unit-Circle

- Sub-Representations for the Circle
- Sub-Representations for the Sphere

Spherical Functions

As in the case of circular functions, we would like to find the sub-representations of the spherical functions – sub-spaces of spherical functions which get rotated back into themselves.

Spherical Functions

As in the case of circular functions, we would like to find the sub-representations of the spherical functions – sub-spaces of spherical functions which get rotated back into themselves.

In this case, the group does not commute, so we do not expect the sub-representations to be one-dimensional.

Homogenous Polynomials

As in the case of circular functions, we will consider spherical functions that are obtained by restricting homogenous polynomials of degree d to the unit sphere:

$$q(x,y,z) = \sum_{j+k+l=d} a_{jkl} \cdot x^j \cdot y^k \cdot z^l$$

Homogenous Polynomials

If R is a rotation:

$$R = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

then R will rotate the polynomial q by:

$$\big(\rho_R(q)\big)(x, y, z) =$$

$$= \sum_{j+k+l=d} a_{jkl} \cdot (ax + dy + gz)^j \cdot (bx + ey + hz)^k \cdot (cx + fy + iz)^l$$

Again, rotations fix homogenous polynomials – mapping homogenous polynomials of degree d back into homogenous polynomials of degree d.

Homogenous Polynomials

As in the 2D case, we know that the restrictions of homogenous polynomials of degree d to the unit sphere contain the restrictions of homogenous polynomials of degree $d-2$ to the unit sphere.

So for any $q(x,y,z) \in HP^d(x,y,z)$, the polynomial $q(x,y,z) \cdot (x^2+y^2+z^2) \in HP^{d+2}(x,y,z)$ will have the same restriction to the unit sphere.

Homogenous Polynomials

Thus, sub-representations can be obtained by considering the restrictions of homogenous polynomials of degree d to the unit sphere, and removing those that were already accounted for at degree $d-2$.

Thus, the dimension of the space obtained from the degree d homogenous polynomials will be:

$$\dim[HP^d(x,y,z)] - \dim[HP^{d-2}(x,y,z)] =$$
$$= \frac{(d+2)\cdot(d+1)}{2} - \frac{d\cdot(d-1)}{2}$$
$$= 2d+1$$

Homogenous Polynomials

Thus, sub-representations can be obtained by considering the restrictions of homogenous polynomials of degree d to the unit sphere, and removing those that were already accounted for at degree $d-2$.

Thus, the dimension of the space obtained from the degree d homogenous polynomials will be:

$$\dim[HP^d(x,y,z)] - \dim[HP^{d-2}(x,y,z)] = 2d+1$$

It turns out that for spherical functions, these are the irreducible representations for the group of rotations.