FFTs in Graphics and Vision

Differential Equations

Outline

A Simple PDE

Solving the PDE

Relationship to the Fourier Transform

Generalizations

Examples

Evolving Systems

In many physical systems, the way that the system changes over time only depends on its current state.

Examples:

- Population growth
- Radioactive decay
- Vibrations of a plucked string
- Heat dissipation
- Advection of particles in a vector field

Evolving Systems

In many physical systems, the way that the system changes over time only depends on its current state.

What we would like to be able to answer is:

- Given the dependency of the change in the system to its current state, and
- Given the initial state of the system,

How will the system evolve over time?

Evolving Systems

A Simple Case:

Consider a 1D system represented by the function $f(x,t)$, where x represents the point in space and t the point in time.

Evolving Systems

A Simple Case:

If the change in the system can be described by:

$$\frac{\partial f(x,t)}{\partial t} = a_0 \cdot f(x,t) + \cdots + a_n \cdot \frac{\partial^n f(x,t)}{\partial x^n}$$

$$= \sum_{p=0}^{n} a_p \cdot \frac{\partial^p f(x,t)}{\partial x^p}$$

and the initial state is defined by:

$$f(x,0) = g(x)$$

How do we compute the state at time t:

$$f(x,t) = ?$$

Outline

A Simple PDE

Solving the PDE

Relationship to the Fourier Transform

Generalizations

Examples

Solving the PDE

General Approach:

1. Solve the 1D equation $h'(t) = \lambda \cdot h(t)$.
2. Find a set of solutions to the 2D PDE.
3. Find the linear combination of solutions that satisfies the initial condition.

Solve the 1D equation $\boldsymbol{h'(t) = \lambda \cdot h(t)}$

This we know how to do:

$$h'(t) = \lambda \cdot h(t)$$
$$\Updownarrow$$
$$h(t) = C \cdot e^{\lambda t}$$

Find a Set of Solutions to the 2D PDE

Approach:

To solve for the function $f(x,t)$ that satisfies:

$$\frac{\partial f(x,t)}{\partial t} = \sum_{p=0}^{n} a_p \cdot \frac{\partial^p f(x,t)}{\partial x^p}$$

we will try to express $f(x,t)$ as the product:

$$f(x,t) = g_\lambda(x) \cdot h_\lambda(t)$$

That is, we will try to solve for $g_\lambda(x)$ and $h_\lambda(t)$ s.t.:

$$\frac{\partial (g_\lambda(x) \cdot h_\lambda(t))}{\partial t} = \sum_{p=0}^{n} a_p \cdot \frac{\partial^p (g_\lambda(x) \cdot h_\lambda(t))}{\partial x^p}$$

Find a Set of Solutions to the 2D PDE

Observation 1:

The map:

$$f(x) \to \sum_{p=0}^{n} a_p \cdot \frac{\partial^p f(x)}{\partial x^p}$$

is linear.

Taking the p-th derivative is a linear operation:

$$\frac{\partial^p\big(\alpha \cdot f(x) + \beta \cdot g(x)\big)}{\partial x^p} = \alpha \cdot \frac{\partial^p f(x)}{\partial x^p} + \beta \cdot \frac{\partial^p g(x)}{\partial x^p}$$

Find a Set of Solutions to the 2D PDE

Observation 1:

The map:

$$f(x) \to \sum_{p=0}^{n} a_p \cdot \frac{\partial^p f(x)}{\partial x^p}$$

is linear.

We will write out this linear operator as:

$$D = \sum_{p=0}^{n} a_k \cdot \frac{\partial^p}{\partial x^p}$$

Find a Set of Solutions to the 2D PDE

Observation 2:

If we can find an eigenvalue/eigenvector of D, we can find a solution to the PDE.

