FFTs in Graphics and Vision

Groups and Representations

Outline

Groups

Representations

Schur's Lemma

Correlation

Groups

A group is a set of elements G with a binary operation (often denoted "·") such that for all $f, g, h \in G$, the following properties are satisfied:

- Closure:
$$g \cdot h \in G$$
- Associativity:
$$f \cdot (g \cdot h) = (f \cdot g) \cdot h$$
- Identity: $\exists 1 \in G$ s.t.:
$$1 \cdot g = g \cdot 1 = g$$
- Inverse: $\forall g \in G\ \exists g^{-1} \in G$ s.t.:
$$g \cdot g^{-1} = g^{-1} \cdot g = 1$$

If it is also true that $f \cdot g = g \cdot f$ for all $f, g \in G$, the group is called commutative, or abelian.

Groups

Examples

Under what binary operations are the following groups, what is the identity element, and what is the inverse:

- Integers?
- Positive real-numbers?
- Points in $\mathbb{R}^2$ modulo $(2\pi, 2\pi)$?
- Vectors in a fixed vector space?
- Invertible linear transformations of a vector space?

Groups

<u>Examples</u>

Are these groups commutative:

- Integers under addition?
- Positive real-numbers under multiplication?
- Points in $\mathbb{R}^2$ modulo $(2\pi, 2\pi)$ under addition?
- Vectors under addition?
- Linear transformations under composition?

Representations

Often, we think of a group as a set of elements that act on some space:

E.g.:

- Invertible linear transformations act on vector spaces
- 2D rotations act on 2D arrays
- 3D rotations act on 3D arrays

A representation is a way of formalizing this…

Representations

A <u>representation</u> of a group G on a vector space V, denoted (ρ, V), is a map ρ that sends every element in G to an invertible linear transformation on V, satisfying:

$$\rho(g \cdot h) = \rho(g) \cdot \rho(h) \quad \forall g, h \in G.$$

<u>Note</u>:

- $\rho(1) = 1$ since:
$$\rho(g) = \rho(g \cdot 1) = \rho(g) \cdot \rho(1)$$
- $\left(\rho(g)\right)^{-1} = \rho(g^{-1})$ since:
$$\rho(1) = \rho(g \cdot g^{-1}) = \rho(g) \cdot \rho(g^{-1})$$

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Analogy:

Linear maps are functions between vector spaces that preserve the vector space structure:

$$L(a \cdot v_1 + b \cdot v_2) = a \cdot L(v_1) + b \cdot L(v_2)$$

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For simplicity, we will write:

$$\rho(g) = \rho_g$$

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Unitary Representations

If the vector space V has a Hermitian inner product, and the representation preserves the inner product:

$$\langle v, w \rangle = \langle \rho_g v, \rho_g w \rangle \qquad \forall g \in G; v, w \in V$$

the representation is called unitary.

Note:
For nice (e.g. finite, compact) we can always define a Hermitian inner product such that the representation is unitary.

Unitary Representations

Examples

- G is the group of invertible $n \times n$ matrices
- V is the space of n-dimensional arrays with the standard inner-product
- ρ is the map:

$$\rho_M(v) = Mv$$

Representation?

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Unitary Representations

<u>Examples</u>

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- V is the space of n-dimensional arrays with the standard inner-product
- ρ is the map:

$$\rho_M(v) = v$$

Representation?

Unitary?

Unitary Representations

Examples

- G is the group of unitary transformations on V
- V is a complex Hermitian inner product space
- ρ is the map:

$$\rho_U(v) = Uv$$

Representation?

Unitary?

Unitary Representations

<u>Examples</u>

- G is the group of 2D/3D rotations
- V is the space of functions on a circle/sphere with the standard inner-product
- ρ is the map:

$$[\rho_R(f)](p) = f(Rp) \quad \forall R \in G$$

Representation?

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<u>Examples</u>

- G is the group of 2D/3D rotations
- V is the space of functions on a circle/sphere with the standard inner-product
- ρ is the map:

$$[\rho_R(f)](p) = f(R^{-1}p) \quad \forall R \in G$$

Representation?

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Unitary Representations

<u>Examples</u>

- G Is the group $\mathbb{R}^2$ modulo $(2\pi, 2\pi)$
- V is the space of continuous, periodic functions in the plane, with the standard Hermitian inner-product
- ρ is the map:

$$\left[\rho_{a,b}(f)\right](x,y) = f(x-a, y-b)$$

Representation?

Unitary?

Big Picture

Our goal is to try to better understand how a group acts on a vector space:

- How translational shifts act on periodic functions,
- How rotations act on functions on a sphere/circle
- Etc.

To do this we would like to simplify the "action" of the group into bite-size chunks.

Unless otherwise stated we will always be assuming that our representations are unitary

Sub-Representation

Given a representation (ρ, V) of a group G, if there exists a subspace $W \subset V$ such that the representation fixes W:

$$\rho_g(w) \in W \qquad \forall g \in G \text{ and } w \in W$$

then we say that W is a sub-representation of V.

Sub-Representation

Maschke's Theorem:

If W is a sub-representation of V, then the perpendicular space $W^\perp$ will also be a sub-representation of V.

$W^\perp$ is defined by the property that every vector in $W^\perp$ is perpendicular to every vector in W:

$$\langle w, w' \rangle = 0 \quad \forall w \in W \text{ and } w' \in W^\perp$$

Sub-Representation

Claim: $W^\perp$ will also be a sub-representation of V.

Proof: (By contradiction)

We would like to show that the representation ρ sends $W^\perp$ back into itself…

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There exist $w' \in W^\perp$, $w \in W$, and $g \in G$ s.t.:

$$\langle w, \rho_g(w') \rangle \neq 0$$

Since ρ is unitary, this implies that:

$$\langle \rho_{g^{-1}}(w), \rho_{g^{-1}}\left(\rho_g(w')\right) \rangle \neq 0$$

$$\Updownarrow$$

$$\langle \rho_{g^{-1}}(w), w' \rangle \neq 0$$

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Since ρ

But this would contradict the assumption that the representation ρ maps W back into itself!

$$\Updownarrow$$

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Sub-Representation

Example:

1. Consider the group of 2D rotations, acting on vectors in 3D by rotating around the y-axis. What are two sub-representations?

Sub-Representation

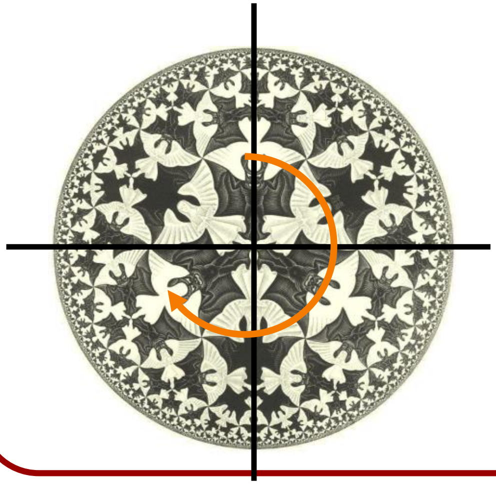
<u>Example</u>:

1. Consider the group of 2D rotations, acting on vectors in 3D by rotating around the y-axis. What are two sub-representations?
   a) The y-axis: The group acts on this sub-space trivially, mapping every vector to itself
   b) The xz-plane: The group acts as a 2D rotation on this 2D space.

Sub-Representation

Example:

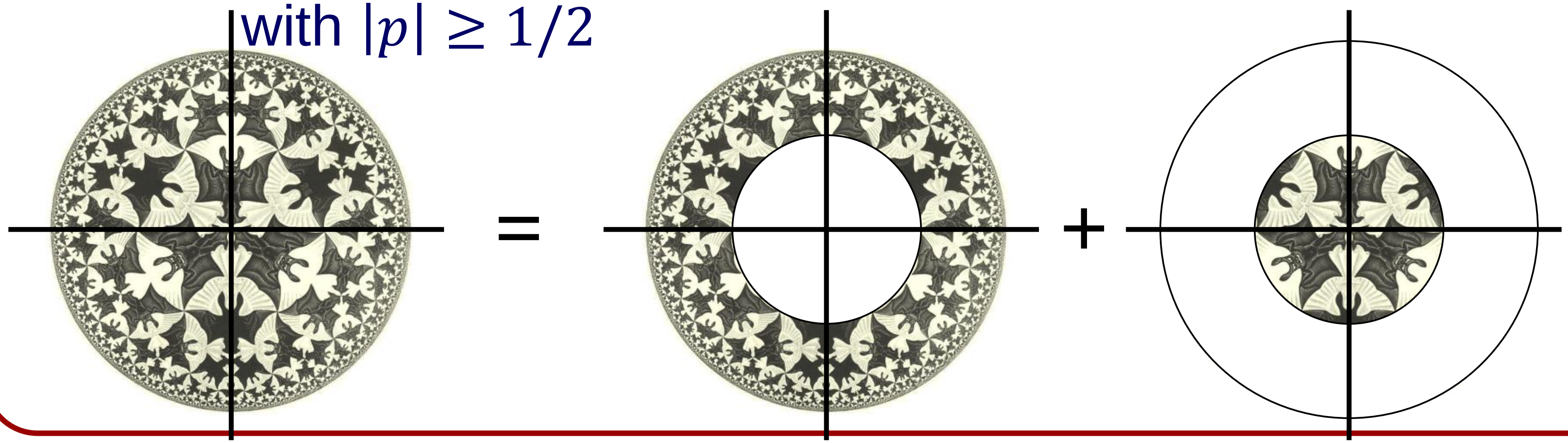
2. Consider the group of 2D rotations, acting on functions on the unit disk.
   What are two sub-representations?



Sub-Representation

Example:

2. Consider the group of 2D rotations, acting on functions on the unit disk.
   What are two sub-representations?
   a) The space of functions that are zero for all points p with $|p| < 1/2$
   b) The space of functions that are zero for all points p with $|p| \geq 1/2$



Irreducible Representations

Given a representation (ρ, V) of a group G, the representation is said to be irreducible if the only subspaces of V that are sub-representations are:

$$W = V \quad \text{and} \quad W = \emptyset$$

Structure Preservation

We had talked about linear transformations as maps between vector spaces, that preserve the underlying vector space structure:

$$L(a \cdot v_1 + b \cdot v_2) = a \cdot L(v_1) + b \cdot L(v_2)$$

We had talked about a representation as a map from a group into the group of invertible linear transforms that preserves the group structure:

$$\rho(g \cdot h) = \rho(g) \cdot \rho(h)$$

It doesn't matter if we perform the vector-space/group operations before or after we apply the map.

