



# FFTs in Graphics and Vision

Differential Equations



# Outline

A Simple PDE

Solving the PDE

Relationship to the Fourier Transform

Generalizations

Examples



# Evolving Systems

In many physical systems, the way that the system changes over time only depends on its current state.

## Examples:

- Population growth
- Radioactive decay
- Vibrations of a plucked string
- Heat dissipation
- Advection of particles in a vector field



# Evolving Systems

In many physical systems, the way that the system changes over time only depends on its current state.

What we would like to be able to answer is:

- Given the dependency of the change in the system to its current state, and
- Given the initial state of the system,

How will the system evolve over time?



# Evolving Systems

## A Simple Case:

Consider a 1D system represented by the function  $f(x, t)$ , where  $x$  represents the point in space and  $t$  the point in time.



# Evolving Systems

## A Simple Case:

If the change in the system can be described by:

$$\begin{aligned}\frac{\partial f(x, t)}{\partial t} &= a_0 \cdot f(x, t) + \cdots + a_n \cdot \frac{\partial^n f(x, t)}{\partial x^n} \\ &= \sum_{p=0}^n a_p \cdot \frac{\partial^p f(x, t)}{\partial x^p}\end{aligned}$$

and the initial state is defined by:

$$f(x, 0) = g(x)$$

How do we compute the state at time  $t$ :

$$f(x, t) = ?$$



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A Simple PDE

**Solving the PDE**

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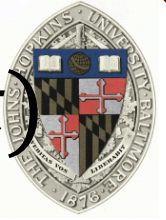
# Solving the PDE

## General Approach:

1. Solve the 1D equation  $h'(t) = \lambda \cdot h(t)$ .
2. Find a set of solutions to the 2D PDE.
3. Find the linear combination of solutions that satisfies the initial condition.



# Solve the 1D equation $h'(t) = \lambda \cdot h(t)$

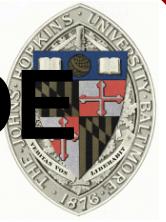


This we know how to do:

$$h'(t) = \lambda \cdot h(t)$$



$$h(t) = C \cdot e^{\lambda t}$$



# Find a Set of Solutions to the 2D PDE

Approach:

To solve for the function  $f(x, t)$  that satisfies:

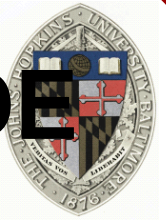
$$\frac{\partial f(x, t)}{\partial t} = \sum_{p=0}^n a_p \cdot \frac{\partial^p f(x, t)}{\partial x^p}$$

we will try to express  $f(x, t)$  as the product:

$$f(x, t) = g_\lambda(x) \cdot h_\lambda(t)$$

That is, we will try to solve for  $g_\lambda(x)$  and  $h_\lambda(t)$  s.t.:

$$\frac{\partial (g_\lambda(x) \cdot h_\lambda(t))}{\partial t} = \sum_{p=0}^n a_p \cdot \frac{\partial^p (g_\lambda(x) \cdot h_\lambda(t))}{\partial x^p}$$



# Find a Set of Solutions to the 2D PDE

Observation 1:

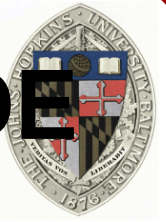
The map:

$$f(x) \rightarrow \sum_{p=0}^n a_p \cdot \frac{\partial^p f(x)}{\partial x^p}$$

is linear.

Taking the  $p$ -th derivative is a linear operation:

$$\frac{\partial^p (\alpha \cdot f(x) + \beta \cdot g(x))}{\partial x^p} = \alpha \cdot \frac{\partial^p f(x)}{\partial x^p} + \beta \cdot \frac{\partial^p g(x)}{\partial x^p}$$



# Find a Set of Solutions to the 2D PDE

Observation 1:

The map:

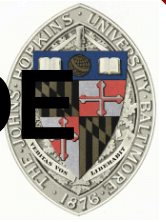
$$f(x) \rightarrow \sum_{p=0}^n a_p \cdot \frac{\partial^p f(x)}{\partial x^p}$$

is linear.

We will write out this linear operator as:

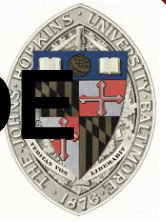
$$D = \sum_{p=0}^n a_k \cdot \frac{\partial^p}{\partial x^p}$$

# Find a Set of Solutions to the 2D PDE



## Observation 2:

If we can find an eigenvalue/eigenvector of  $D$ , we can find a solution to the PDE.