Find a Set of Solutions to the 2D PDE

Observation 2:

Suppose that $g_\lambda(x)$ is an eigenvector of D with eigenvalue λ:

$$D\big(g_\lambda(x)\big) = \lambda \cdot g_\lambda(x)$$

We want to find $h_\lambda(t)$ such that:

$$\frac{\partial(g_\lambda(x) \cdot h_\lambda(t))}{\partial t} = \sum_{p=0}^{n} a_p \cdot \frac{\partial^p(g_\lambda(x) \cdot h_\lambda(t))}{\partial x^p}$$

Find a Set of Solutions to the 2D PDE

Observation 2:

$$\frac{\partial(g_\lambda(x)\cdot h_\lambda(t))}{\partial t}=\sum_{p=0}^{n} a_p\cdot\frac{\partial^p(g_\lambda(x)\cdot h_\lambda(t))}{\partial x^p}$$

Using the fact that $g_\lambda(x)$ does not depend on t we can re-write the left-hand side as:

$$\text{LHS}=g_\lambda(x)\cdot\frac{\partial h_\lambda(t)}{\partial t}$$

Find a Set of Solutions to the 2D PDE

Observation 2:

$$\frac{\partial(g_\lambda(x)\cdot h_\lambda(t))}{\partial t} = \sum_{p=0}^{n} a_p \cdot \frac{\partial^p (g_\lambda(x)\cdot h_\lambda(t))}{\partial x^p}$$

Using the fact that $h_\lambda(t)$ does not depend on x we can re-write the right-hand side as:

$$\begin{aligned} RHS &= h_\lambda(t)\cdot \sum_{p=0}^{n} a_p \cdot \frac{\partial^p g_\lambda(x)}{\partial x^p} \\ &= h_\lambda(t)\cdot D\big(g_\lambda(x)\big) \\ &= \lambda \cdot h_\lambda(t)\cdot g_\lambda(x) \end{aligned}$$

Find a Set of Solutions to the 2D PDE

Observation 2:

So now we are left with the problem of solving for the function $h_\lambda(t)$ such that:

$$g_\lambda(x) \cdot \frac{\partial h_\lambda(t)}{\partial t} = \lambda \cdot h_\lambda(t) \cdot g_\lambda(x)$$

$$\Downarrow$$

$$\frac{\partial h_\lambda(t)}{\partial t} = \lambda \cdot h_\lambda(t)$$

$$\Downarrow$$

$$h_\lambda(t) = C \cdot e^{\lambda t}$$

Find a Set of Solutions to the 2D PDE

Observation 2:

⇒ If $g_\lambda(x)$ is an eigenvector of the linear operator D with eigenvalue λ, then:

$$f_\lambda(x,t) = e^{\lambda t} \cdot g_\lambda(x)$$

must be a solution to the differential equation:

$$\frac{\partial f(x,t)}{\partial t} = \sum_{p=0}^{n} a_p \cdot \frac{\partial^p f(x,t)}{\partial x^p}$$

Satisfying the Initial Condition

Observation 3:

If $f_1(x,t)$ and $f_2(x,t)$ are solutions to the (partial) differential equation:

$$\frac{\partial f(x,t)}{\partial t} = \sum_{p=0}^{n} a_p \cdot \frac{\partial^p f(x,t)}{\partial x^p}$$

then any linear combination of the two must also be a solution.

Satisfying the Initial Condition

- For any eigenvector $g_\lambda(x)$ with eigenvalue λ:
$$f_\lambda(x,t) = e^{\lambda t} \cdot g_\lambda(x)$$
is a solution.
- Any linear combination of solutions is a solution.

$\Rightarrow$ Any function $f(x,t)$ expressable as:
$$f(x,t) = \sum_\lambda c_\lambda \cdot e^{\lambda t} \cdot g_\lambda(x)$$
must be a solution to the PDE.

