FFTs in Graphics and Vision

More Math Review

Outline

Inner Product Spaces

- Real Inner Products
- Hermitian Inner Products
- Orthogonal Transforms
- Unitary Transforms
- Function Spaces

Inner Product Spaces

Given a real vector space V, a *real inner product* is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ that takes a pair of vectors and returns a real value.

Inner Product Spaces

An inner product is a map from $V \times V$ into the real numbers that is:

1. Linear: For all $u, v, w \in V$ and $\lambda \in \mathbb{R}$:
$$\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$$
$$\langle \lambda v, w \rangle = \lambda \langle v, w, \rangle$$

2. Symmetric: For all $v, w \in V$:
$$\langle v, w \rangle = \langle w, v \rangle$$

3. Positive Definite: For all $v \in V$:
$$\langle v, v \rangle \geq 0$$
$$\langle v, v \rangle = 0 \Leftrightarrow v = 0$$

Inner Product Spaces

An inner product defines a notion of distance on a vector space by setting:

$$D(v,w) = \sqrt{\langle v-w, v-w \rangle} = \|v-w\|$$

Inner Product Spaces

Examples:

1. On the space of n-dimensional arrays, the standard inner product is:

$$\langle (a_1,\cdots,a_n),(b_1,\cdots,b_n)\rangle = a_1\cdot b_1+\cdots+a_n\cdot b_n$$
$$= (a_1,\cdots,a_n)\cdot(b_1,\cdots b_n)^t$$

Inner Product Spaces

Examples:

1. On the space of n-dimensional arrays, the standard inner product is:
$$\begin{aligned}\langle (a_1,\cdots,a_n),(b_1,\cdots,b_n)\rangle &= a_1\cdot b_1+\cdots+a_n\cdot b_n\\ &= (a_1,\cdots,a_n)\cdot(b_1,\cdots b_n)^t\end{aligned}$$

2. On the space of continuous, real-valued functions, defined on a circle, the standard inner product is:
$$\langle f,g\rangle = \int_0^{2\pi} f(\theta)\cdot g(\theta)d\theta$$

Inner Product Spaces

Examples:

3. Suppose we have the space of n-dimensional arrays, and suppose we have a matrix:

$$M = \begin{pmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & \ddots & \vdots \\ M_{n1} & \cdots & M_{nn} \end{pmatrix}$$

Does the map:

$$\langle v, w \rangle_M = v^t M w$$

define an inner product?

Inner Product Spaces

Examples:

3. Does the map:

$$\langle v, w \rangle_M = v^t M w$$

define an inner product?

- Is it linear?
- Is it symmetric?
- Is it positive definite?

Inner Product Spaces

Examples:

$$\langle v, w \rangle_M = v^t M w$$

- Is it linear?

$$\begin{aligned}\langle u + v, w \rangle_M &= (u + v)^t M w \\ &= (u^t + v^t) M w \\ &= u^t M w + v^t M w \\ &= \langle u, w \rangle_M + \langle v, w \rangle_M\end{aligned}$$

Inner Product Spaces

Examples:

$$\langle v, w \rangle_M = v^t M w$$

- Is it linear?

$$\langle u + v, w \rangle_M = \langle u, w \rangle_M + \langle v, w \rangle_M$$

$$\begin{aligned}\langle \lambda v, w \rangle_M &= (\lambda v)^t M w \\ &= \lambda v^t M w \\ &= \lambda \langle v, w \rangle_M\end{aligned}$$

Inner Product Spaces

Examples:

$$\langle v, w \rangle_M = v^t M w$$

- Is it linear? Yes

$$\langle u + v, w \rangle_M = \langle u, w \rangle_M + \langle v, w \rangle_M$$

$$\langle \lambda v, w \rangle_M = \lambda \langle v, w \rangle_M$$

Inner Product Spaces

Examples:

$$\langle v, w \rangle_M = v^t M w$$

- Is it symmetric?

$$\begin{aligned} \langle w, v \rangle_M &= w^t M v \\ &= (w^t M v)^t \\ &= v^t M^t w \\ &= \langle v, w \rangle_{M^t} \end{aligned}$$

Inner Product Spaces

Examples:

$$\langle v, w \rangle_M = v^t M w$$

- Is it symmetric? Only if M is ($M = M^t$)

$$\langle w, v \rangle_M = \langle v, w \rangle_{M^t}$$

Inner Product Spaces

<u>Examples</u>:

$$\langle v, w \rangle_M = v^t M w$$

- Is it positive definite?