Structure Preservation

Given a representation (ρ, V) a group G, what does it mean for a map $\Phi: V \rightarrow V$ to preserve the representation structure?

- Since Φ is a map between vector spaces, it should preserve the vector space structure:
  $\Rightarrow$ Φ is a linear transformation.
- Φ should also preserve the group action structure:

$$\Phi\left(\rho_g(v)\right) = \rho_g\left(\Phi(v)\right)$$

Such a map is called G-linear.

Structure Preservation

Claim:

If $\Phi: V \to V$ is G-linear, then both the kernel and the image of Φ are sub-representations.

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Proof:

If $v \in \text{Kernel}(\Phi)$ then, for $g \in G$ we have:

$$0 = \Phi(v) = \rho_g\big(\Phi(v)\big)$$
$$= \Phi\left(\rho_g(v)\right)$$
$$\Updownarrow$$
$$\rho_g(v) \in \text{Kernel}(\Phi)$$

Structure Preservation

Claim:

If $\Phi: V \to V$ is G-linear, then both the kernel and the image of Φ are sub-representations.

Proof:

If $w = \Phi(v) \in \text{Image}(\Phi)$ then, for $g \in G$ we have:

$$\begin{aligned} \rho_g(w) &= \rho_g\big(\Phi(v)\big) \\ &= \Phi\left(\rho_g(v)\right) \\ &\in \text{Image}(\Phi) \end{aligned}$$

Schur’s Lemma

Given an irreducible representation (ρ, V) of a group G, if Φ is G-linear than Φ is scalar multiplication:

$$\Phi = \lambda \cdot \mathrm{Id}.$$

Schur's Lemma

Proof:

1. Since Φ is a linear transformation, it has a (complex) eigenvalue λ.

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1. Since Φ is a linear transformation, it has a (complex) eigenvalue λ.

2. Since Φ is G-linear, so is $(\Phi - \lambda \cdot \mathrm{Id.})$:

$$
\begin{aligned}
(\Phi - \lambda \cdot \mathrm{Id.})\big(\rho_g(v)\big) &= \Phi\big(\rho_g(v)\big) - \lambda \cdot \rho_g(v) \\
&= \rho_g\big(\Phi(v)\big) - \rho_g(\lambda \cdot v) \\
&= \rho_g\big((\Phi - \lambda \cdot \mathrm{Id.})(v)\big)
\end{aligned}
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Proof:

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5. Since (ρ, V) is irreducible and the kernel of $(\Phi - \lambda \cdot \mathrm{Id.})$ is not empty, $W = V$.

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5. Since (ρ, V) is irreducible and the kernel of $(\Phi - \lambda \cdot \mathrm{Id.})$ is not empty, $W = V$.

6. Since the kernel is the entire vector space:
$$(\Phi - \lambda \cdot \mathrm{Id.}) = 0 \quad \Leftrightarrow \quad \Phi = \lambda \cdot \mathrm{Id.}$$

Schur's Lemma (Corollary)

Corollary:

All irreducible representations of commutative groups must be one-dimensional.

Schur's Lemma (Corollary)

Proof:

1. Fix some element $h \in G$.

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1. Fix some element $h \in G$.
2. Since G is commutative, ρ_h must be G-linear:

$$\begin{aligned}\rho_g\big(\rho_h(v)\big) &= \rho_{g\cdot h}(v)\\ &= \rho_{h\cdot g}(v)\\ &= \rho_h\big(\rho_g(v)\big)\end{aligned}$$

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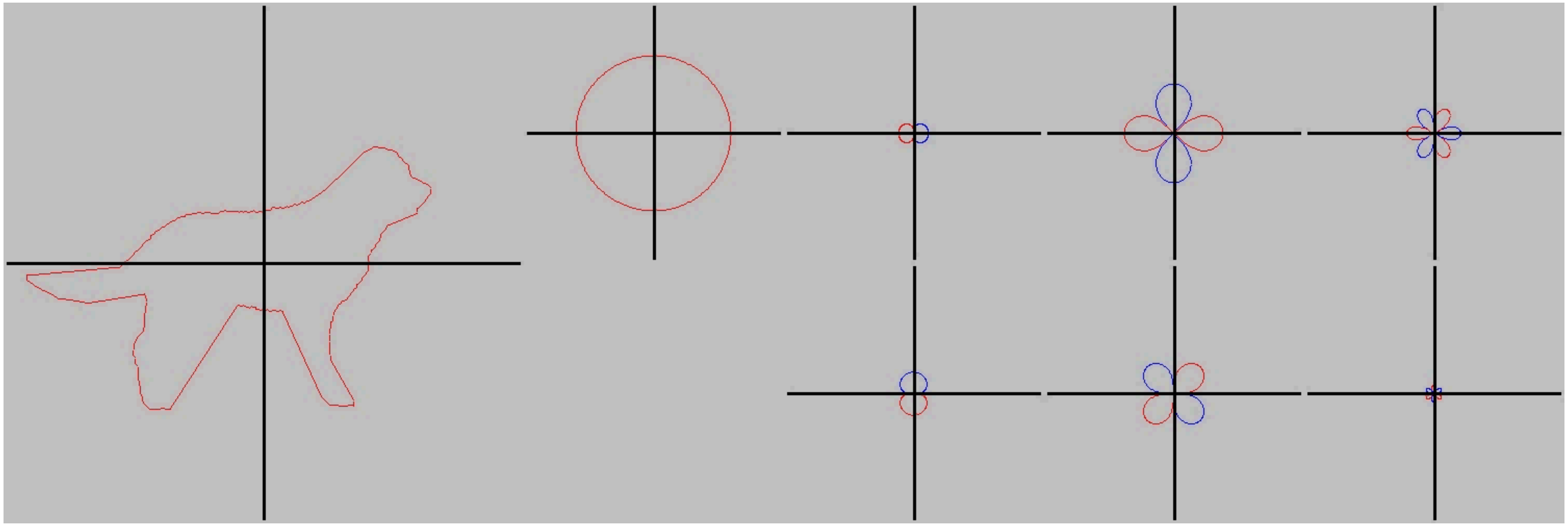
3. Since (ρ, V) is irreducible, $\rho_h = \lambda \cdot \mathrm{Id}$.
4. Since this is true for any $h \in G$, any subspace $W \subset V$ is a sub-representation.
5. Since V is irreducible, V is one-dimensional.

Example:

Since 2D rotations commute, we can decompose the space of functions on a circle into1D subspaces that rotate into themselves.



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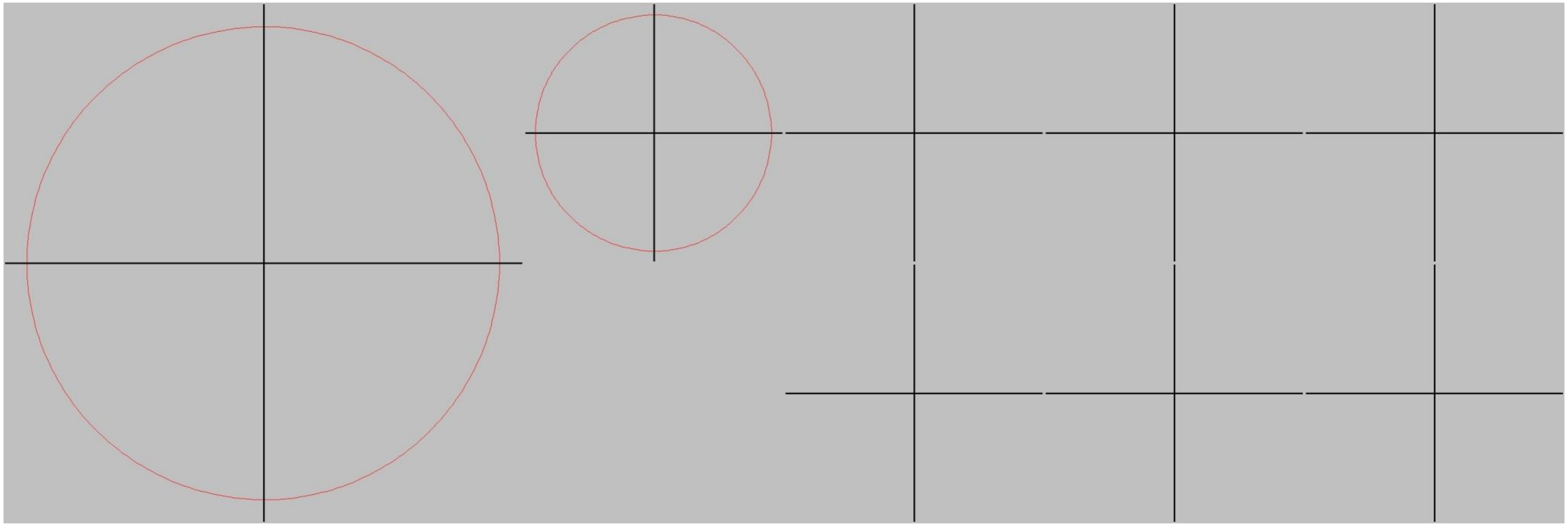