Satisfying the Initial Condition

To satisfy the initial value conditions, we need to choose c_λ so that the function:

$$f(x,t) = \sum_\lambda c_\lambda \cdot e^{\lambda t} \cdot g_\lambda(x)$$

satisfies the initial value conditions:

$$f(x,0) = g(x)$$

But this implies that:

$$g(x) = \sum_\lambda c_\lambda \cdot g_\lambda(x)$$

Satisfying the Initial Condition

To satisfy the initial value conditions, we need to choose c_λ so that the function:

$$f(x,t) = \sum_\lambda c_\lambda \cdot e^{\lambda t} \cdot g_\lambda(x)$$

sat

But

Satisfying the initial value conditions is equivalent to finding the coefficients of $g(x)$ with respect to the functions $\{g_\lambda(x)\}$

$$g(x) = \sum_\lambda c_\lambda \cdot g_\lambda(x)$$

Outline

A Simple PDE

Solving the PDE

Relationship to the Fourier Transform

Generalizations

Examples

Relationship to the Fourier Transform

Recall that the Fourier decomposition expresses a circular function $f(\theta)$ as a sum of complex exponentials of different frequencies:

$$f(\theta) = \sum_{k=-\infty}^{\infty} \hat{f}(k) \cdot \frac{e^{ik\theta}}{\sqrt{2\pi}}$$

Relationship to the Fourier Transform

The complex exponentials are the eigenvectors of the derivative operator:

$$\frac{\partial}{\partial\theta}e^{ik\theta} = ik \cdot e^{ik\theta}$$

$$\frac{\partial^2}{\partial\theta^2}e^{ik\theta} = -k^2 \cdot e^{ik\theta}$$

$$\vdots$$

$$\frac{\partial^n}{\partial\theta^n}e^{ik\theta} = (ik)^n \cdot e^{ik\theta}$$

Relationship to the Fourier Transform

So, if we are given the linear map:

$$D = \sum_{p=0}^{n} a_p \cdot \frac{\partial^p}{\partial \theta^p}$$

it will act on $e^{ik\theta}$ as:

$$D\left(e^{ik\theta}\right) = \sum_{p=0}^{n} a_p \cdot (ik)^p \cdot e^{ik\theta}$$

Relationship to the Fourier Transform

So, if we are given the linear map:

$$D = \sum_{l=0}^{n} a_0 \cdot \frac{\partial^l}{\partial\theta^l}$$

it acts on $e^{ik\theta}$ as:

$$D\left(e^{ik\theta}\right) = \left(\sum_{p=0}^{n} a_l \cdot (ik)^p\right) \cdot e^{ik\theta}$$

$\Rightarrow e^{ik\theta}$ is an eigenvector with eigenvalue:

$$\lambda_k = \sum_{p=0}^{n} a_p \cdot (ik)^p$$

Relationship to the Fourier Transform

In particular, this implies that:

$$f_k(\theta,t) = e^{\lambda_k t} \cdot e^{ik\theta}$$

are solutions to the partial differential equation:

$$\frac{\partial f(\theta,t)}{\partial t} = \sum_{p=0}^{n} a_p \cdot \frac{\partial^p f(\theta,t)}{\partial \theta^p}$$

⇒ The solutions to the PDE will be of the form:

$$f(\theta,t) = \sum_{k=-\infty}^{\infty} c_k \cdot e^{\lambda_k t} \cdot \frac{e^{ik\theta}}{\sqrt{2\pi}}$$

Relationship to the Fourier Transform

$$f(\theta,t)=\sum_{k=-\infty}^{\infty} c_k\cdot e^{\lambda_k t}\cdot\frac{e^{ik\theta}}{\sqrt{2\pi}}$$

To satisfy the initial condition:

$$f(\theta,0)=g(\theta)$$

we need to solve for the values of c_k such that:

$$f(\theta,0)=\sum_{k=-\infty}^{\infty} c_k\cdot\frac{e^{ik\theta}}{\sqrt{2\pi}}$$