If M is symmetric, there exists an orthogonal basis $\{v_1, \ldots, v_n\}$ w.r.t which M is diagonal:

$$M = B^t \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & & 0 & 0 \\ \vdots & & \ddots & \vdots & \\ 0 & 0 & & \lambda_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix} B$$

Inner Product Spaces

Examples:

$$\langle v, w \rangle_M = v^t M w$$

- Is it positive definite?

If we express v in terms of this basis:

$$v = a_1 v_1 + \cdots + a_n v_n$$

then:

$$\langle v, v \rangle_M = \lambda_1 a_1^2 + \cdots + \lambda_n a_n^2$$

Inner Product Spaces

Examples:

$$\langle v, w\rangle_M = v^t M w$$

- Is it positive definite? Only if $\lambda_i > 0$ for all i.

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then:

$$\langle v, v\rangle_M = \lambda_1 a_1^2 + \cdots + \lambda_n a_n^2$$

Inner Product Spaces

Examples:

4. On the space of continuous, real-valued functions, defined on a circle, does the map:
$$\langle f,g\rangle = \int_0^{2\pi} f(\theta)\cdot g(\theta)\cdot \omega(\theta)\,d\theta$$
define an inner product?

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define an inner product? No

Inner Product Spaces

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What if $\omega(\theta) > 0$?

Inner Product Spaces

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define an inner product? No

What if $\omega(\theta) > 0$? Yes

Hermitian Inner Product Spaces

Given a complex vector space V, a *Hermitian inner product* is a function $\langle\cdot,\cdot\rangle: V \times V \to \mathbb{C}$ that takes a pair of vectors and returns a complex value.

Hermitian Inner Product Spaces

A Hermitian inner product is a map from $V \times V$ into $\mathbb{C}$ that is:

1. Linear: For all $u, v, w \in V$ and any $\lambda \in \mathbb{C}$:
$$\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$$
$$\langle \lambda v, w \rangle = \lambda \langle v, w \rangle$$

2. Conjugate Symmetric: For all $u, v \in V$:
$$\langle v, w \rangle = \overline{\langle w, v \rangle}$$

3. Positive Definite: For all $v \in V$:
$$\langle v, v \rangle \geq 0$$
$$\langle v, v \rangle = 0 \Leftrightarrow v = 0$$

Inner Product Spaces

As in the real case, a Hermitian inner product defines a notion of distance on a complex vector space by setting:

$$D(v,w) = \sqrt{\langle v-w, v-w\rangle} = \|v-w\|$$

Hermitian Inner Product Spaces

Examples:

1. On complex-valued, n-dimensional arrays, the standard Hermitian inner product is:

$$\langle (a_1, \cdots, a_n), (b_1, \cdots, b_n) \rangle = a_1 \cdot \overline{b_1} + \cdots + a_n \cdot \overline{b_n}$$
$$= (a_1, \cdots, a_n) \cdot \left(\overline{b_1}, \cdots, \overline{b_n}\right)^t$$

Hermitian Inner Product Spaces

<u>Examples</u>:

1. On complex-valued, n-dimensional arrays, the standard Hermitian inner product is:
$$\langle (a_1,\cdots,a_n),(b_1,\cdots,b_n)\rangle = a_1\cdot\overline{b_1}+\cdots+a_n\cdot\overline{b_n}$$
$$= (a_1,\cdots,a_n)\cdot\left(\overline{b_1},\cdots,\overline{b_n}\right)^t$$

2. On the space of continuous, complex-valued functions, defined on a circle, the standard Hermitian inner product is:
$$\langle f,g\rangle = \int_0^{2\pi} f(\theta)\cdot\overline{g(\theta)}\,d\theta$$

Structure Preservation

Recall:

If we have an n-dimensional vector space V then a linear map L is a function from V to V that preserves the linear structure:

$$L(a \cdot v_1 + b \cdot v_2) = a \cdot L(v_1) + b \cdot L(v_2)$$

for all $v, w \in V$ and all scalars a and b.