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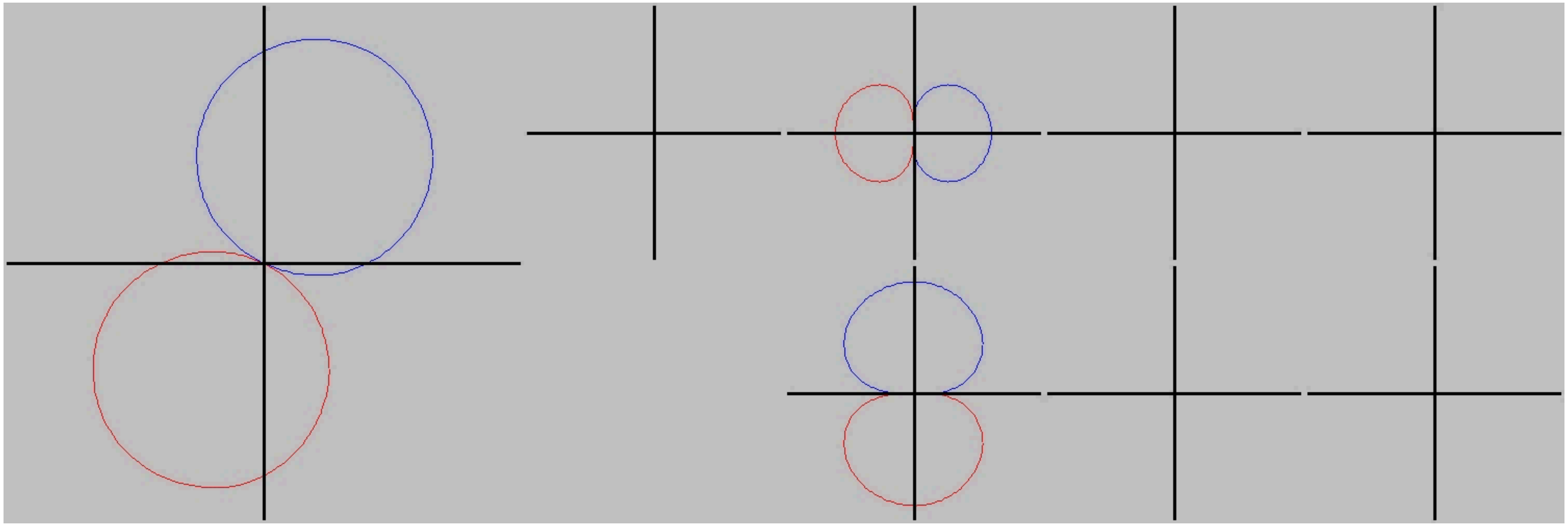

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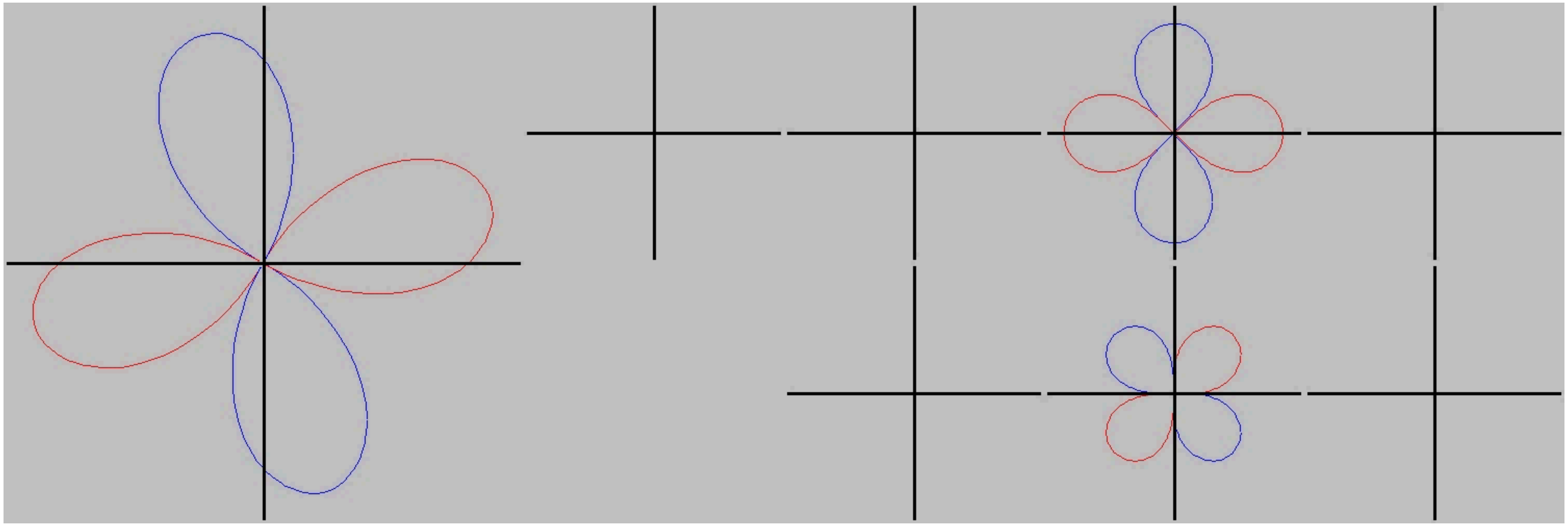

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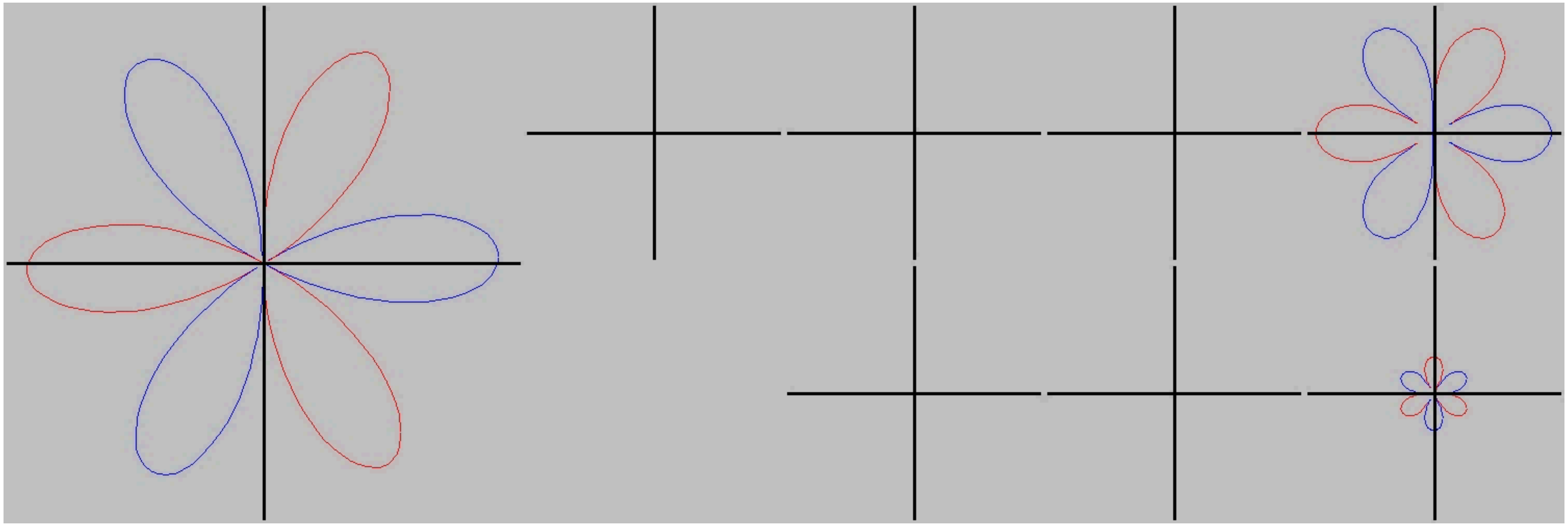
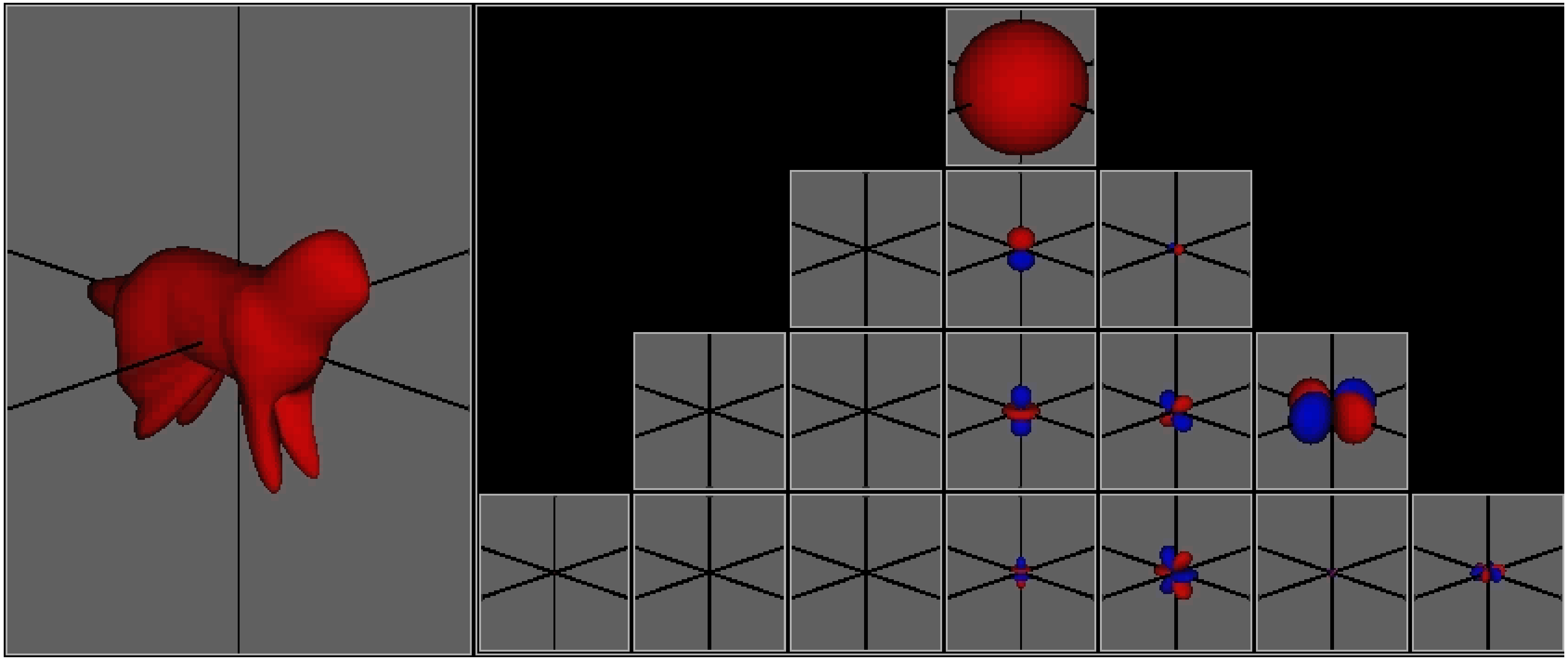


Example:

3D rotations don't commute, so the space of spherical functions need not be decomposable into 1D irreducible sub-representations.