# Find a Set of Solutions to the 2D PDE

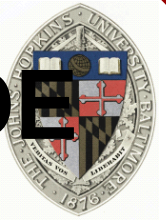
## Observation 2:

Suppose that  $g_\lambda(x)$  is an eigenvector of  $D$  with eigenvalue  $\lambda$ :

$$D(g_\lambda(x)) = \lambda \cdot g_\lambda(x)$$

We want to find  $h_\lambda(t)$  such that:

$$\frac{\partial(g_\lambda(x) \cdot h_\lambda(t))}{\partial t} = \sum_{p=0}^n a_p \cdot \frac{\partial^p(g_\lambda(x) \cdot h_\lambda(t))}{\partial x^p}$$



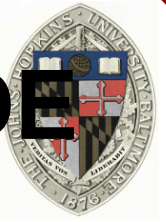
# Find a Set of Solutions to the 2D PDE

Observation 2:

$$\frac{\partial(g_\lambda(x) \cdot h_\lambda(t))}{\partial t} = \sum_{p=0}^n a_p \cdot \frac{\partial^p(g_\lambda(x) \cdot h_\lambda(t))}{\partial x^p}$$

Using the fact that  $g_\lambda(x)$  does not depend on  $t$  we can re-write the left-hand side as:

$$\text{LHS} = g_\lambda(x) \cdot \frac{\partial h_\lambda(t)}{\partial t}$$



# Find a Set of Solutions to the 2D PDE

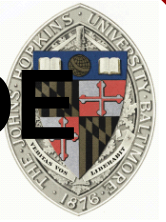
Observation 2:

$$\frac{\partial(g_\lambda(x) \cdot h_\lambda(t))}{\partial t} = \sum_{p=0}^n a_p \cdot \frac{\partial^p(g_\lambda(x) \cdot h_\lambda(t))}{\partial x^p}$$

Using the fact that  $h_\lambda(t)$  does not depend on  $x$  we can re-write the right-hand side as:

$$\begin{aligned} RHS &= h_\lambda(t) \cdot \sum_{p=0}^n a_p \cdot \frac{\partial^p g_\lambda(x)}{\partial x^p} \\ &= h_\lambda(t) \cdot D(g_\lambda(x)) \\ &= \lambda \cdot h_\lambda(t) \cdot g_\lambda(x) \end{aligned}$$





# Find a Set of Solutions to the 2D PDE

## Observation 2:

So now we are left with the problem of solving for the function  $h_\lambda(t)$  such that:

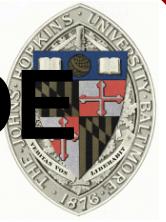
$$g_\lambda(x) \cdot \frac{\partial h_\lambda(t)}{\partial t} = \lambda \cdot h_\lambda(t) \cdot g_\lambda(x)$$

$\Downarrow$

$$\frac{\partial h_\lambda(t)}{\partial t} = \lambda \cdot h_\lambda(t)$$

$\Downarrow$

$$h_\lambda(t) = C \cdot e^{\lambda t}$$



# Find a Set of Solutions to the 2D PDE

## Observation 2:

⇒ If  $g_\lambda(x)$  is an eigenvector of the linear operator  $D$  with eigenvalue  $\lambda$ , then:

$$f_\lambda(x, t) = e^{\lambda t} \cdot g_\lambda(x)$$

must be a solution to the differential equation:

$$\frac{\partial f(x, t)}{\partial t} = \sum_{p=0}^n a_p \cdot \frac{\partial^p f(x, t)}{\partial x^p}$$



# Satisfying the Initial Condition

## Observation 3:

If  $f_1(x, t)$  and  $f_2(x, t)$  are solutions to the (partial) differential equation:

$$\frac{\partial f(x, t)}{\partial t} = \sum_{p=0}^n a_p \cdot \frac{\partial^p f(x, t)}{\partial x^p}$$

then any linear combination of the two must also be a solution.



# Satisfying the Initial Condition

- For any eigenvector  $g_\lambda(x)$  with eigenvalue  $\lambda$ :  
$$f_\lambda(x, t) = e^{\lambda t} \cdot g_\lambda(x)$$
is a solution.
- Any linear combination of solutions is a solution.