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for all $v, w \in V$ and all scalars a and b.

If L is invertible, then we can think of L as a function that "renames" all the elements in V while preserving the underlying vector space structure.

Structure Preservation

Orthogonal Transformations:

For a real vector space V that has an inner product, we would also like to consider those functions that "rename" the elements of V while preserving the underlying structure.

Structure Preservation

Orthogonal Transformations:

For a real vector space V that has an inner product, we would also like to consider those functions that “rename” the elements of V while preserving the underlying structure.

If R is such a function, then:

- R must be an invertible linear operator, in order to preserve the underlying vector space structure.
- R must also preserve the underlying inner product.

Structure Preservation

Orthogonal Transformations:

For a real inner-product space V, a linear operator R is called orthogonal if it preserves the inner product:

$$\langle R(v), R(w)\rangle = \langle v, w\rangle$$

for all $v, w \in V$.

Structure Preservation

Example:

On the space of real-valued, n-dimensional arrays, a matrix is orthogonal if:

$$\langle R(v), R(w)\rangle = \langle v, w\rangle$$
$$\Updownarrow$$
$$(Rv)^t(Rw) = v^t w$$
$$\Updownarrow$$
$$v^t R^t R w = v^t w$$
$$\Updownarrow$$
$$R^t = R^{-1}$$

Structure Preservation

Example:

On the space of real-valued, n-dimensional arrays, a matrix is orthogonal if:

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Structure Preservation

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On the space of real-valued, n-dimensional arrays, a matrix is orthogonal if:

$$R^t = R^{-1}$$

Note: The determinant of an orthogonal matrix always has absolute value 1:

$$\begin{aligned}[\det(R)]^2 &= \det(R) \cdot \det(R^t) \\ &= \det(R) \cdot \det(R^{-1}) \\ &= \det(RR^{-1}) \\ &= 1\end{aligned}$$

Structure Preservation

Example:

On the space of real-valued, n-dimensional arrays, a matrix is orthogonal if:

$$R^t = R^{-1}$$

Note: The determinant of an orthogonal matrix always has absolute value 1.

If the determinant of an orthogonal matrix is equal to 1, the matrix is called a rotation.

Orthogonal Matrices and Eigenvalues

If R is an orthogonal transformation and R has an eigenvalue λ, then $|\lambda| = 1$.

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If R is an orthogonal transformation and R has an eigenvalue λ, then $|\lambda| = 1$.

To see this, let v be the e.vector corresponding to the e.value λ. Since R is orthogonal, we have:

$$\begin{aligned}\langle v, v\rangle &= \langle Rv, Rv\rangle \\ &= \langle \lambda v, \lambda v\rangle \\ &= \lambda^2 \langle v, v\rangle\end{aligned}$$

so $\lambda^2 = 1$.

Structure Preservation

Unitary Transformations:

For a complex vector space V, an invertible linear operator R is called unitary if it preserves the hermitian inner product:

$$\langle Rv, Rw \rangle = \langle v, w \rangle$$

for all $v, w \in V$.