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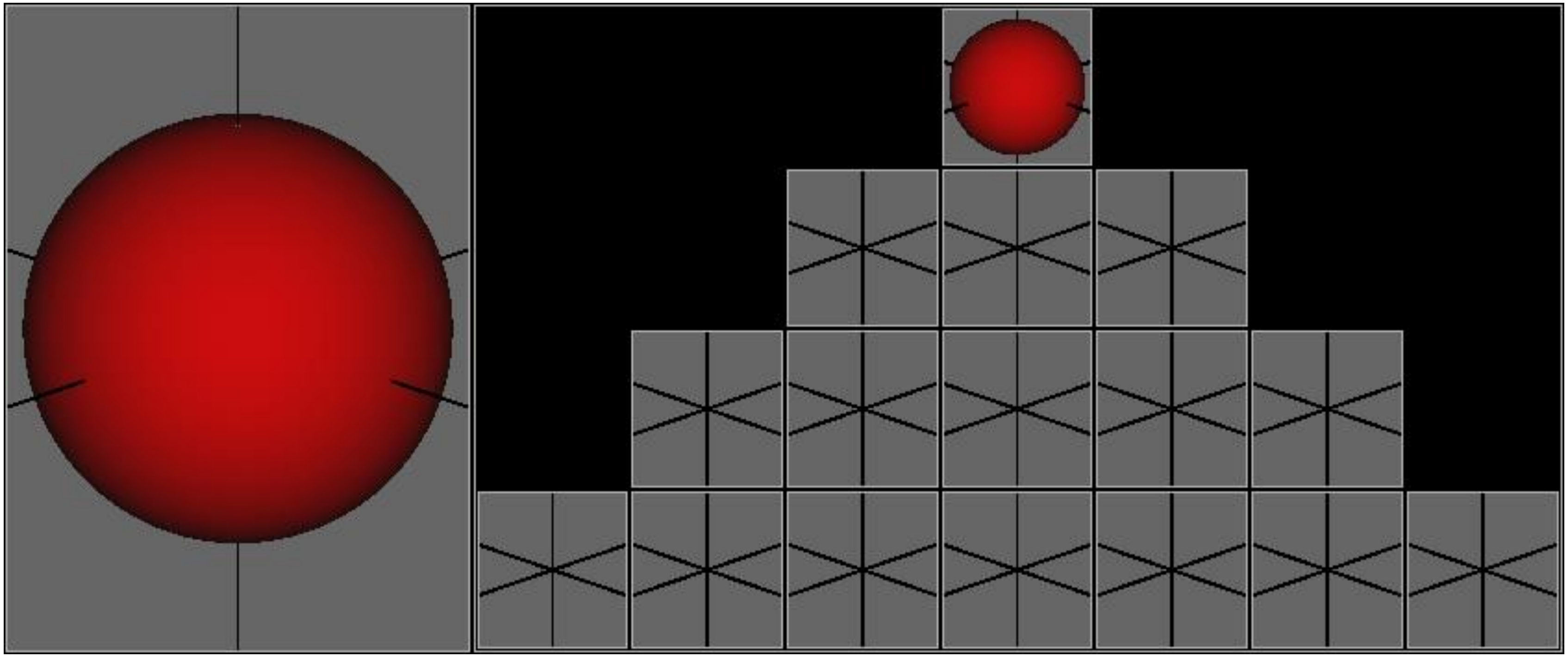

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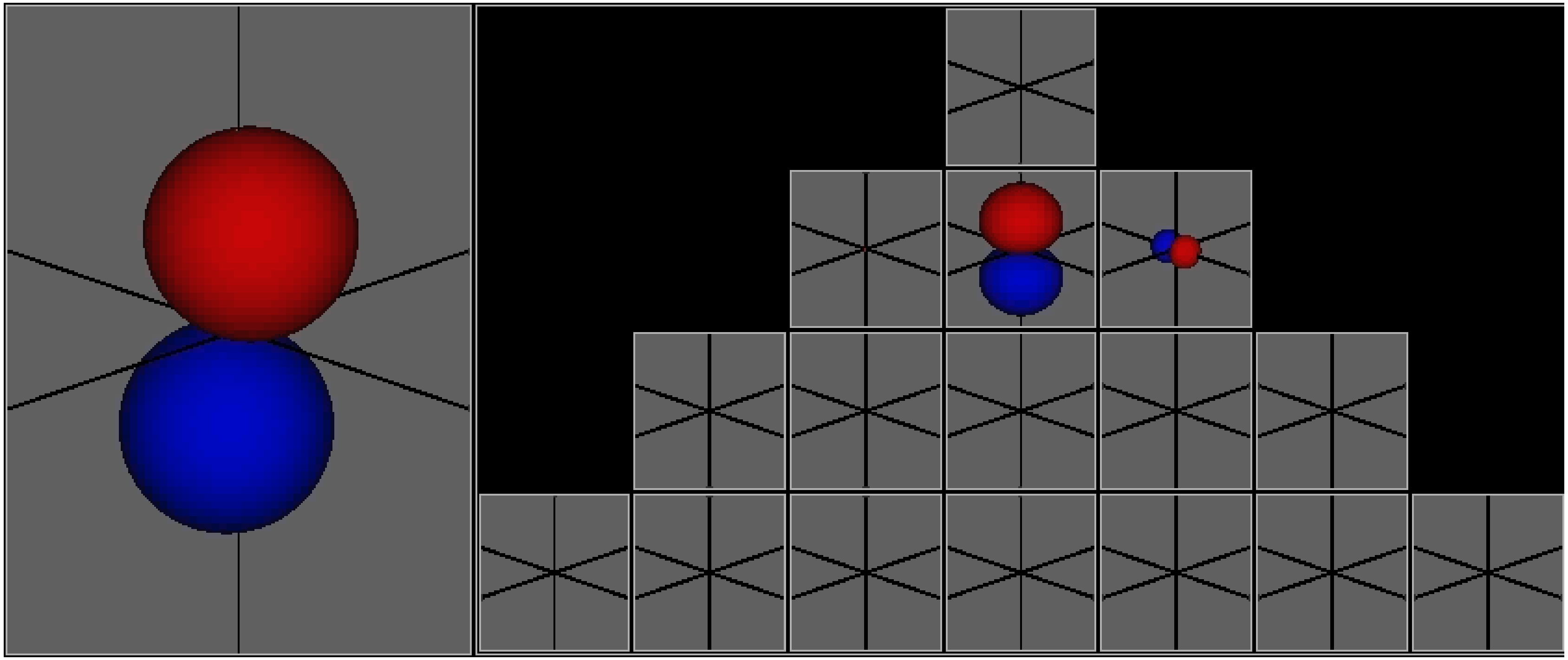

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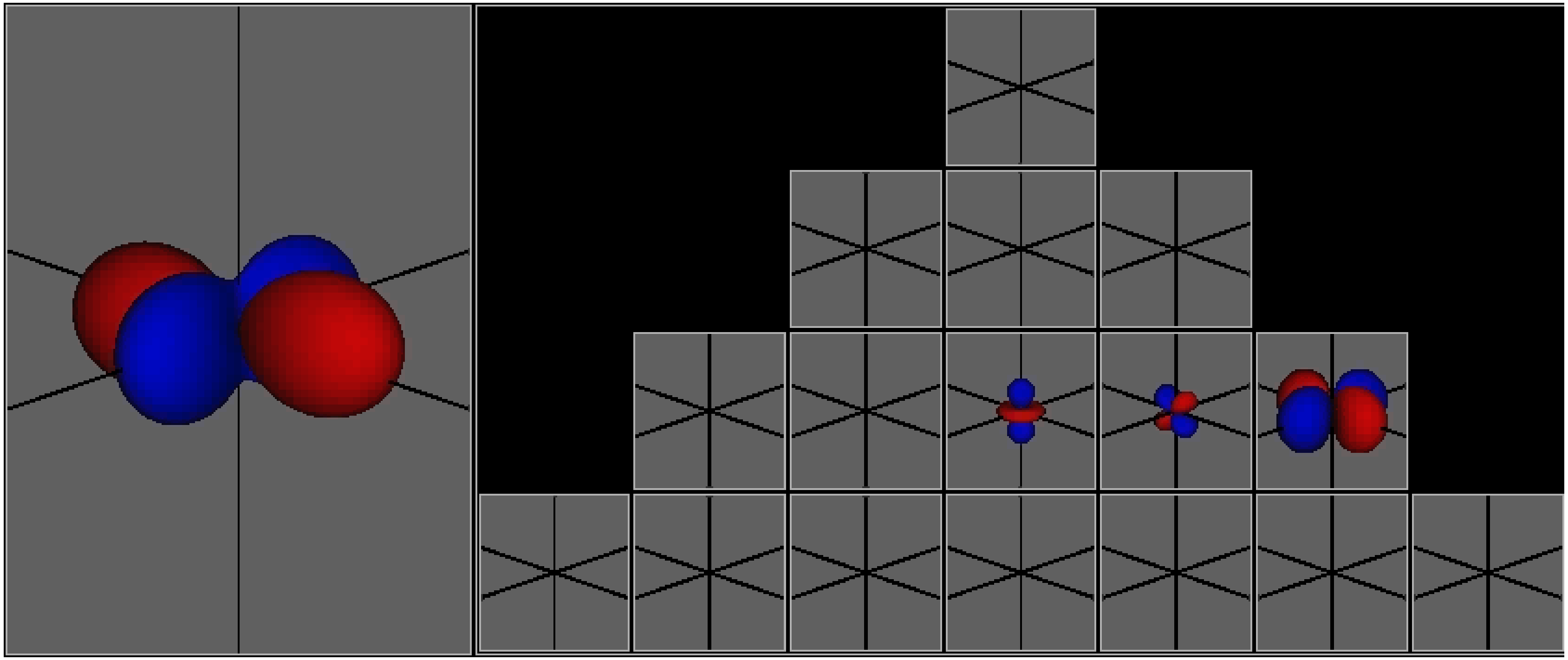