⇒ Any function  $f(x, t)$  expressable as:

$$f(x, t) = \sum_{\lambda} c_{\lambda} \cdot e^{\lambda t} \cdot g_{\lambda}(x)$$

must be a solution to the PDE.



# Satisfying the Initial Condition

To satisfy the initial value conditions, we need to choose  $c_\lambda$  so that the function:

$$f(x, t) = \sum_{\lambda} c_{\lambda} \cdot e^{\lambda t} \cdot g_{\lambda}(x)$$

satisfies the initial value conditions:

$$f(x, 0) = g(x)$$

But this implies that:

$$g(x) = \sum_{\lambda} c_{\lambda} \cdot g_{\lambda}(x)$$



# Satisfying the Initial Condition

To satisfy the initial value conditions, we need to choose  $c_\lambda$  so that the function:

$$f(x, t) = \sum_{\lambda} c_{\lambda} \cdot e^{\lambda t} \cdot g_{\lambda}(x)$$

satisfies the initial conditions.

Satisfying the initial value conditions is equivalent to finding the coefficients of  $g(x)$  with respect to the functions  $\{g_{\lambda}(x)\}$

But

$$g(x) = \sum_{\lambda} c_{\lambda} \cdot g_{\lambda}(x)$$



# Outline

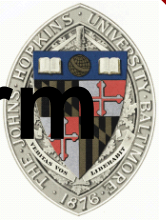
A Simple PDE

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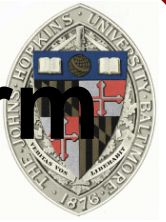


# Relationship to the Fourier Transform

Recall that the Fourier decomposition expresses a circular function  $f(\theta)$  as a sum of complex exponentials of different frequencies:

$$f(\theta) = \sum_{k=-\infty}^{\infty} \hat{f}(k) \cdot \frac{e^{ik\theta}}{\sqrt{2\pi}}$$

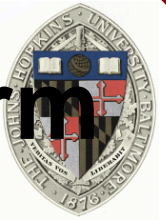




# Relationship to the Fourier Transform

The complex exponentials are the eigenvectors of the derivative operator:

$$\begin{aligned}\frac{\partial}{\partial \theta} e^{ik\theta} &= ik \cdot e^{ik\theta} \\ \frac{\partial^2}{\partial \theta^2} e^{ik\theta} &= -k^2 \cdot e^{ik\theta} \\ &\vdots \\ \frac{\partial^n}{\partial \theta^n} e^{ik\theta} &= (ik)^n \cdot e^{ik\theta}\end{aligned}$$



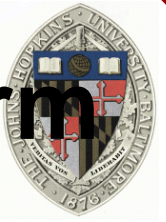
# Relationship to the Fourier Transform

So, if we are given the linear map:

$$D = \sum_{p=0}^n a_p \cdot \frac{\partial^p}{\partial \theta^p}$$

it will act on  $e^{ik\theta}$  as:

$$D(e^{ik\theta}) = \sum_{p=0}^n a_p \cdot (ik)^p \cdot e^{ik\theta}$$



# Relationship to the Fourier Transform

So, if we are given the linear map:

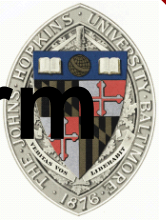
$$D = \sum_{l=0}^n a_l \cdot \frac{\partial^l}{\partial \theta^l}$$

it acts on  $e^{ik\theta}$  as:

$$D(e^{ik\theta}) = \left( \sum_{p=0}^n a_p \cdot (ik)^p \right) \cdot e^{ik\theta}$$

$\Rightarrow e^{ik\theta}$  is an eigenvector with eigenvalue:

$$\lambda_k = \sum_{p=0}^n a_p \cdot (ik)^p$$



# Relationship to the Fourier Transform

In particular, this implies that:

$$f_k(\theta, t) = e^{\lambda_k t} \cdot e^{ik\theta}$$

are solutions to the partial differential equation:

$$\frac{\partial f(\theta, t)}{\partial t} = \sum_{p=0}^n a_p \cdot \frac{\partial^p f(\theta, t)}{\partial \theta^p}$$

⇒ The solutions to the PDE will be of the form:

$$f(\theta, t) = \sum_{k=-\infty}^{\infty} c_k \cdot e^{\lambda_k t} \cdot \frac{e^{ik\theta}}{\sqrt{2\pi}}$$