Structure Preservation

Example:

On the space of complex-valued, n-dimensional arrays, a matrix is unitary if:

$$\langle Rv, Rw\rangle = \langle v, w\rangle$$

$$\Updownarrow$$

$$(Rv)^t\overline{(Rw)} = v^t\overline{w}$$

$$\Updownarrow$$

$$v^t R^t \overline{R}\,\overline{w} = v^t\overline{w}$$

$$\Updownarrow$$

$$\overline{R}^t = R^{-1}$$

Structure Preservation

Example:

On the space of complex-valued, n-dimensional arrays, a matrix is unitary if:

$$\bar{R}^t = R^{-1}$$

Structure Preservation

Example:

On the space of complex-valued, n-dimensional arrays, a matrix is unitary if:

$$\bar{R}^t = R^{-1}$$

Note: The determinant of a unitary matrix always has norm 1:

$$\begin{aligned} \|\det(R)\|^2 &= \det(R) \cdot \det(\bar{R}^t) \\ &= \det(R) \cdot \det(R^{-1}) \\ &= \det(RR^{-1}) \\ &= 1 \end{aligned}$$

Unitary Matrices and Eigenvalues

If R is a unitary transformation and R has an eigenvalue λ, then $\|\lambda\| = 1$.

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To see this, let v be the e.vector corresponding to the e.value λ. Since R is unitary, we have:

$$\begin{aligned}\langle v, v\rangle &= \langle Rv, Rv\rangle \\ &= \langle \lambda v, \lambda v\rangle \\ &= \lambda\bar{\lambda}\langle v, v\rangle \\ &= \|\lambda\|^2\langle v, v\rangle\end{aligned}$$

so $\|\lambda\| = 1$.

Function Spaces

In this course, the vector spaces we will be looking at most often are the vector spaces of functions defined on some domain:

- Continuous functions on the unit circle (S^1)
- Continuous functions on the unit disk (D^2)
- Continuous, periodic functions on the plane ($\mathbb{R}^2$)
- Continuous functions on the unit sphere (S^2)
- Continuous functions on the unit ball (B^3)

Function Spaces

Continuous functions on the unit circle (S^1):

This is the set of points (x, y) s.t. $x^2 + y^2 = 1$.

If we have functions $f(x, y)$ and $g(x, y)$ then:

$$\langle f, g\rangle = \int_{p\in S^1} f(p) \cdot \bar{g}(p)\, dp$$

Function Spaces

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If we have functions $f(x, y)$ and $g(x, y)$ then:

$$\langle f, g \rangle = \int_{p \in S^1} f(p) \cdot \bar{g}(p)\, dp$$

Or, we can represent points on the circle in terms of angle $\theta \in [0,2\pi)$:

$$\theta \to (\cos\theta, \sin\theta)$$

For functions $f(\theta)$ and $g(\theta)$ the inner product is:

$$\langle f, g \rangle = \int_0^{2\pi} f(\theta) \cdot \bar{g}(\theta)\, d\theta$$

Function Spaces

Continuous functions on the unit disk (D^2):

This is the set of points (x, y) s.t. $x^2 + y^2 \leq 1$.

If we have functions $f(x, y)$ and $g(x, y)$ then:

$$\langle f, g \rangle = \int_{p \in D^2} f(p) \cdot \bar{g}(p) \, dp$$

Function Spaces

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If we have functions $f(x, y)$ and $g(x, y)$ then:

$$\langle f, g \rangle = \int_{p \in D^2} f(p) \cdot \bar{g}(p)\, dp$$

Or, we can represent points on the circle in terms of radius $r \in [0,1]$ and angle $\theta \in [0,2\pi)$:

$$(r, \theta) \rightarrow (r \cdot \cos\theta, r \cdot \sin\theta)$$

For functions $f(r, \theta)$ and $g(r, \theta)$ the inner product is:

$$\langle f, g \rangle = \int_0^{2\pi} \int_0^1 f(r, \theta) \cdot \bar{g}(r, \theta) \cdot r\, dr\ d\theta$$

Function Spaces

Continuous, periodic functions on the plane ($\mathbb{R}^2$):

This is the set of functions $f(x,y)$ s.t.:

$$f(x,y) = f(x+2\pi, y) = f(x, y+2\pi)$$

If we have functions $f(x,y)$ and $g(x,y)$ then:

$$\langle f, g \rangle = \int_0^{2\pi} \int_0^{2\pi} f(x,y) \cdot \bar{g}(x,y)\, dy\ dx$$

Function Spaces

Continuous functions on the unit sphere (S^2):

This is the set of points (x, y, z) s.t. $x^2 + y^2 + z^2 = 1$.

