Variance Analysis for Monte Carlo Integration

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Monte Carlo Integration

\[ \int_{[0,1]^2} f(x) \, dx \]
Monte Carlo Integration

\[ \int_{[0,1]^2} f(x) \, dx \approx \frac{1}{N} \sum_{i=1}^{N} f(x_i) \]
Light Simulation: on Surface
Light Simulation: on Surface
Light Simulation: Participating Media
Regular Sampling Pattern

Euclidean  Spherical  Hemispherical

[Marques et al. 2013]
Purely Random Sampling Pattern

Euclidean    Spherical    Hemispherical
Jittered Sampling Pattern

Euclidean  Spherical  Hemispherical
Poisson Disk Sampling Pattern

Euclidean  Spherical  Hemispherical
Ambient Occlusion

Geometric Aliasing

Image Plane
Ambient Occlusion

Geometric Aliasing

Image Plane
Error: Structure Artifacts

Hemisphere

Same Hemispherical pointset at all hitpoints

Image Plane
Error: Noise

With rotation

Image Plane
Error: Structures to Noise

No rotation

With rotation
Homogeneous Sampling Pattern

Purely random samples
Homogeneous Sampling Pattern

Statistically invariant properties over the domain

Purely random samples
Homogeneous Sampling Pattern

Statistically invariant properties over the domain

Widesense stationary

[Dippe and Wold 1985]

Purely random samples
Homogeneous Sampling Pattern

Statistically invariant properties over the domain

Widesense stationary

[Dippe and Wold 1985]

All sampling patterns derived from white noise

Purely random samples
Regular Samples

Realisation 1  
Realisation 2  
Realisation 3  

Homogenization by Random Translation

Homogenized

Multiple realisations
Jittered Samples

1. Regular
2. Realisation 1
3. Realisation 2
4. Realisation 3

5. Homogenization by Random Translation

6. Homogenized
7. Multiple realisations
Homogeneous Samples

Realisation 1

Realisation 2

Realisation 3

Homogenization by Random Rotation

Regular

Homogenized

Multiple realisations
Error in Terms of Variance

Error = Bias^2 + Variance
Error in Terms of Variance

Error = Bias^2 + Variance

Homogeneous Sampling:

Bias $\rightarrow$ Zero
Error in Terms of Variance

Error = \text{Bias}^2 + \text{Variance}

Homogeneous Sampling:

Bias \rightarrow Zero

Implies:

Error = \text{Variance}
Nature of Noise

Purely Random
MSE: $4.79 \times 10^{-3}$

Jittered
MSE: $8.56 \times 10^{-4}$

Regular
MSE: $3.95 \times 10^{-4}$

(96 hemispherical samples)
Variance in Integration

Homogeneous Sampling Patterns
How can we characterize sampling patterns?

Variance in Integration

Homogeneous Sampling Patterns
Previous Work on Fourier Analysis of Sampling Patterns

- Many prior works [Dippé and Wold 1985], [Cook 1986], [Ulichney 1987]
Previous Work on Fourier Analysis of Sampling Patterns

- Many prior works [Dippé and Wold 1985], [Cook 1986], [Ulichney 1987]
- Error relates to the frequency content of samples, [Durand 2011]
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- Many prior works [Dippé and Wold 1985], [Cook 1986], [Ulichney 1987]
- Error relates to the frequency content of samples, [Durand 2011]
- Relates variance directly to the variance of Samples’ Fourier Coefficients [Subr and Kautz 2013]
Regular Sampling Pattern

Samples

Power Spectrum
Purely Random Sampling Pattern

Samples

Power Spectrum
Poisson Disk Sampling Pattern

Samples

Power Spectrum
Jittered Sampling Pattern

Samples

Power Spectrum
Radial Averging of Power Spectrum

(Jittered Sampling Pattern)
Mean Angular Power Spectrum

Jittered Sampling Pattern
Variance in Integration

Homogeneous Samples
+ Frequency content (Power Spectra)
Variance Formulation in Euclidean Domain

\[ \text{Var}(I_N) = \frac{\mu(T^d)\mu(S^{d-1})}{N} \int_0^\infty \rho^{d-1} \tilde{P}_S(\rho) \tilde{P}_F(\rho) d\rho \]
Variance Formulation in Euclidean Domain

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\[ \tilde{P}_S(\rho) = \text{(Jittered Sampling Pattern)} \]
Variance Formulation in Spherical Domain

\[ \text{Var}(I_N) = \frac{\mu(S^2)}{N} \sum_{l=1}^{\infty} (2l + 1) \tilde{P}_S(l) \tilde{P}_F(l) \]
Variance Formulation in Spherical Domain

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\[ \tilde{P}_S(\omega) = \omega = l/\alpha \sqrt{N} \]