# Relationship to the Fourier Transform

$$f(\theta, t) = \sum_{k=-\infty}^{\infty} c_k \cdot e^{\lambda_k t} \cdot \frac{e^{ik\theta}}{\sqrt{2\pi}}$$

To satisfy the initial condition:

$$f(\theta, 0) = g(\theta)$$

we need to solve for the values of  $c_k$  such that:

$$f(\theta, 0) = \sum_{k=-\infty}^{\infty} c_k \cdot \frac{e^{ik\theta}}{\sqrt{2\pi}}$$

$\Rightarrow c_k$  must be the  $k$ -th Fourier coefficients of  $g(\theta)$ :

$$c_k = \hat{g}(k)$$



# Relationship to the Fourier Transform

The solution to the PDE is the function  $f(\theta, t)$  whose  $k$ -th Fourier coefficient at time  $t$  is the modulation of the  $k$ -th Fourier coefficient of  $g(\theta)$  by a function of  $t$ :

$$\hat{f}_t(k) = \hat{g}(k) \cdot e^{\lambda_k t}$$



# Relationship to the Fourier Transform

To implement this, we start off by:

- Computing the Fourier coefficients of  $g(\theta)$

Then, at each time  $t$ , we:

- Compute the modulated Fourier coefficients:

$$\hat{f}_t(k) = \hat{g}(k) \cdot e^{\lambda_k t}$$

- And compute the inverse Fourier transform.



# Outline

A Simple PDE

Solving the PDE

Relationship to the Fourier Transform

Generalizations

- Higher dimensions
- Second order time derivatives

Examples





# 2D Systems

In the case that the system is 2D, we want to consider functions of the form  $f(\theta, \phi, t)$ .

The linear partial differential equation becomes:

$$\frac{\partial f(\theta, \phi, t)}{\partial t} = \sum_{p=0}^m \sum_{q=0}^n a_{pq} \cdot \frac{\partial^p \partial^q f(\theta, \phi, t)}{\partial \theta^p \partial \phi^q}$$

The initial state becomes:

$$f(\theta, \phi, 0) = g(\theta, \phi)$$

And the challenge is to compute the state of the system at an arbitrary point in time:

$$f(\theta, \phi, t) = ?$$



# 2D Systems

As in the 1D case, we can use the fact that the a periodic 2D function  $f(\theta, \phi)$  can be expressed in terms of its Fourier decomposition:

$$f(\theta, \phi) = \sum_{k, l=-\infty}^{\infty} \hat{f}(k, l) \cdot \frac{e^{ik\theta}}{\sqrt{2\pi}} \cdot \frac{e^{il\phi}}{\sqrt{2\pi}}$$



# 2D Systems

And again, we use the fact that the complex exponentials are the eigenvectors of the partial derivative operator:

$$\frac{\partial}{\partial \theta} e^{ik\theta} \cdot e^{il\phi} = (ik) \cdot e^{ik\theta} \cdot e^{il\phi}$$

$$\frac{\partial}{\partial \phi} e^{ik\theta} \cdot e^{il\phi} = (il) \cdot e^{ik\theta} \cdot e^{il\phi}$$

$\vdots$

$$\frac{\partial^m \partial^n}{\partial \theta^m \partial \phi^n} e^{ik\theta} \cdot e^{il\phi} = (ik)^m \cdot (il)^n \cdot e^{ik\theta} \cdot e^{il\phi}$$



# 2D Systems

Thus, given the linear map:

$$D = \sum_{p=0}^m \sum_{q=0}^n a_{pq} \cdot \frac{\partial^p \partial^q}{\partial \theta^p \partial \phi^q}$$

Eigenvectors of this map are:

$$f_{kl}(\theta, \phi) = e^{ik\theta} \cdot e^{il\phi}$$

And the associated eigenvalues are:

$$\lambda_{kl} = \sum_{p=0}^m \sum_{q=0}^n a_{pq} \cdot (ik)^p \cdot (il)^q$$



## 2D Systems

Given these eigenvectors, we can proceed as before, obtaining a solution to the differential equation for each eigenvector:

$$f_{kl}(\theta, \phi, t) = e^{\lambda_{kl}t} \cdot e^{ik\theta} \cdot e^{il\phi}$$

And we can satisfy the initial condition:

$$f(\theta, \phi, 0) = g(\theta, \phi)$$

by setting the Fourier coefficients of  $f(\theta, \phi, t)$  at time  $t$  equal to:

$$\hat{f}_t(k, l) = \hat{g}(k, l) \cdot e^{\lambda_{kl}t}$$



# Outline

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- Higher dimensions
- Second order time derivatives