If we have functions $f(x, y, z)$ and $g(x, y, z)$ then:

$$\langle f, g \rangle \int_{p \in S^2} f(p) \cdot \bar{g}(p)\, dp$$

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Or, we can represent points on the sphere in terms of spherical angle $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$:

$$(\theta, \phi) \rightarrow (\sin\theta \cdot \cos\phi, \cos\theta, \sin\theta \cdot \sin\phi)$$

For functions $f(\theta, \phi)$ and $g(\theta, \phi)$ the inner product is:

$$\langle f, g \rangle = \int_0^{2\pi} \int_0^{\pi} f(\theta, \phi) \cdot \bar{g}(\theta, \phi) \cdot \sin(\theta)\ d\theta\ d\phi$$

Function Spaces

Continuous functions on the unit ball (B^3):

This is the set of points (x, y, z) s.t. $x^2 + y^2 + z^2 \leq 1$.

If we have functions $f(x, y, z)$ and $g(x, y, z)$ then:

$$\langle f, g \rangle \int_{p \in B^3} f(p) \cdot \bar{g}(p)\, dp$$

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Or, representing points in the ball in terms of radius $r \in [0,1]$ and spherical angle $\theta \in [0, \pi]$, $\phi \in [0,2\pi)$:

$$(r, \theta, \phi) \to (r \cdot \sin\theta \cdot \cos\phi\,, r \cdot \cos\theta, r \cdot \sin\theta \cdot \sin\phi)$$

For functions $f(r, \theta, \phi)$ and $g(r, \theta, \phi)$ then:

$$\langle f, g \rangle = \int_0^{2\pi} \int_0^{\pi} \int_0^{1} f(r, \theta, \phi) \cdot \bar{g}(r, \theta, \phi) \cdot r^2 \cdot \sin(\theta)\ dr\ d\theta\ d\phi$$

Function Spaces

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If we consider the space of continuous, complex-valued functions on the unit circle:

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Function Spaces

Examples

If we consider the space of continuous, complex-valued functions on the unit circle:

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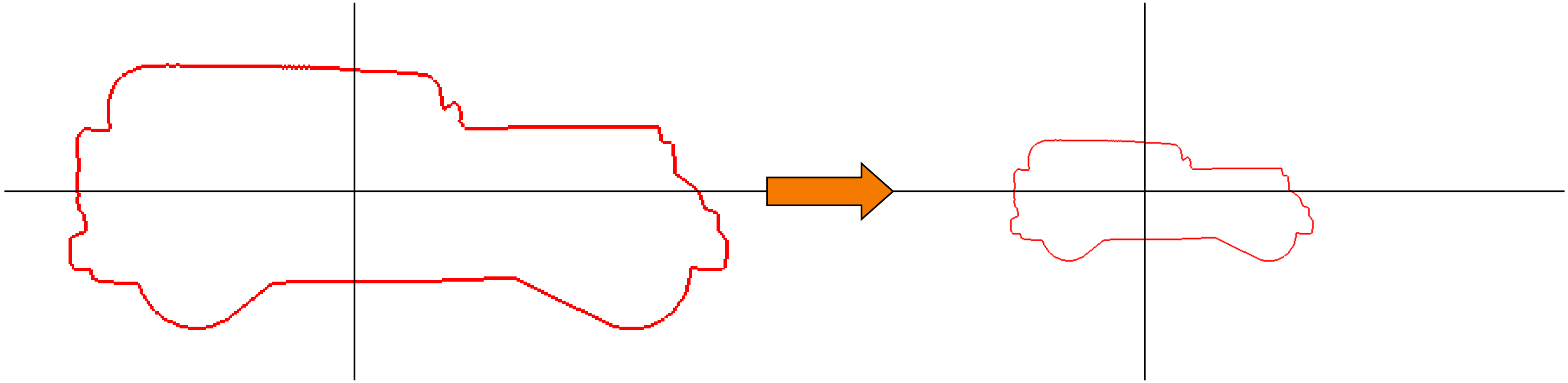