$\Rightarrow c_k$ must be the k-th Fourier coefficients of $g(\theta)$:

$$c_k=\hat{g}(k)$$

Relationship to the Fourier Transform

The solution to the PDE is the function $f(\theta, t)$ whose k-th Fourier coefficient at time t is the modulation of the k-th Fourier coefficient of $g(\theta)$ by a function of t:

$$\hat{f}_t(k) = \hat{g}(k) \cdot e^{\lambda_k t}$$

Relationship to the Fourier Transform

To implement this, we start off by:

- Computing the Fourier coefficients of $g(\theta)$

Then, at each time t, we:

- Compute the modulated Fourier coefficients:
$$\hat{f}_t(k) = \hat{g}(k) \cdot e^{\lambda_k t}$$
- And compute the inverse Fourier transform.

Outline

A Simple PDE

Solving the PDE

Relationship to the Fourier Transform

Generalizations

- Higher dimensions
- Second order time derivatives

Examples

2D Systems

In the case that the system is 2D, we want to consider functions of the form $f(\theta,\phi,t)$.

The linear partial differential equation becomes:

$$\frac{\partial f(\theta,\phi,t)}{\partial t}=\sum_{p=0}^{m}\sum_{q=0}^{n}a_{pq}\cdot\frac{\partial^p\partial^q f(\theta,\phi,t)}{\partial\theta^p\partial\phi^q}$$

The initial state becomes:

$$f(\theta,\phi,0)=g(\theta,\phi)$$

And the challenge is to compute the state of the system at an arbitrary point in time:

$$f(\theta,\phi,t)=?$$

2D Systems

As in the 1D case, we can use the fact that the a periodic 2D function $f(\theta,\phi)$ can be expressed in terms of its Fourier decomposition:

$$f(\theta,\phi)=\sum_{k,l=-\infty}^{\infty}\hat{f}(k,l)\cdot\frac{e^{ik\theta}}{\sqrt{2\pi}}\cdot\frac{e^{il\phi}}{\sqrt{2\pi}}$$

2D Systems

And again, we use the fact that the complex exponentials are the eigenvectors of the partial derivative operator:

$$\frac{\partial}{\partial\theta} e^{ik\theta} \cdot e^{il\phi} = (ik) \cdot e^{ik\theta} \cdot e^{il\phi}$$

$$\frac{\partial}{\partial\phi} e^{ik\theta} \cdot e^{il\phi} = (il) \cdot e^{ik\theta} \cdot e^{il\phi}$$

$$\vdots$$

$$\frac{\partial^m \partial^n}{\partial\theta^m \partial\phi^n} e^{ik\theta} \cdot e^{il\phi} = (ik)^m \cdot (il)^n \cdot e^{ik\theta} \cdot e^{il\phi}$$

2D Systems

Thus, given the linear map:

$$D=\sum_{p=0}^{m}\sum_{q=0}^{n}a_{pq}\cdot\frac{\partial^p\partial^q}{\partial\theta^p\partial\phi^q}$$

Eigenvectors of this map are:

$$f_{kl}(\theta,\phi)=e^{ik\theta}\cdot e^{il\phi}$$

And the associated eigenvalues are:

$$\lambda_{kl}=\sum_{p=0}^{m}\sum_{q=0}^{n}a_{pq}\cdot(ik)^p\cdot(il)^q$$

2D Systems

Given these eigenvectors, we can proceed as before, obtaining a solution to the differential equation for each eigenvector:

$$f_{kl}(\theta,\phi,t) = e^{\lambda_{kl}t} \cdot e^{ik\theta} \cdot e^{il\phi}$$