# Second Order Time Derivatives

What if the change in the system is characterized by the second derivative with respect to time:

$$\frac{\partial^2 f(\theta, t)}{\partial t^2} = \sum_{p=0}^n a_p \cdot \frac{\partial^p f(\theta, t)}{\partial \theta^p}$$



# Second Order Time Derivatives

Recall that the solution to the original PDE was derived by solving the (first-order time) derivative:

$$h'(t) = \lambda \cdot h(t)$$

In this case, we need to solve the second-order time derivative:

$$h''(t) = \lambda \cdot h(t)$$

$\Downarrow$

$$h(t) = e^{\sqrt{\lambda}t} \quad \text{and} \quad h(t) = e^{-\sqrt{\lambda}t}$$





# Second Order Time Derivatives

As before, if  $g_\lambda(\theta)$  is an eigenvector of the linear operator  $D$  with eigenvalue  $\lambda$ , then the functions:

$$f_\lambda^+(\theta, t) = e^{\sqrt{\lambda}t} \cdot g_\lambda(\theta)$$

$$f_\lambda^-(\theta, t) = e^{-\sqrt{\lambda}t} \cdot g_\lambda(\theta)$$

will both be solutions to the PDE.



# Second Order Time Derivatives

Note that in this case, one eigenvector of  $D$  gives us two solutions to the differential equation.



We have more functions with which we can satisfy the initial boundary conditions.



We can specify more stringent boundary conditions.



# Second Order Time Derivatives

In practice, this amounts to specifying two boundary conditions:

- Initial value conditions:

$$f(\theta, 0) = g(\theta)$$

- Initial derivative conditions:

$$\frac{\partial}{\partial t} f(\theta, 0) = h(\theta)$$



# Second Order Time Derivatives

## Intuitively:

In the first-order case we are given the velocity at every point. If we know the initial positions, this is enough to know where things end up.

In the second-order case we are given the acceleration at every point. In order to know where things end up, we need to know both the initial position and the initial velocity.



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## Examples

- The Laplacian
- The 2D heat equation
- The 2D wave equation



# The Laplacian

Given a function  $f$  in 1D, how do we interpret its second derivative:

$$\Delta f = \frac{\partial f}{\partial x^2}$$



# The Laplacian

The first derivative of  $f(x)$  is approximated by looking at the difference between the value of  $f$  at  $x$  and the value of  $f$  at a neighboring point:

$$f'(x) \approx f(x+1) - f(x)$$

The Laplacian of  $f(x)$  is approximated by applying the process to the derivative of  $f(x)$ :

$$\begin{aligned} f''(x) &\approx f'(x) - f'(x-1) \\ &\approx (f(x+1) - f(x)) - (f(x) - f(x-1)) \\ &\approx f(x+1) + f(x-1) - 2f(x) \end{aligned}$$



# The Laplacian

The first derivative of  $f(x)$  is approximated by looking at the difference between the value of  $f$  at  $x$  and the value of  $f$  at a neighboring point:

$$f'(x) \approx f(x+1) - f(x)$$

The Laplacian of  $f(x)$  is approximated by applying the process to the derivative of  $f(x)$ :

$$f''(x) \approx 2 \left( \frac{f(x+1) + f(x-1)}{2} - f(x) \right)$$

i.e. it is a measure of the difference between the value of  $f$  at  $x$  and the average value of  $f$  at the neighbors of  $x$ .





# The Laplacian

The same interpretation holds for the Laplacian of a 2D function  $f$ :

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

i.e. The Laplacian of a function is a measure of how the value of  $f$  at a point  $(x, y)$  differs from the average of the values of its neighbors.