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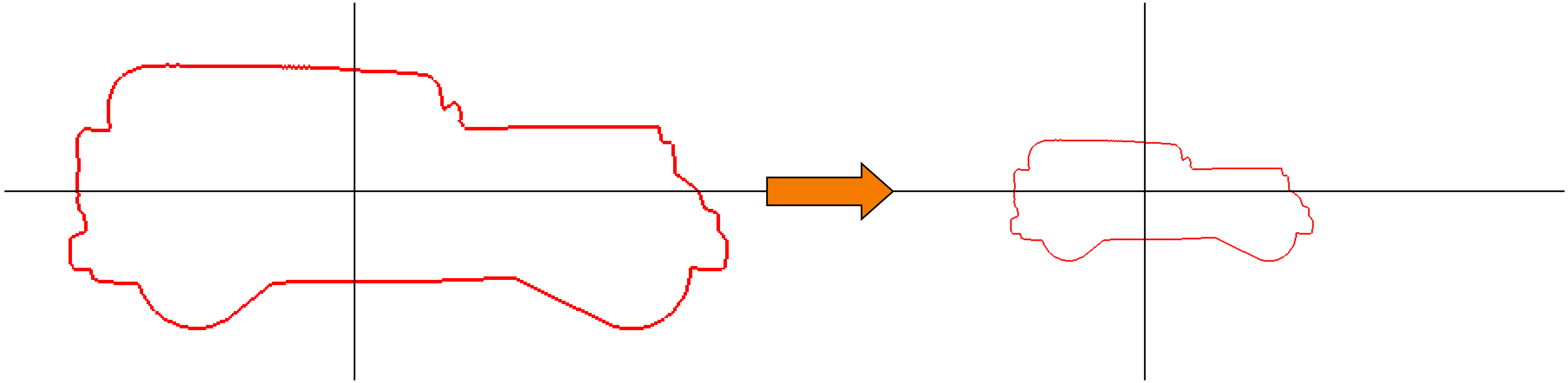

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Function Spaces

Examples

If we consider the space of continuous, complex-valued functions on the unit circle:

- For any 2D rotation R is the transformation:

  $$f(p) \to f(R^{-1}p)$$

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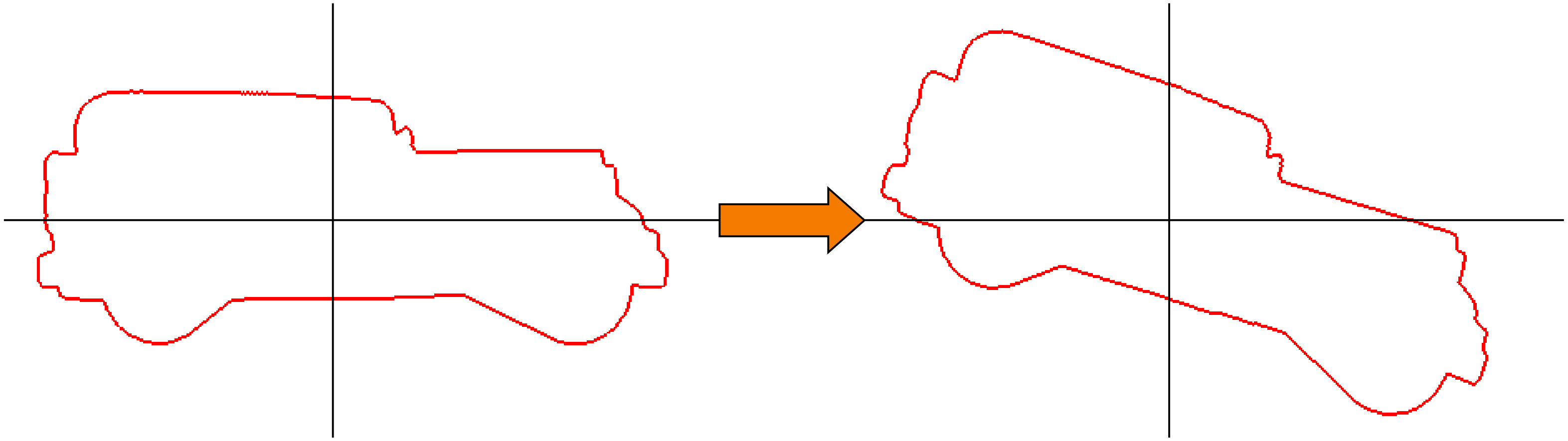