And we can satisfy the initial condition:

$$f(\theta,\phi,0) = g(\theta,\phi)$$

by setting the Fourier coefficients of $f(\theta,\phi,t)$ at time t equal to:

$$\hat{f}_t(k,l) = \hat{g}(k,l) \cdot e^{\lambda_{kl}t}$$

Outline

A Simple PDE

Solving the PDE

Relationship to the Fourier Transform

Generalizations

- Higher dimensions
- Second order time derivatives

Second Order Time Derivatives

What if the change in the system is characterized by the second derivative with respect to time:

$$\frac{\partial^2 f(\theta,t)}{\partial t^2} = \sum_{p=0}^{n} a_p \cdot \frac{\partial^p f(\theta,t)}{\partial \theta^p}$$

Second Order Time Derivatives

Recall that the solution to the original PDE was derived by solving the (first-order time) derivative:

$$h'(t) = \lambda \cdot h(t)$$

In this case, we need to solve the second-order time derivative:

$$h''(t) = \lambda \cdot h(t)$$

$$\Downarrow$$

$$h(t) = e^{\sqrt{\lambda}t} \quad \text{and} \quad h(t) = e^{-\sqrt{\lambda}t}$$

Second Order Time Derivatives

As before, if $g_\lambda(\theta)$ is an eigenvector of the linear operator D with eigenvalue λ, then the functions:

$$f_\lambda^+(\theta,t) = e^{\sqrt{\lambda}t} \cdot g_\lambda(\theta)$$

$$f_\lambda^-(\theta,t) = e^{-\sqrt{\lambda}t} \cdot g_\lambda(\theta)$$

will both be solutions to the PDE.

Second Order Time Derivatives

Note that in this case, one eigenvector of D gives us two solutions to the differential equation.

↓

We have more functions with which we can satisfy the initial boundary conditions.

↓

We can specify more stringent boundary conditions.

Second Order Time Derivatives

In practice, this amounts to specifying two boundary conditions:

- Initial value conditions:

$$f(\theta, 0) = g(\theta)$$

- Initial derivative conditions:

$$\frac{\partial}{\partial t} f(\theta, 0) = h(\theta)$$

Second Order Time Derivatives

Intuitively:

In the first-order case we are given the velocity at every point. If we know the initial positions, this is enough to know where things end up.

In the second-order case we are given the acceleration at every point. In order to know where things end up, we need to know both the initial position and the initial velocity.

Outline

A Simple PDE

Solving the PDE

Relationship to the Fourier Transform

Generalizations

Examples

- The Laplacian
- The 2D heat equation
- The 2D wave equation

The Laplacian

Given a function f in 1D, how do we interpret its second derivative:

$$\Delta f = \frac{\partial f}{\partial x^2}$$

The Laplacian

The first derivative of $f(x)$ is approximated by looking at the difference between the value of f at x and the value of f at a neighboring point:

$$f'(x) \approx f(x+1) - f(x)$$

The Laplacian of $f(x)$ is approximated by applying the process to the derivative of $f(x)$:

$$\begin{aligned} f''(x) &\approx f'(x) - f'(x-1) \\ &\approx \big(f(x+1) - f(x)\big) - \big(f(x) - f(x-1)\big) \\ &\approx f(x+1) + f(x-1) - 2f(x) \end{aligned}$$

The Laplacian

The first derivative of $f(x)$ is approximated by looking at the difference between the value of f at x and the value of f at a neighboring point:

$$f'(x) \approx f(x+1) - f(x)$$

The Laplacian of $f(x)$ is approximated by applying the process to the derivative of $f(x)$:

$$f''(x) \approx 2\left(\frac{f(x+1)+f(x-1)}{2} - f(x)\right)$$

i.e. it is a measure of the difference between the value of f at x and the average value of f at the neighbors of x.

The Laplacian

The same interpretation holds for the Laplacian of a 2D function f:

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

i.e. The Laplacian of a function is a measure of how the value of f at a point (x, y) differs from the average of the values of its neighbors.