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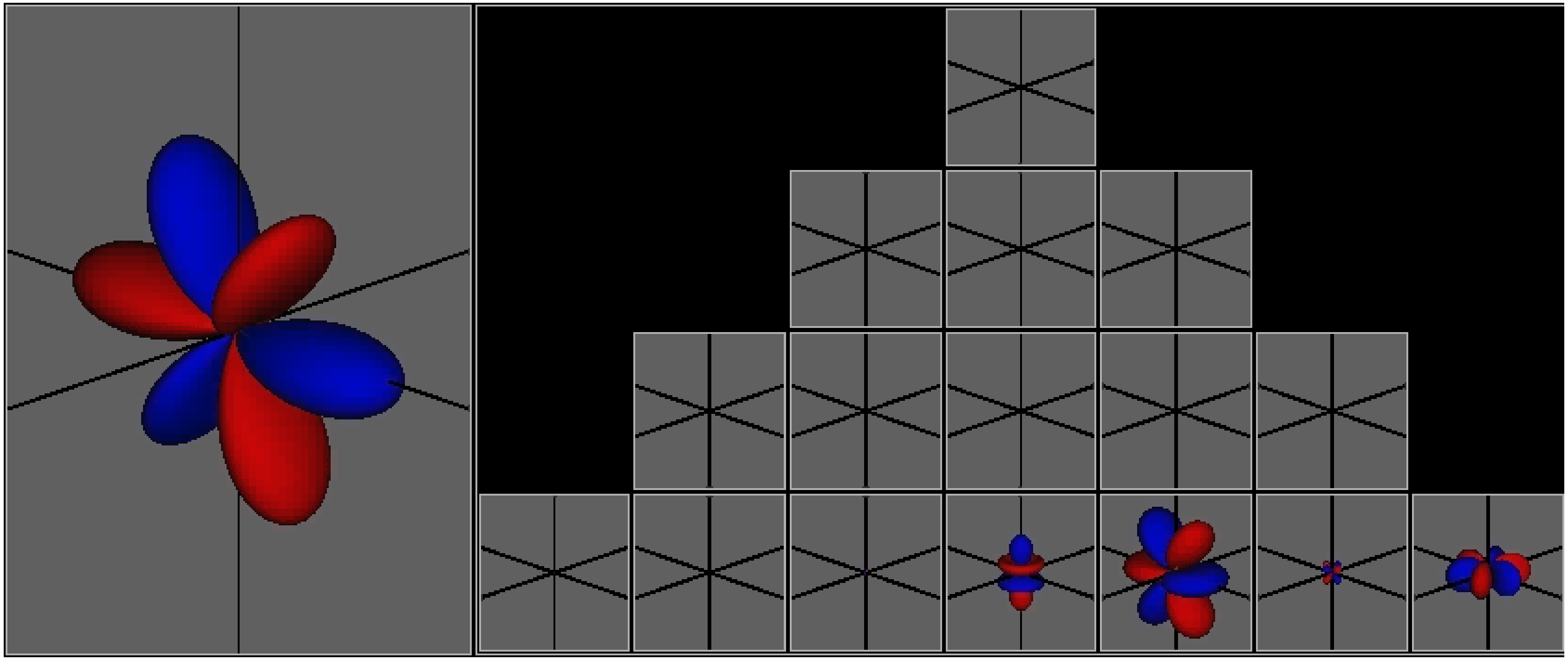
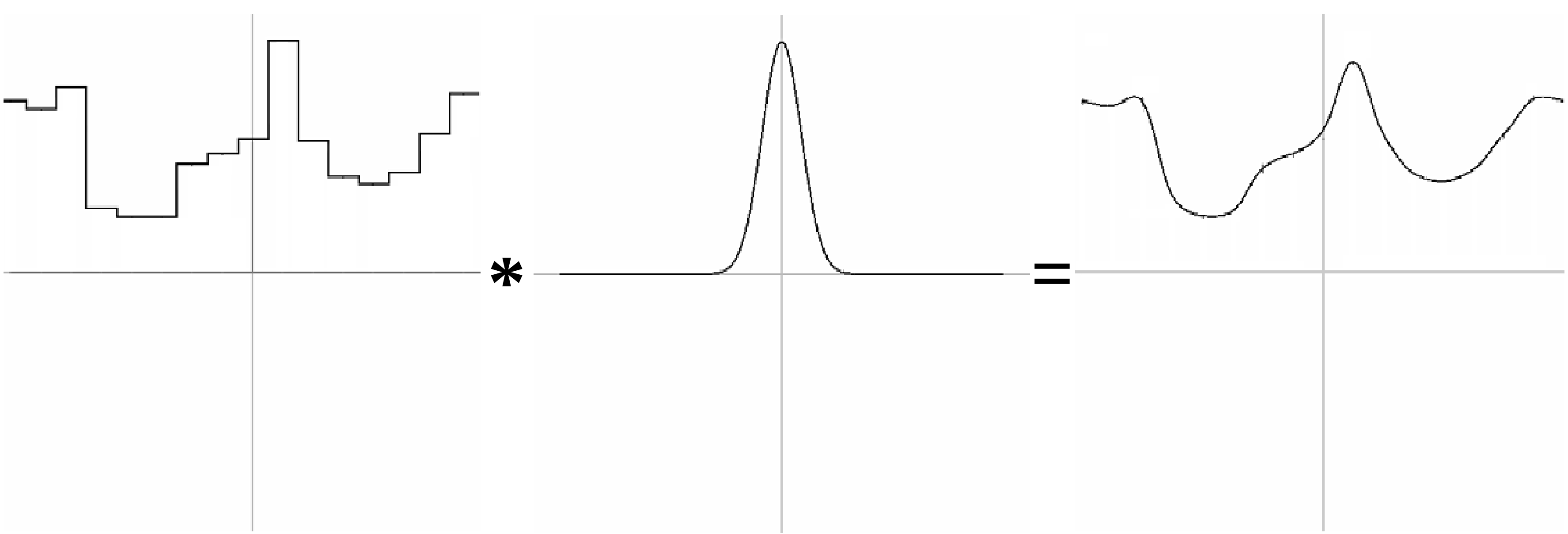


Why do we care?

In signal/image/voxel processing, we are often interested in applying a filter to some initial data.

E.g. Smoothing:



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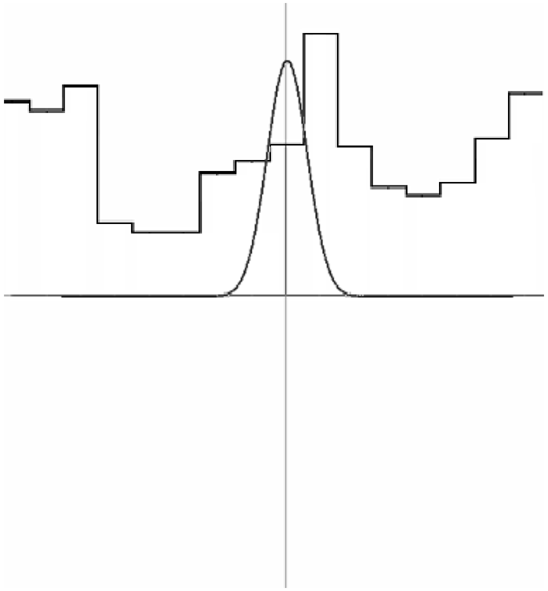

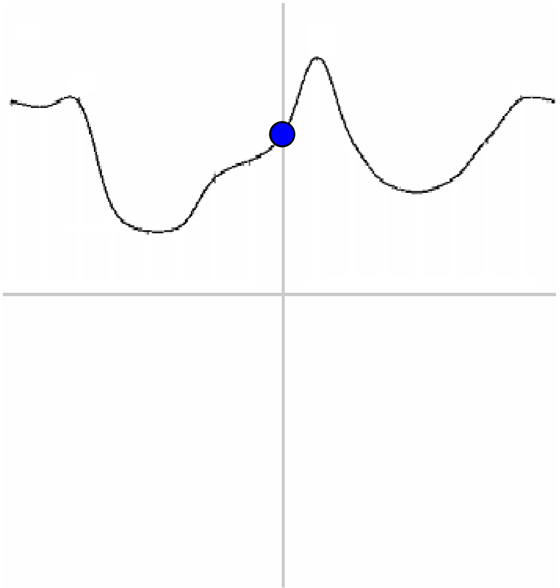

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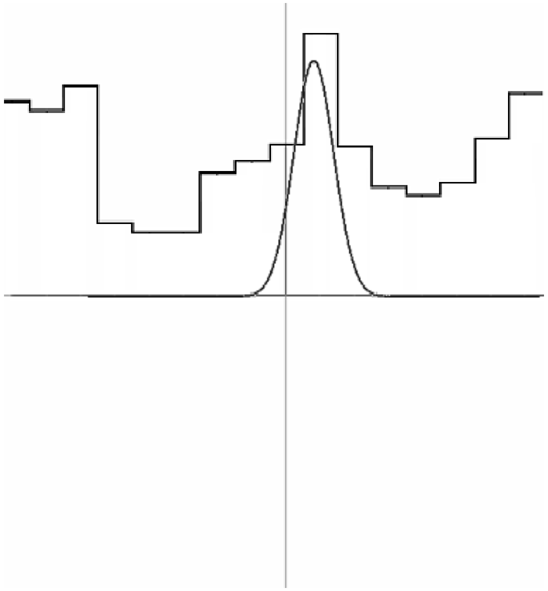