(Jittered Sampling Pattern)
Variance: Product of $\mathcal{P}_s(\cdot)$ and $\mathcal{P}_f(\cdot)$
Variance: Product of $\mathcal{P}_S(\cdot)$ and $\mathcal{P}_F(\cdot)$

\[
Var(I_N) \propto \frac{1}{N} \int_0^\infty \rho^{d-1} \mathcal{P}_S(\rho) \mathcal{P}_F(\rho) d\rho
\]
Variance: Product of $\mathcal{P}_S(\cdot)$ and $\mathcal{P}_F(\cdot)$

Euclidean

$$Var(I_N) \propto \frac{1}{N} \int_0^\infty \rho^{d-1} \mathcal{P}_S(\rho) \mathcal{P}_F(\rho) d\rho$$

Spherical

$$Var(I_N) \propto \frac{1}{N} \sum_{l=1}^{\infty} (2l + 1) \mathcal{P}_S(l) \mathcal{P}_F(l)$$
Dependence on Number of Samples

For a given number of samples

Integrand Power Spectrum

Sampling Power Spectrum
Dependence on Number of Samples

For a given number of samples
Dependence on Number of Samples

For a given number of samples

Increasing the number of samples

Integrand Power Spectrum

Sampling Power Spectrum

Product

Scaled Power Spectrum
\[
Var(I_N) = \frac{\mu(T^d)\mu(S^{d-1})}{N} \int_0^\infty \rho^{d-1} \tilde{P}_S(\rho) \tilde{P}_F(\rho) d\rho
\]

Euclidean
$$\text{Var}(I_N) = \frac{\mu(T^d)\mu(S^{d-1})}{N} \int_0^\infty \rho^{d-1} \hat{P}_S(\rho) \hat{P}_F(\rho) d\rho$$

Euclidean
\[ Var(I_N) = \frac{\mu(T^d) \mu(S^{d-1})}{N} \int_0^\infty \rho^{d-1} \tilde{P}_S(\rho) \tilde{P}_F(\rho) d\rho \]
Integrand Power Spectrum

\[ \hat{P}_F(\rho) = \begin{cases} 
c_F & \rho < \rho_0 
c'_F \rho^{-d-1} & \text{otherwise}
\end{cases} \]

where, \( c_F \) and \( c'_F \) are constants

[Brandolini et al 2001, Mean square decay of Fourier transforms in euclidean and non euclidean spaces]
Theoretical Convergence Analysis

\[ b = 0 \]

\[ \alpha \sqrt{\frac{d}{N}} \]

\[ O \left( \frac{1}{N} \right) \]

- \( b \) = degree of the polynomial
- \( d \) = dimensions
- \( N \) = number of samples
Theoretical Convergence Analysis

\[ \Phi_{S}(\rho) \]

\[ \bar{P}_{S}(\rho) \]

\[ Var(I_{N}) = O \left( \frac{1}{N} \right) \]

\[ O \left( \frac{1}{N^{d/b}} \right) \]

\[ b = \text{degree of the polynomial} \]

\[ d = \text{dimensions} \]

\[ N = \text{number of samples} \]
Theoretical Convergence Analysis

\[ b = 0 \]

\[ 0 < b \leq 1 \]

\[ b \geq 1 \]

\[ \hat{P}_S(\rho) \]

\[ \text{Var}(I_N) \]

\[ O \left( \frac{1}{N} \right) \]

\[ O \left( \frac{1}{N^{d+1/N^b}} \right) \]

\[ O \left( \frac{1}{N^{d+1/N^b}} \right) \]

\( b = \text{degree of the polynomial} \)

\( d = \text{dimensions} \)

\( N = \text{number of samples} \)
Theoretical Convergence Analysis

\( b = 0 \)  \( 0 < b \leq 1 \)  \( b \geq 1 \)  \( b \to \infty \)

\( O \left( \frac{1}{N} \right) \)  \( O \left( \frac{1}{N^{d/\sqrt{N}}^b} \right) \)  \( O \left( \frac{1}{N^{d/\sqrt{N}}} \right) \)  \( O \left( \frac{1}{N^{d/\sqrt{N}}} \right) \)

\( b \) = degree of the polynomial  \( d \) = dimensions  \( N \) = number of samples
Power Spectrum Bounds
Power Spectrum Bounds

With Bounds

Jittered
Convergence Rate Analysis

Variance convergence rate: $O \left( \frac{1}{N \sqrt{N}} \right)$

Power Spectrum

Convergence rate

Variance

Jittered
Convergence Rate Analysis

Power Spectrum

Convergence rate

Variance convergence rate: $O\left(\frac{1}{N\sqrt{N}}\right)$
Convergence Rate Analysis
Jittered vs Poisson Disk Sampling
Why is jittered sampling better than Poisson Disk sampling?
Power Spectra