Outline

A Simple PDE

Solving the PDE

Relationship to the Fourier Transform

Generalizations

Examples

- The Laplacian
- The 2D heat equation
- The 2D wave equation

Newton's Law of Cooling

"*The rate of heat loss of a body is proportional to the difference in temperatures between the body and its surroundings.*"

Newton's Law of Cooling

"*The rate of heat loss of a body is proportional to the difference in temperatures between the body and its surroundings.*"

Translating this into the PDE setting, if $f(\theta, \phi, t)$ is the heat at position (x, y) at time t, then:

$$\frac{\partial f}{\partial t} = \eta \cdot \Delta f$$

Newton's Law of Cooling

$$\frac{\partial f}{\partial t} = \eta \cdot \Delta f$$

In this case, the linear operator D is defined by:

$$D = \eta \cdot \Delta$$

And the complex exponential:

$$e^{ik\theta} \cdot e^{il\phi}$$

is an eigenvector with eigenvalue:

$$\lambda_{kl} = -\eta \cdot (k^2 + l^2)$$

Newton's Law of Cooling

$$\frac{\partial f}{\partial t} = \eta \cdot \Delta f$$

Thus, the solution to this equation:

$$f(\theta, \phi, 0) = g(\theta, \phi)$$

is the function whose (k, l)-th Fourier coefficient at time t is:

$$\hat{f}_t(k, l) = \hat{g}(k, l) \cdot e^{-\eta \cdot (k^2 + l^2) \cdot t}$$

Newton's Law of Cooling

$$\hat{f}_t(k,l) = \hat{g}(k,l) \cdot e^{-\eta\cdot(k^2+l^2)\cdot t}$$

This looks remarkably like what we get when implement Gaussian smoothing…

Outline

A Simple PDE

Solving the PDE

Relationship to the Fourier Transform

Generalizations

Examples

- The Laplacian
- The 2D heat equation
- The 2D wave equation

The 2D Wave Equation

Consider a square rubber sheet.

If we displace the points on the sheet, then at any point (θ, ϕ), the neighbors of (θ, ϕ) will exert a force to pull the point towards them.

The force that exerted on (θ, ϕ) is proportional to the distance of (θ, ϕ) from its neighbors.

The 2D Wave Equation

If the height of the point (θ, ϕ) is given by the function $f(\theta, \phi)$ then the force at (θ, ϕ) is:

$$F(\theta, \phi) = \eta \cdot \Delta f(\theta, \phi) \quad \text{w/ } \eta > 0$$

Using the fact that Force = Mass · Acceleration, the PDE for the height at time t is:

$$\frac{\partial^2 f}{\partial t^2} = \eta \cdot \Delta f$$

The 2D Wave Equation

$$\frac{\partial^2 f}{\partial t^2} = \eta \cdot \Delta f$$

We would like to solve this equation, subject to the constraints that at the initial time-step:

- The height at each point is given by:

$$f(\theta, \phi, 0) = g(\theta, \phi)$$

- The sheet is not moving:

$$\frac{\partial}{\partial t} f(\theta, \phi, 0) = 0$$

The 2D Wave Equation

$$\frac{\partial^2 f}{\partial t^2} = \eta \cdot \Delta f$$

Again, the linear operator D is defined by:

$$D = \eta \cdot \Delta$$

And the complex exponential:

$$e^{ik\theta} \cdot e^{il\phi}$$

is an eigenvector with eigenvalue:

$$\lambda_{kl} = -\eta \cdot (k^2 + l^2)$$

The 2D Wave Equation

$$\frac{\partial^2 f}{\partial t^2} = \eta \cdot \Delta f$$

Thus, the solutions to this equation are:

$$f_{kl}^{+}(\theta, \phi, t) = e^{i\sqrt{\eta\cdot(k^2+l^2)}t} \cdot e^{ik\theta} \cdot e^{il\phi}$$

$$f_{kl}^{-}(\theta, \phi, t) = e^{-i\sqrt{\eta\cdot(k^2+l^2)}t} \cdot e^{ik\theta} \cdot e^{il\phi}$$