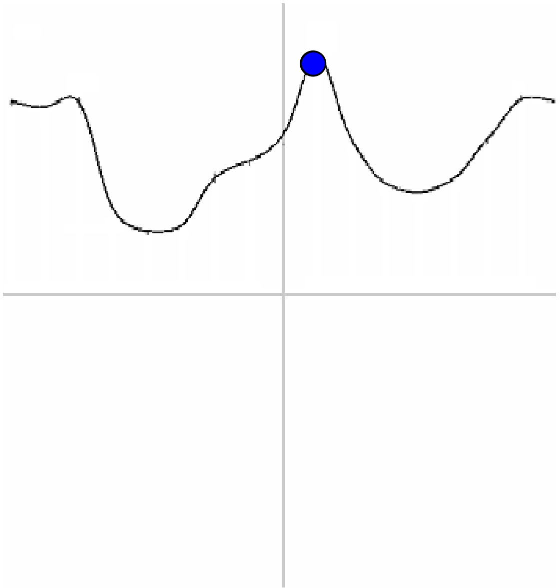

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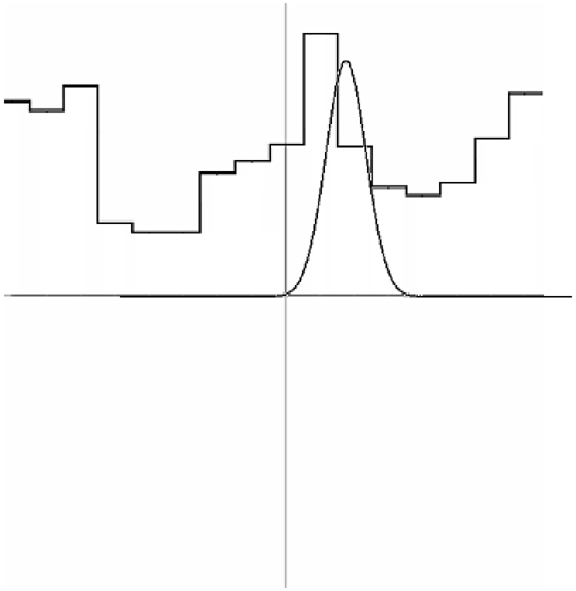

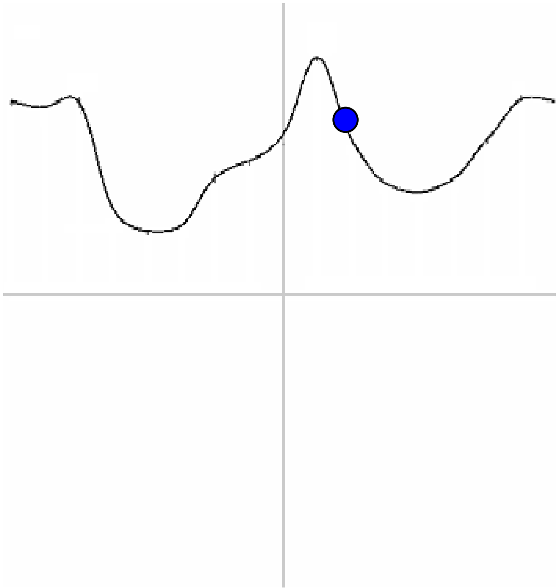

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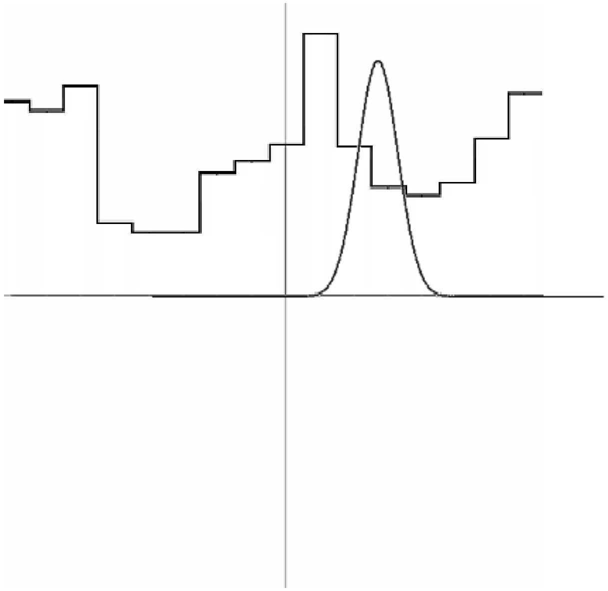

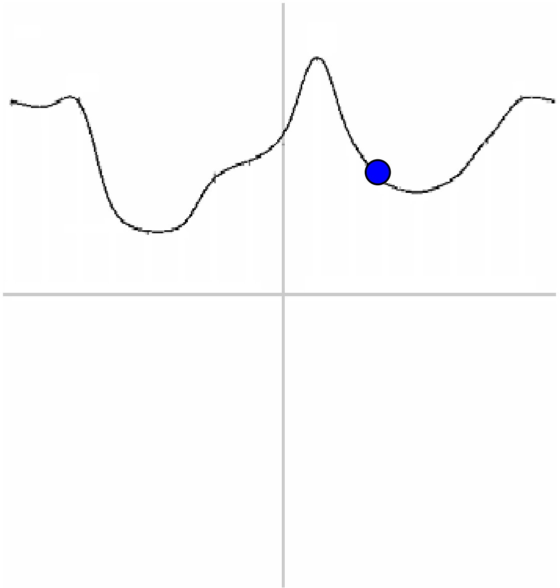

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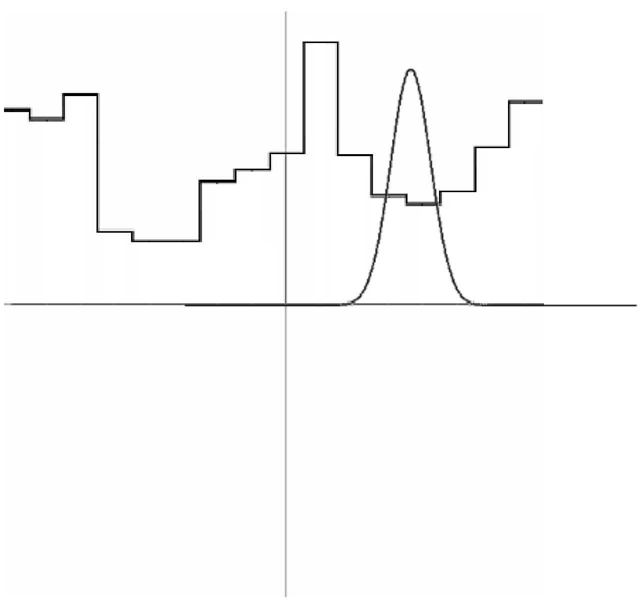

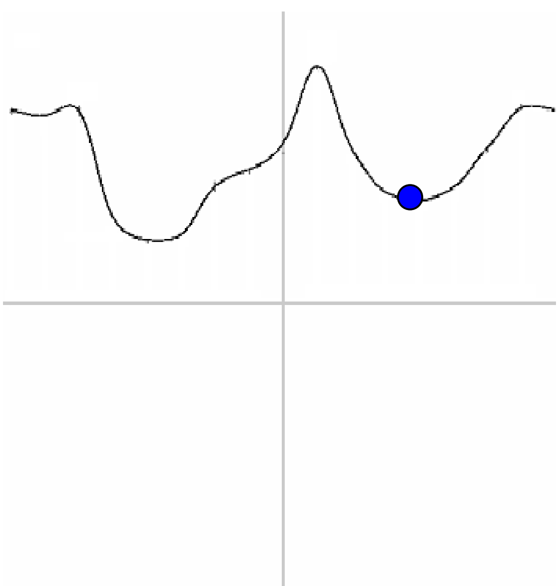

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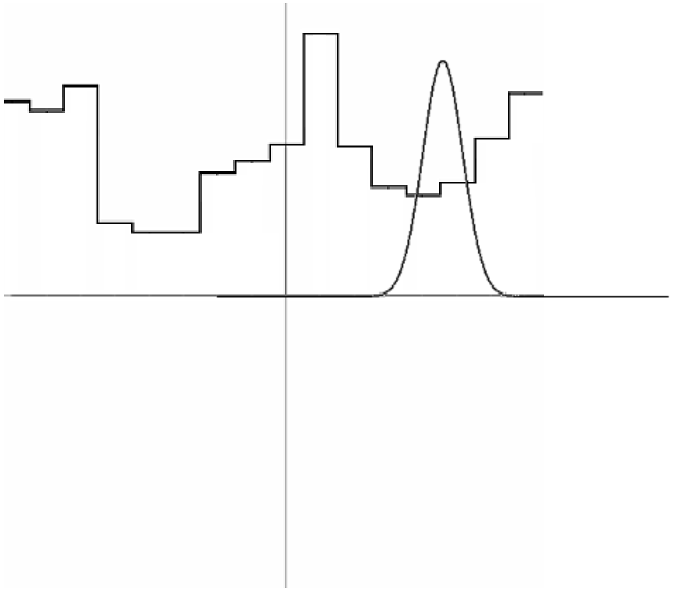