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- The Laplacian
- **The 2D heat equation**
- The 2D wave equation



# Newton's Law of Cooling

*"The rate of heat loss of a body is proportional to the difference in temperatures between the body and its surroundings."*



# Newton's Law of Cooling

*“The rate of heat loss of a body is proportional to the difference in temperatures between the body and its surroundings.”*

Translating this into the PDE setting, if  $f(\theta, \phi, t)$  is the heat at position  $(x, y)$  at time  $t$ , then:

$$\frac{\partial f}{\partial t} = \eta \cdot \Delta f$$



# Newton's Law of Cooling

$$\frac{\partial f}{\partial t} = \eta \cdot \Delta f$$

In this case, the linear operator  $D$  is defined by:

$$D = \eta \cdot \Delta$$

And the complex exponential:

$$e^{ik\theta} \cdot e^{il\phi}$$

is an eigenvector with eigenvalue:

$$\lambda_{kl} = -\eta \cdot (k^2 + l^2)$$



# Newton's Law of Cooling

$$\frac{\partial f}{\partial t} = \eta \cdot \Delta f$$

Thus, the solution to this equation:

$$f(\theta, \phi, 0) = g(\theta, \phi)$$

is the function whose  $(k, l)$ -th Fourier coefficient at time  $t$  is:

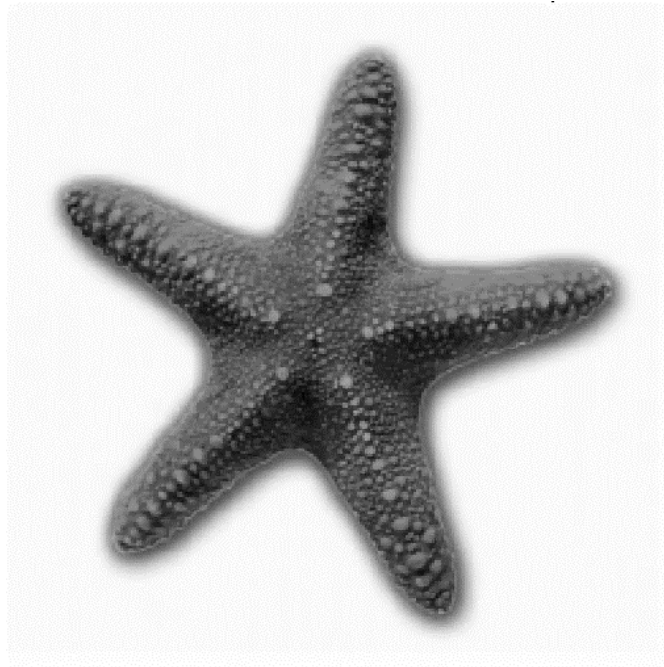
$$\hat{f}_t(k, l) = \hat{g}(k, l) \cdot e^{-\eta \cdot (k^2 + l^2) \cdot t}$$



# Newton's Law of Cooling

$$\hat{f}_t(k, l) = \hat{g}(k, l) \cdot e^{-\eta \cdot (k^2 + l^2) \cdot t}$$

This looks remarkably like what we get when implement Gaussian smoothing...





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- **The 2D wave equation**





# The 2D Wave Equation

Consider a square rubber sheet.

If we displace the points on the sheet, then at any point  $(\theta, \phi)$ , the neighbors of  $(\theta, \phi)$  will exert a force to pull the point towards them.

The force that exerted on  $(\theta, \phi)$  is proportional to the distance of  $(\theta, \phi)$  from its neighbors.



# The 2D Wave Equation

If the height of the point  $(\theta, \phi)$  is given by the function  $f(\theta, \phi)$  then the force at  $(\theta, \phi)$  is:

$$F(\theta, \phi) = \eta \cdot \Delta f(\theta, \phi) \quad \text{w/ } \eta > 0$$

Using the fact that Force = Mass · Acceleration, the PDE for the height at time  $t$  is:

$$\frac{\partial^2 f}{\partial t^2} = \eta \cdot \Delta f$$



# The 2D Wave Equation

$$\frac{\partial^2 f}{\partial t^2} = \eta \cdot \Delta f$$

We would like to solve this equation, subject to the constraints that at the initial time-step:

- The height at each point is given by:

$$f(\theta, \phi, 0) = g(\theta, \phi)$$

- The sheet is not moving:

$$\frac{\partial}{\partial t} f(\theta, \phi, 0) = 0$$



# The 2D Wave Equation

$$\frac{\partial^2 f}{\partial t^2} = \eta \cdot \Delta f$$

Again, the linear operator  $D$  is defined by:

$$D = \eta \cdot \Delta$$

And the complex exponential:

$$e^{ik\theta} \cdot e^{il\phi}$$

is an eigenvector with eigenvalue:

$$\lambda_{kl} = -\eta \cdot (k^2 + l^2)$$



# The 2D Wave Equation

$$\frac{\partial^2 f}{\partial t^2} = \eta \cdot \Delta f$$

Thus, the solutions to this equation are:

$$f_{kl}^+(\theta, \phi, t) = e^{i\sqrt{\eta \cdot (k^2 + l^2)}t} \cdot e^{ik\theta} \cdot e^{il\phi}$$

$$f_{kl}^-(\theta, \phi, t) = e^{-i\sqrt{\eta \cdot (k^2 + l^2)}t} \cdot e^{ik\theta} \cdot e^{il\phi}$$