Poisson Disk

Jittered
Power Spectra: Low Frequency Region

Poisson Disk $o\left(\frac{1}{N}\right)$

Jittered $o\left(\frac{1}{N\sqrt{N}}\right)$

Constant Offset

Approaching Zero
CCVT [Balzer et al. 2009]

Variance Convergence Rate: $O \left( \frac{1}{N\sqrt{N}} \right)$
CCVT [Balzer et al. 2009]

Variance Convergence Rate: $O\left(\frac{1}{N\sqrt{N}}\right)$
Our mathematical model can be used to tailor new sampling patterns.
Novel Contributions

- Frequency analysis of spherical and hemispherical samples using spherical harmonics
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- Unified closed form variance expression that can be used to design new sampling patterns
Novel Contributions

- Frequency analysis of spherical and hemispherical samples using spherical harmonics
- Unified closed form variance expression that can be used to design new sampling patterns
- Analysis tool to theoretically compute and bound variance convergence rates of any stochastic sampler
Future Work

- Extend our mathematical framework to adaptive sampling strategies
Future Work

- Extend our mathematical framework to adaptive sampling strategies
- Explore how we can extend our mathematical model to deterministic sampling patterns
Future Work

- Extend our mathematical framework to adaptive sampling strategies
- Explore how we can extend our mathematical model to deterministic sampling patterns
- Use our framework to construct new sampling patterns with the best convergence speed and with lowest variance even for small number of samples
Our tools will be made public very soon.

http://liris.cnrs.fr/variance
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Thank you for your attention.

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\text{Var}(I_N) \propto \frac{1}{N} \int_0^\infty \rho^{d-1} \tilde{P}_S(\rho) \tilde{P}_F(\rho) \, d\rho
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\[
\text{Var}(I_N) \propto \frac{1}{N} \sum_{l=1}^\infty (2l + 1) \tilde{P}_S(l) \tilde{P}_F(l)
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Thank you for your attention.

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\text{Var}(I_N) \propto \frac{1}{N} \int_0^\infty \rho^{d-1} \tilde{P}_S(\rho) \tilde{P}_F(\rho) d\rho
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Convergence Rate Analysis

Convergence rate

Power Spectrum

Convergence rate

\[ O \left( \frac{1}{N} \right) \]

\[ O \left( \frac{1}{N \sqrt{N}} \right) \]
Step Sampling Pattern

\[ O(N^{-1.5}) \]
Convergence Rate Analysis

BNOT de Goes et al. 2012

CCVT Balzer et al. 2009

FPO Schlomer et al. 2011

Purely random samples
Light Simulation: on Surface
Light Simulation: on Surface
Light Simulation: Participating media

Light Source → Sphere → Image Plane

Eye
Ambient Occlusion

Structural Artifacts

Image Plane
Ambient Occlusion

Structural Artifacts

Image Plane
Euclidean 2D

Spherical
Regular samples

Structural Artifacts

Image Plane
Jittered Sampling Pattern

Samples

Power Spectrum
Radial averaging of Power Spectrum

Jittered sampling Pattern
Dependence on Number of Samples

For a given number of samples
Dependence on Number of Samples

For a given number of samples

Increase in number of samples
Dependence on Number of Samples

For a given number of samples
Dependence on Number of Samples

For a given number of samples:
- Integrand Power Spectrum
- Sampling Power Spectrum

Increase in number of samples:
- Integrand Power Spectrum
- Scaled Power Spectrum
Low Frequency zone
Low Frequency zone
Low Frequency zone

For a given number of samples
For a given number of samples

Increase in number of samples

Low Frequency zone
Regular Sampling Pattern

Samples

Power Spectrum
Variance Analysis for Monte Carlo Integration

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Previous Work
Previous Work

Durand [2011]
A Frequency analysis of Monte-Carlo and other numerical integration schemes

Error relates to the frequency content of samples
Previous Work

Durand [2011]
A Frequency analysis of Monte-Carlo and other numerical integration schemes

Subr and Kautz [2013]
Fourier analysis of stochastic sampling strategies for assessing bias and variance in integration

Error relates to the frequency content of samples

Relates variance directly to the variance of Samples’ Fourier Coefficients
Integrand Power Spectrum

\[ \tilde{P}_F(\rho) = \begin{cases} 
    c_F & \rho < \rho_0 \\
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where, \( c_F \) and \( c'_F \) are constants

Brandolini et al. [2001] Mean square decay of Fourier transforms in euclidean and non euclidean spaces.
Convergence Rate Analysis

**Poisson Disk**

- Power Spectrum
- Convergence rate

**Jittered**

- Power Spectrum
- Convergence rate

\[ O\left(\frac{1}{N}\right) \]

\[ O\left(\frac{1}{N\sqrt{N}}\right) \]
Power Spectrum Bounds

Poisson Disk
Power Spectrum Bounds

Poisson Disk

With Bounds

Poisson Disk

With Bounds