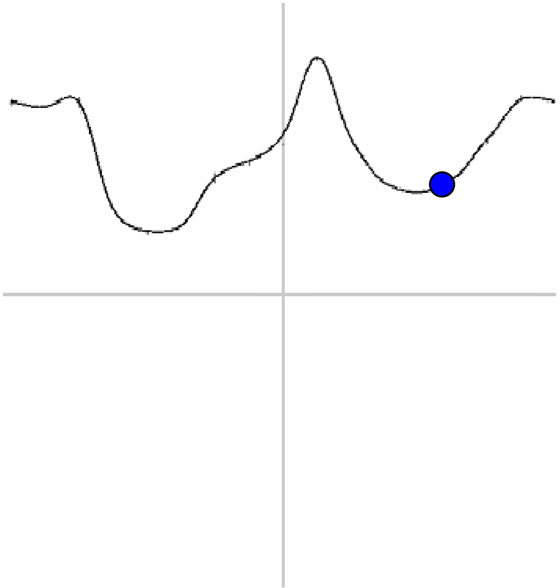

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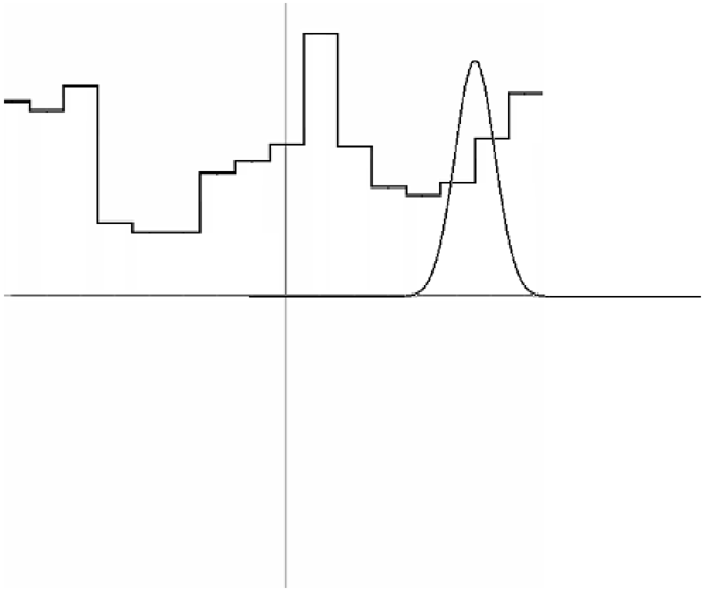
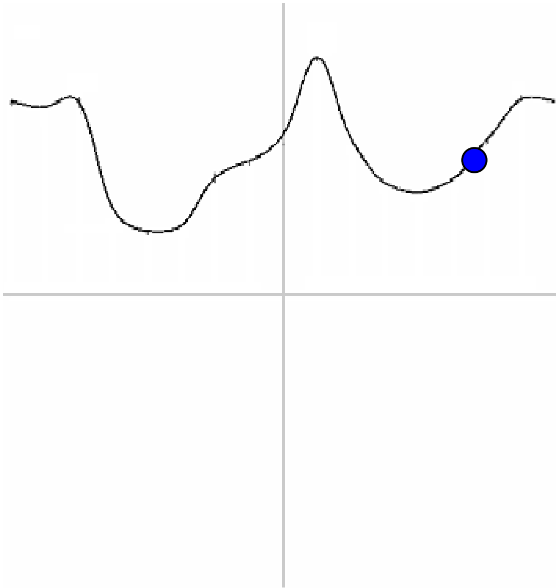
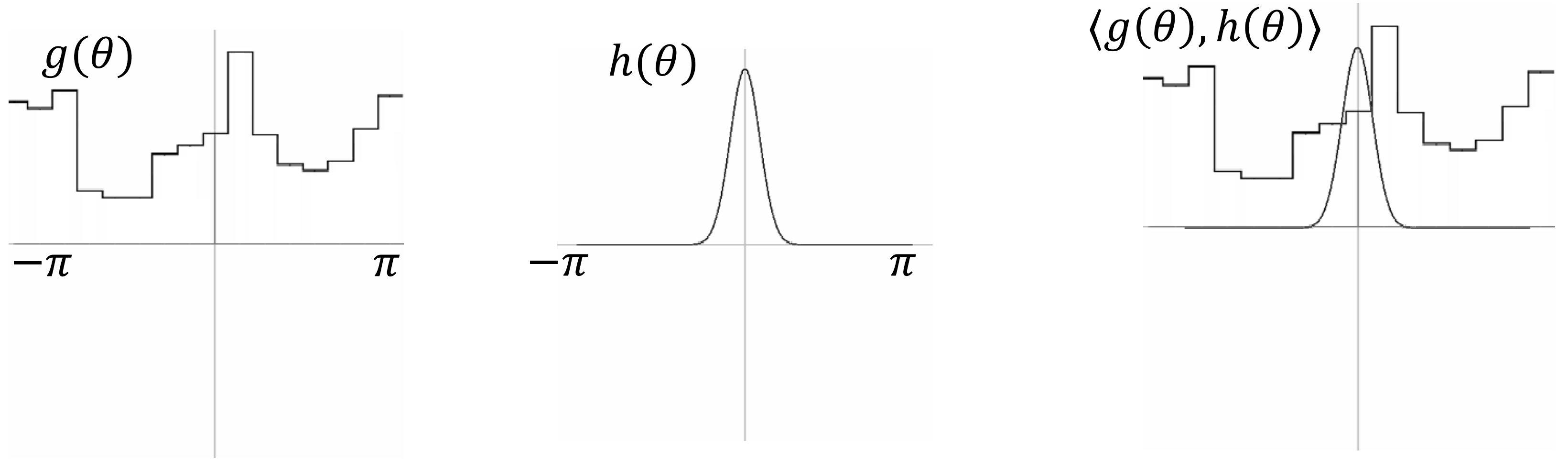




Smoothing

What we are really doing is computing a moving inner product:



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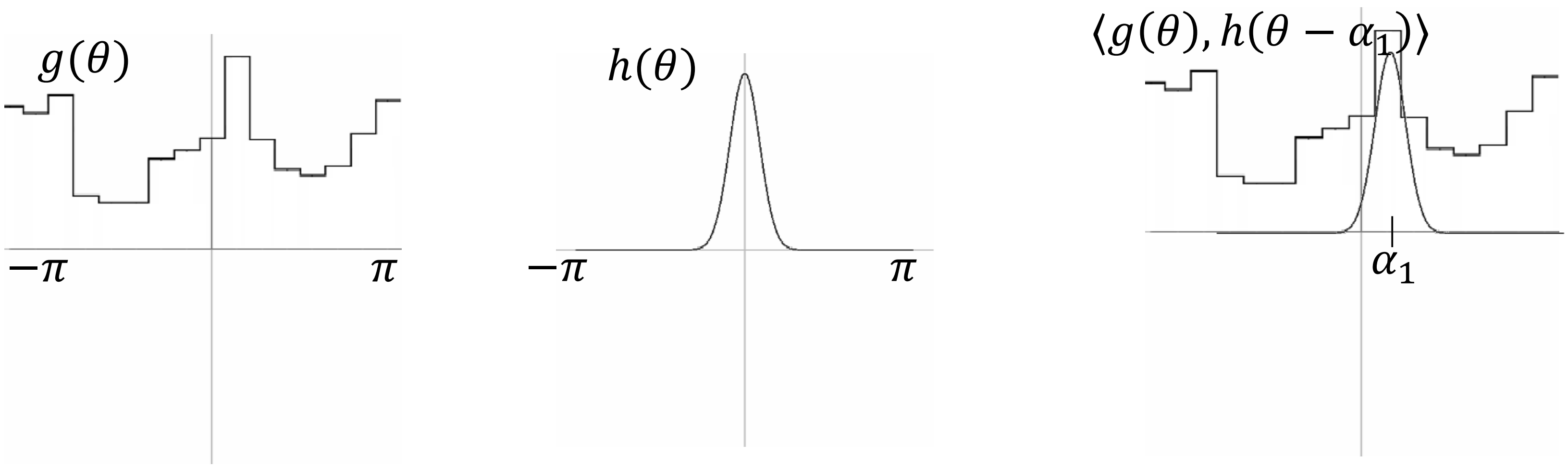
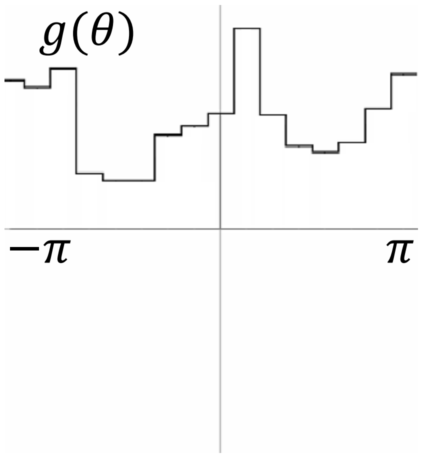
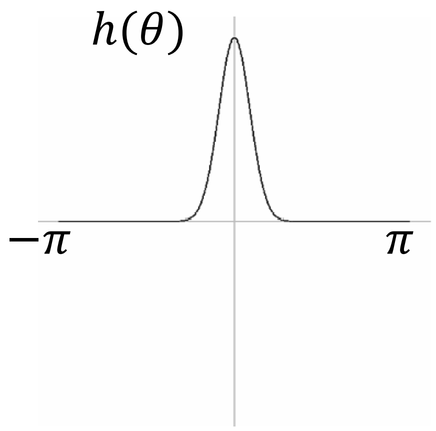
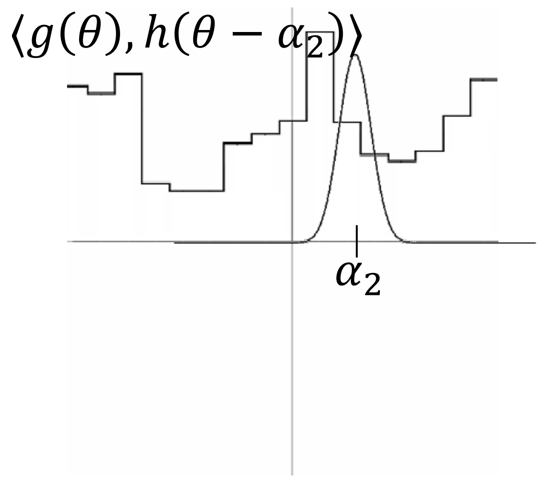


Smoothing

What we are really doing is computing a moving inner product:







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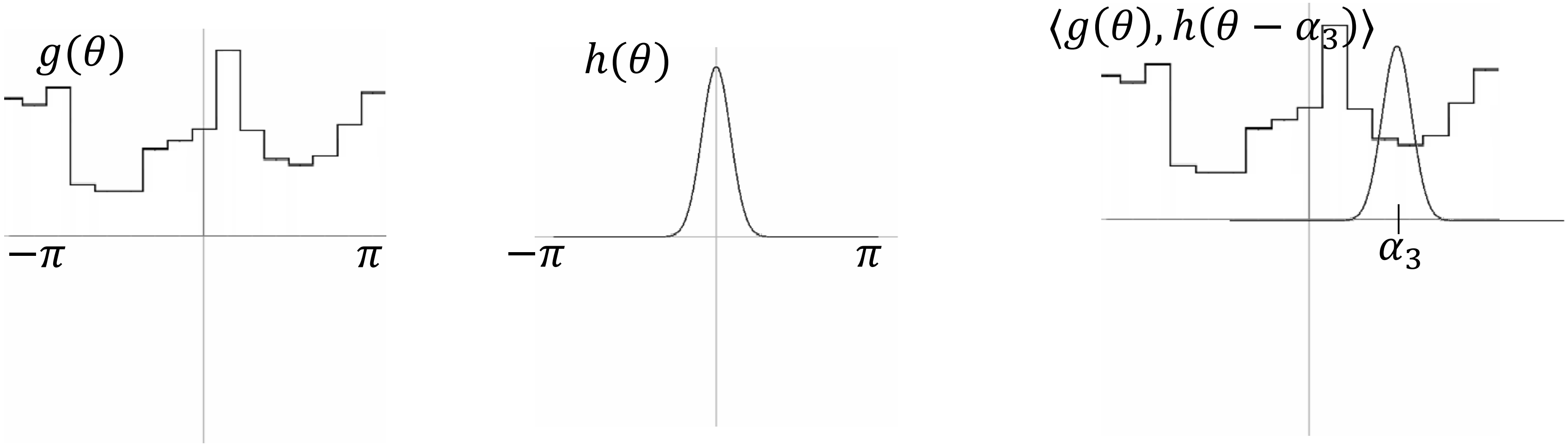
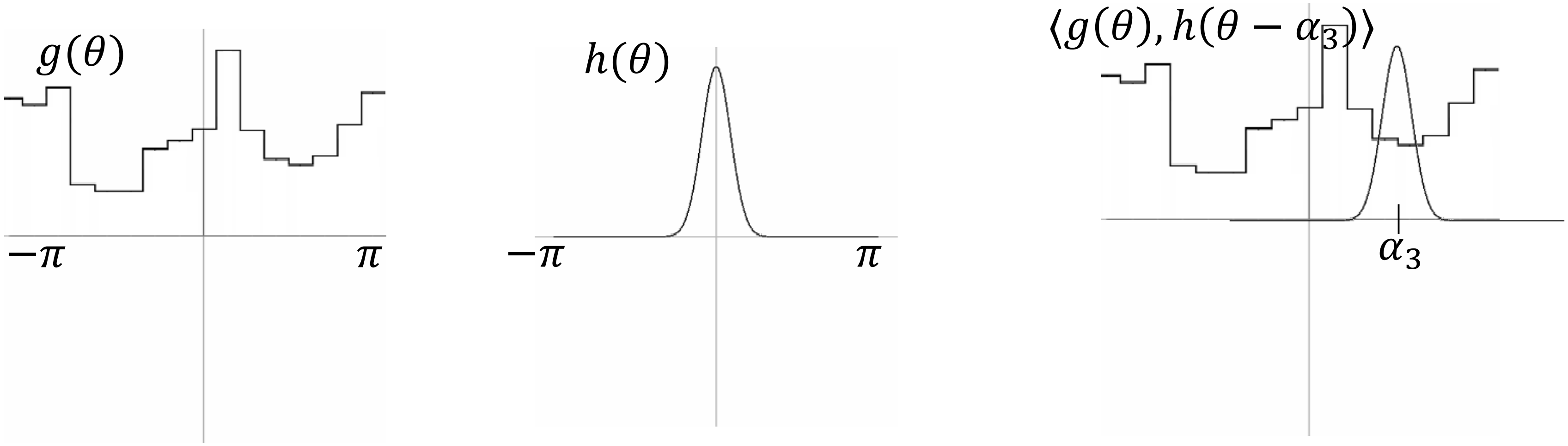


Smoothing

We can write out the operation of smoothing a signal g by a filter h as:

$$(g \star h)(\alpha) = \langle g, \rho_\alpha(h) \rangle$$

where ρ_α is the linear transformation that translates a periodic function by α.



Smoothing

We can think of this as a representation:

- V is the space of periodic functions on the line
- G is the group of real numbers in $[0,2\pi)$
- ρ_α is the representation translating a function by α.

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This is a representation of a commutative group…

Smoothing

⇒ There exist orthogonal one-dimensional (complex) subspaces $V_1, \cdots, V_n \subset V$ that are the irreducible representations of V.

Setting $\phi_i \in V_i$ to be a unit-vector, we know that the group acts on ϕ_i by scalar multiplication:

$$\rho_\alpha(\phi_i) = \lambda_i(\alpha) \cdot \phi_i$$

Note:

Since the V_i are orthogonal, the basis $\{\phi_1, \cdots \phi_n\}$ is orthonormal.

Smoothing

Setting $\phi_i \in V_i$ to be a unit-vector, we know that the group acts on ϕ_i by scalar multiplication:

$$\rho_\alpha(\phi_i) = \lambda_i(\alpha) \cdot \phi_i$$

We can write out the functions $g, h \in V$ as:

$$g(\theta) = \hat{g}_1 \cdot \phi_1(\theta) + \cdots + \hat{g}_n \cdot \phi_n(\theta)$$
$$h(\theta) = \hat{h}_1 \cdot \phi_1(\theta) + \cdots + \hat{h}_n \cdot \phi_n(\theta)$$

with $\hat{g}_i, \hat{h}_i \in \mathbb{C}$.

Smoothing

Then the moving dot-product can be written as:

$$(g \star h)(\alpha) = \langle g, \rho_\alpha(h) \rangle$$

Smoothing

$$(g \star h)(\alpha) = \langle g, \rho_\alpha(h) \rangle$$

Expanding in the basis $\{\phi_1, \cdots, \phi_n\}$:

$$(g \star h)(\alpha) = \left\langle \sum_{j=1}^{n} \hat{g}_j \phi_j, \rho_\alpha\left(\sum_{k=1}^{n} \hat{h}_k \phi_k\right)\right\rangle$$

Smoothing

$$(g \star h)(\alpha) = \langle \sum_{j=1}^{n} \hat{g}_j \phi_j , \rho_\alpha \left(\sum_{k=1}^{n} \hat{h}_k \phi_k \right) \rangle$$

By linearity of ρ_α:

$$(g \star h)(\alpha) = \langle \sum_{j=1}^{n} \hat{g}_j \phi_j , \sum_{k=1}^{n} \hat{h}_k \rho_\alpha(\phi_k) \rangle$$

Smoothing

$$(g \star h)(\alpha) = \langle \sum_{j=1}^{n} \hat{g}_j \phi_j , \sum_{k=1}^{n} \hat{h}_k \rho_\alpha(\phi_k) \rangle$$

By linearity of the inner product in the first term:

$$(g \star h)(\alpha) = \sum_{j=1}^{n} \hat{g}_j \langle \phi_j , \sum_{k=1}^{n} \hat{h}_k \rho_\alpha(\phi_k) \rangle$$

Smoothing

$$(g \star h)(\alpha) = \sum_{j=1}^{n} \hat{g}_j \langle \phi_j, \sum_{j=1}^{n} \hat{h}_k \rho_\alpha(\phi_k) \rangle$$

By conjugate-linearity in the second term:

$$(g \star h)(\alpha) = \sum_{j,k=1}^{n} \hat{g}_j \overline{\hat{h}}_k \langle \phi_j, \rho_\alpha(\phi_k) \rangle$$

Smoothing

$$(g \star h)(\alpha) = \sum_{j,k=1}^{n} \hat{g}_j \bar{\hat{h}}_k \langle \phi_j, \rho_\alpha(\phi_k) \rangle$$

Because ρ_α is scalar multiplication in V_i:

$$(g \star h)(\alpha) = \sum_{j,k=1}^{n} \hat{g}_j \bar{\hat{h}}_k \langle \phi_j, \lambda_k(\alpha) \phi_k \rangle$$

Smoothing

$$(g \star h)(\alpha) = \sum_{j,k=1}^{n} \hat{g}_j \bar{\hat{h}}_k \langle \phi_j, \lambda_k(\alpha)\phi_k \rangle$$

Again, by conjugate-linearity in the second term:

$$(g \star h)(\alpha) = \sum_{j,k=1}^{n} \hat{g}_j \bar{\hat{h}}_k \overline{\lambda_k(\alpha)} \langle \phi_j, \phi_k \rangle$$

Smoothing

$$(g \star h)(\alpha) = \sum_{j,k=1}^{n} \hat{g}_j \bar{\hat{h}}_k \overline{\lambda_k(\alpha)} \langle \phi_j, \phi_k \rangle$$

And finally, by the orthonormality of $\{\phi_1, \cdots, \phi_n\}$:

$$(g \star h)(\alpha) = \sum_{j=1}^{n} \hat{g}_j \bar{\hat{h}}_j \overline{\lambda_j(\alpha)}$$

Smoothing

$$(g \star h)(\alpha) = \sum_{j=1}^{n} \hat{g}_j \bar{\hat{h}}_j \overline{\lambda_j(\alpha)}$$

This implies that we can compute the moving dot-product by multiplying the coefficients of g and h.

Convolution/Correlation in the spatial domain is multiplication in the frequency domain!

Smoothing

What is $\lambda_j(\alpha)$?

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What is $\lambda_j(\alpha)$?

Since the representation is unitary, $|\lambda_j(\alpha)| = 1$.

$$\Downarrow$$

$$\exists \tilde{\lambda}_j : [0,2\pi) \to \mathbb{R} \quad \text{s.t.} \quad \lambda_j(\alpha) = e^{i\tilde{\lambda}_j(\alpha)}$$

Smoothing

What is $\lambda_j(\alpha)$?

$$\lambda_j(\alpha) = e^{i\tilde{\lambda}_j(\alpha)} \text{ for some } \tilde{\lambda}_j : [0,2\pi) \to \mathbb{R}.$$

Since it's a representation:

$$\Downarrow$$

$$\lambda_j(\alpha+\beta) = \lambda_j(\alpha)\cdot\lambda_j(\beta) \quad \forall \alpha,\beta \in [0,2\pi)$$

$$\Downarrow$$

$$\tilde{\lambda}_j(\alpha+\beta) = \tilde{\lambda}_j(\alpha) + \tilde{\lambda}_j(\beta)$$

$$\Downarrow$$

$$\exists \kappa_j \in \mathbb{R} \quad s.t. \quad \tilde{\lambda}_j(\alpha) = \kappa_j \cdot \alpha$$

Smoothing

What is $\lambda_j(\alpha)$?

$$\lambda_j(\alpha) = e^{i\kappa_j\alpha} \text{ for some } \kappa_j \in \mathbb{R}.$$

Since it's a representation:

$$\Downarrow$$

$$1 = \lambda_j(0) = \lambda_j(2\pi) = e^{i\kappa_j 2\pi}$$

$$\Downarrow$$

$$\kappa_j \in \mathbb{Z}$$

Smoothing

Thus, the correlation of the signals $g, h: S^1 \to \mathbb{C}$ can be expressed as:

$$(g \star h)(\alpha) = \sum_{j=1}^{n} \hat{g}_j \bar{\hat{h}}_j e^{-i\kappa_j \alpha}$$

where $\kappa_j \in \mathbb{Z}$.