And a general solution takes the form:

$$f(\theta, \phi, t) = \sum_{k,l} \left(A_{kl} \cdot e^{i\sqrt{\eta\cdot(k^2+l^2)}t} + B_{kl} \cdot e^{-i\sqrt{\eta\cdot(k^2+l^2)}t} \right) \cdot \frac{e^{ik\theta}}{\sqrt{2\pi}} \cdot \frac{e^{il\phi}}{\sqrt{2\pi}}$$

The 2D Wave Equation

$$f(\theta,\phi,t)=\sum_{k,l}\left(A_{kl}\cdot e^{i\sqrt{\eta\cdot(k^2+l^2)}t}+B_{kl}\cdot e^{-i\sqrt{\eta\cdot(k^2+l^2)}t}\right)\cdot\frac{e^{ik\theta}}{\sqrt{2\pi}}\cdot\frac{e^{il\phi}}{\sqrt{2\pi}}$$

To satisfy the initial value condition:

$$f(\theta,\phi,0)=g(\theta,\phi)$$

we need to have:

$$g(\theta,\phi)=\sum_{k,l}(A_{kl}+B_{kl})\cdot\frac{e^{ik\theta}}{\sqrt{2\pi}}\cdot\frac{e^{il\phi}}{\sqrt{2\pi}}$$

$$\Downarrow$$

$$\hat{g}(k,l)=A_{kl}+B_{kl}$$

The 2D Wave Equation

$$f(\theta,\phi,t)=\sum_{k,l}\left(A_{kl}\cdot e^{i\sqrt{\eta\cdot(k^2+l^2)}t}+B_{kl}\cdot e^{-i\sqrt{\eta\cdot(k^2+l^2)}t}\right)\cdot\frac{e^{ik\theta}}{\sqrt{2\pi}}\cdot\frac{e^{il\phi}}{\sqrt{2\pi}}$$

To satisfy the initial derivative condition:

$$\frac{\partial}{\partial t}f(\theta,\phi,0)=0$$

we need to have:

$$0=\sum_{k,l}(A_{kl}-B_{kl})\cdot\left(i\sqrt{\eta\cdot(k^2+l^2)}\right)\cdot\frac{e^{ik\theta}}{\sqrt{2\pi}}\cdot\frac{e^{il\phi}}{\sqrt{2\pi}}$$

$$\Downarrow$$

$$A_{kl}=B_{kl}$$

The 2D Wave Equation

$$f(\theta,\phi,t) = \sum_{k,l}\left(A_{kl}\cdot e^{i\sqrt{\eta\cdot(k^2+l^2)}t} + B_{kl}\cdot e^{-i\sqrt{\eta\cdot(k^2+l^2)}t}\right)\cdot\frac{e^{ik\theta}}{\sqrt{2\pi}}\cdot\frac{e^{il\phi}}{\sqrt{2\pi}}$$

$$\hat{g}(k,l) = A_{kl} + B_{kl} \quad \text{and} \quad A_{kl} = B_{kl}$$

Putting all of this together, the solution to the 2D wave equation, with initial position $g(\theta,\phi)$ and zero initial derivative is the function $f(\theta,\phi,t)$ whose Fourier coefficients at time t are equal to:

$$\hat{f}_t(k,l) = \hat{g}(k,l)\cdot\frac{e^{i\sqrt{\eta\cdot(k^2+l^2)}t} + e^{-i\sqrt{\eta\cdot(k^2+l^2)}t}}{2}$$

$$= \hat{g}(k,l)\cos\left(\sqrt{\eta\cdot(k^2+l^2)}t\right)$$

The 2D Wave Equation

$$\hat{f}_t(k,l) = \hat{g}(k,l)\cos\left(\sqrt{\eta\cdot(k^2+l^2)}t\right)$$

This looks <u>nothing</u> like what we get when implement Gaussian smoothing…