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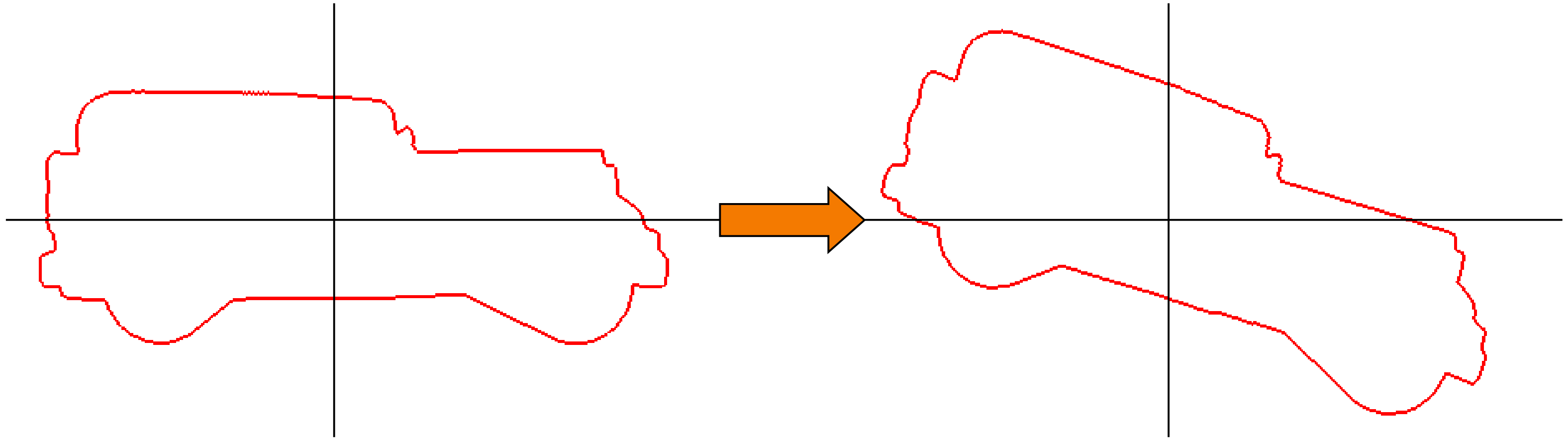

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Function Spaces

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If we consider the space of continuous, periodic, complex-valued functions on the plane:

- For any 2D point (x_0, y_0), is the transformation:

  $$f(x,y) \to f(x-x_0, y-y_0)$$

  a linear transformation?

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  a linear transformation? **Yes**

- Is it unitary? **Yes**

Function Spaces

<u>Examples</u>

If we consider the space of continuous, infinitely-differentiable, periodic, complex-valued functions on the plane:

- Is differentiation with respect to x:
$$f(x,y) \rightarrow \frac{\partial}{\partial x} f(x,y)$$
a linear transformation?

Function Spaces

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Function Spaces

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Function Spaces

Examples

If we consider the space of continuous, complex-valued functions on the sphere:

- For any rotation R, is the transformation:
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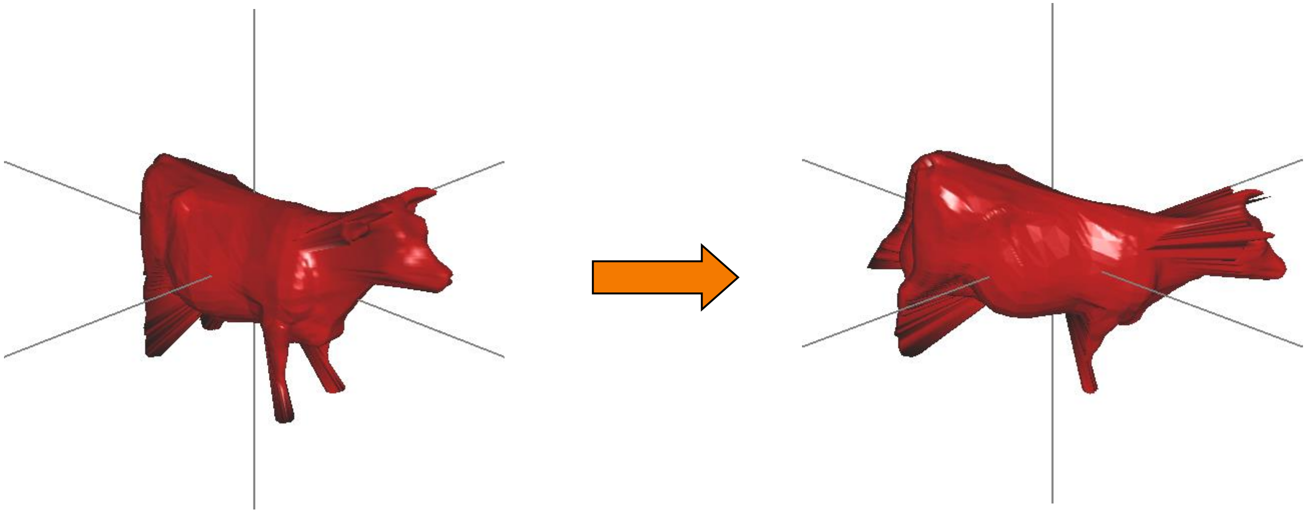

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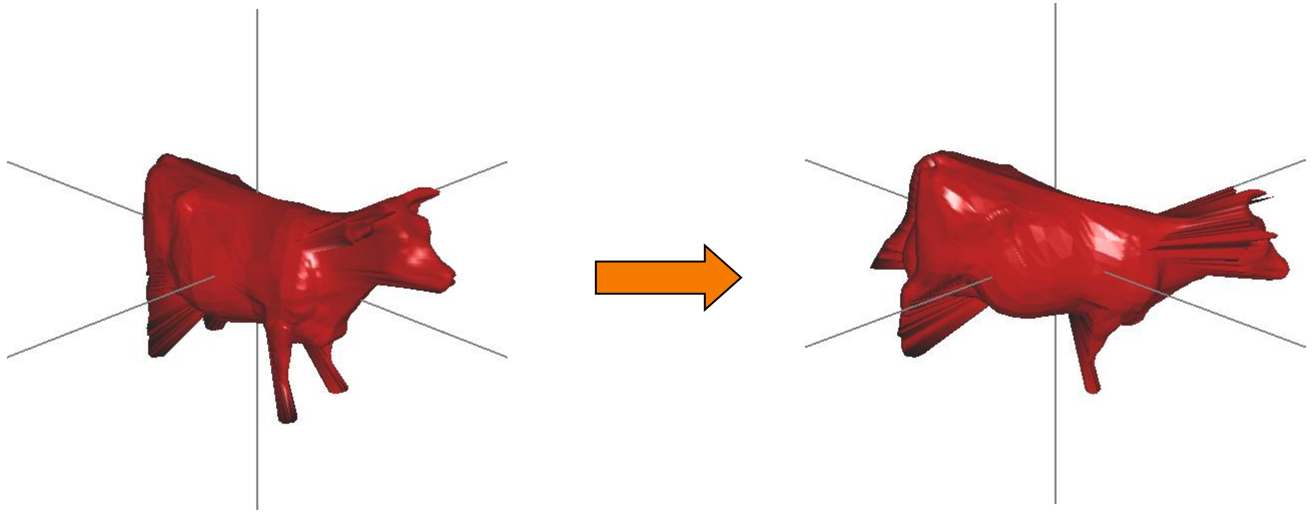

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