And a general solution takes the form:

$$f(\theta, \phi, t) = \sum_{k,l} \left( A_{kl} \cdot e^{i\sqrt{\eta \cdot (k^2 + l^2)}t} + B_{kl} \cdot e^{-i\sqrt{\eta \cdot (k^2 + l^2)}t} \right) \cdot \frac{e^{ik\theta}}{\sqrt{2\pi}} \cdot \frac{e^{il\phi}}{\sqrt{2\pi}}$$



# The 2D Wave Equation

$$f(\theta, \phi, t) = \sum_{k,l} \left( A_{kl} \cdot e^{i\sqrt{\eta \cdot (k^2 + l^2)}t} + B_{kl} \cdot e^{-i\sqrt{\eta \cdot (k^2 + l^2)}t} \right) \cdot \frac{e^{ik\theta}}{\sqrt{2\pi}} \cdot \frac{e^{il\phi}}{\sqrt{2\pi}}$$

To satisfy the initial value condition:

$$f(\theta, \phi, 0) = g(\theta, \phi)$$

we need to have:

$$g(\theta, \phi) = \sum_{k,l} (A_{kl} + B_{kl}) \cdot \frac{e^{ik\theta}}{\sqrt{2\pi}} \cdot \frac{e^{il\phi}}{\sqrt{2\pi}}$$

$\Downarrow$

$$\hat{g}(k, l) = A_{kl} + B_{kl}$$



# The 2D Wave Equation

$$f(\theta, \phi, t) = \sum_{k,l} \left( A_{kl} \cdot e^{i\sqrt{\eta \cdot (k^2 + l^2)}t} + B_{kl} \cdot e^{-i\sqrt{\eta \cdot (k^2 + l^2)}t} \right) \cdot \frac{e^{ik\theta}}{\sqrt{2\pi}} \cdot \frac{e^{il\phi}}{\sqrt{2\pi}}$$

To satisfy the initial derivative condition:

$$\frac{\partial}{\partial t} f(\theta, \phi, 0) = 0$$

we need to have:

$$0 = \sum_{k,l} (A_{kl} - B_{kl}) \cdot \left( i\sqrt{\eta \cdot (k^2 + l^2)} \right) \cdot \frac{e^{ik\theta}}{\sqrt{2\pi}} \cdot \frac{e^{il\phi}}{\sqrt{2\pi}}$$

$\Downarrow$

$$A_{kl} = B_{kl}$$



# The 2D Wave Equation

$$f(\theta, \phi, t) = \sum_{k,l} \left( A_{kl} \cdot e^{i\sqrt{\eta \cdot (k^2 + l^2)}t} + B_{kl} \cdot e^{-i\sqrt{\eta \cdot (k^2 + l^2)}t} \right) \cdot \frac{e^{ik\theta}}{\sqrt{2\pi}} \cdot \frac{e^{il\phi}}{\sqrt{2\pi}}$$
$$\hat{g}(k, l) = A_{kl} + B_{kl} \quad \text{and} \quad A_{kl} = B_{kl}$$

Putting all of this together, the solution to the 2D wave equation, with initial position  $g(\theta, \phi)$  and zero initial derivative is the function  $f(\theta, \phi, t)$  whose Fourier coefficients at time  $t$  are equal to:

$$\begin{aligned} \hat{f}_t(k, l) &= \hat{g}(k, l) \cdot \frac{e^{i\sqrt{\eta \cdot (k^2 + l^2)}t} + e^{-i\sqrt{\eta \cdot (k^2 + l^2)}t}}{2} \\ &= \hat{g}(k, l) \cos \left( \sqrt{\eta \cdot (k^2 + l^2)}t \right) \end{aligned}$$





# The 2D Wave Equation

$$\hat{f}_t(k, l) = \hat{g}(k, l) \cos\left(\sqrt{\eta \cdot (k^2 + l^2)}t\right)$$

This looks nothing like what we get when implement Gaussian smoothing...

