Shading and Visibility

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(601.457/657)

HB 13.2 -- 13.8, 14.5
FvDFH 15.4, 15.5, 15.6, 15.7.1, 16.2
Announcements

We will have a midterm next Wednesday (10/14/20)

- Content will be everything we cover through this week
3D Rendering Pipeline (for direct illumination)

- 3D Primitives
  - 3D Modeling Coordinates
  - Modeling Transformation
  - 3D World Coordinates
  - Viewing Transformation
  - 3D Camera Coordinates
  - Lighting
  - 3D Camera Coordinates
  - Projection Transformation
  - 2D Screen Coordinates
  - Clipping
  - 2D Screen Coordinates
  - Viewport Transformation
  - 2D Image Coordinates
  - Scan Conversion
  - 2D Image Coordinates
  - Image

3D Model

2D Window

2D Screen
3D Rendering Pipeline (for direct illumination)

3D Primitives

Modeling Transformation
3D Modeling Coordinates

Viewing Transformation
3D World Coordinates

3D Camera Coordinates

Lighting

Projection Transformation

3D Camera Coordinates

2D Screen Coordinates

Clipping

Viewport Transformation

2D Screen Coordinates

2D Image Coordinates

Scan Conversion

3D Image Coordinates

Image

2D Window

3D Model

2D Screen
3D Rendering Pipeline (for direct illumination)

3D Primitives
  → 3D Modeling Coordinates

Modeling Transformation
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Viewing Transformation
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Projection Transformation
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Viewport Transformation
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Scan Conversion
  → 2D Image Coordinates

Image
3D Rendering Pipeline (for direct illumination)

3D Primitives → 3D Modeling Coordinates
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    Scan Conversion
    → 2D Image Coordinates

3D Model

2D Window

2D Screen
Overview

• Scan conversion
  ◦ Figure out which pixels to fill

• Shading
  ◦ Determine a color for each filled pixel

• Depth test
  ◦ Determine when the color of a pixel comes from the front-most primitive
Polygon Shading

- Can take advantage of spatial coherence
  - Illumination calculations for pixels covered by same primitive are related to each other

\[
I = I_E + K_A I_{AL} + \sum_i \left( K_D \langle \vec{N}, \vec{L}_i \rangle I_i + K_S \langle \vec{V}, \vec{R}_i \rangle^n I_i \right)
\]
Polygon Shading Algorithms

- Flat Shading
- Gouraud Shading
- Phong Shading
Flat Shading

- Take advantage of spatial coherence
  - Make the lighting equation constant over the surface of each primitive

<table>
<thead>
<tr>
<th></th>
<th>Surface Normal</th>
<th>Light Direction</th>
<th>View Direction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Emissive</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Ambient</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Diffuse</td>
<td>+</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>Specular</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
</tbody>
</table>

\[ I = I_E + K_A I_{AL} + \sum_i \left(K_D \langle \hat{N}, \hat{L}_i \rangle I_i + K_S \langle \hat{V}, \hat{R}_i \rangle^n I_i \right) \]
Flat Shading

- Take advantage of spatial coherence
  - Make the lighting equation constant over the surface of each primitive
    - If the normal is constant over the primitive, and
    - If the light is directional,
      \[ I = I_E + K_A I_{AL} + \sum_i \left( K_D \langle \vec{N}, \vec{L}_i \rangle I_i + K_S \langle \vec{V}, \vec{R}_i \rangle^n I_i \right) \]

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Flat Shading

• Take advantage of spatial coherence
  ◦ Make the lighting equation constant over the surface of each primitive

  • If the normal is constant over the primitive, and
  • If the light is directional,
  ⇒ The diffuse component is the same for all points on the primitive

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</table>
| **I** = **I_E** + **K_A** **I_AL** + ∑ (∑ _i_ **K_D** (N, **L_i**) **I_i** + **K_S** (V, **R_i**)_n **I_i**)
Flat Shading

• Take advantage of spatial coherence
  ◦ Make the lighting equation constant over the surface of each primitive

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- If the normal is constant over the primitive, and
- If the light is directional,
  ⇒ The diffuse component is the same for all points on the primitive

- If the normal is constant over the primitive,
- If the light is directional, and
- If the direction to the viewer is constant over the primitive
  ⇒ The specular component is the same for all points on the primitive

\[
I = I_E + K_A I_{AL} + \sum_i \left( K_D \langle \vec{N}, \vec{L}_i \rangle I_i + K_S \langle \vec{V}, \vec{R}_i \rangle^n I_i \right)
\]
Flat Shading

• Take advantage of spatial coherence
  ◦ Make the lighting equation constant over the surface of each primitive
    • If the normal is constant over the primitive, and
    • If the light is directional,
    ⇒ The diffuse component is the same for all points on the primitive
    • If the normal is constant over the primitive, and
    • If the light is directional, and
    • If the direction to the viewer is constant over the primitive
    ⇒ The specular component is the same for all points on the primitive
Flat Shading

- Illuminate as though all light sources are directional, the polygon is flat, and the camera uses parallel projection
  - $\langle \mathbf{N}, \mathbf{L}_i \rangle$ constant over surface
  - $\langle \mathbf{V}, \mathbf{R}_i \rangle$ constant over surface
  - $I_i$ constant over surface

$$I = I_E + K_A I_{AL} + \sum_i \left( K_D \langle \mathbf{N}, \mathbf{L}_i \rangle I_i + K_S \langle \mathbf{V}, \mathbf{R}_i \rangle^n I_i \right)$$
Flat Shading

- One lighting calculation **per polygon**
  - Assign all pixels inside each polygon the same color
Flat Shading

- Objects look like they are composed of polygons
  - OK for faceted objects
  - Not so good for smooth surfaces

Although this is the “simplest” lighting model, it is tricky to implement this on the graphics card.
Polygon Shading Algorithms

- Flat Shading
- **Gouraud Shading**
- Phong Shading
Gouraud Shading

- Represent a polygonal mesh with a normal at each vertex

\[ I = I_E + K_A I_{AL} + \sum_i \left( K_D \langle \vec{N}, \vec{L}_i \rangle I_i + K_S \langle \vec{V}, \vec{R}_i \rangle^n I_i \right) \]
Gouraud Shading

- One lighting calculation per vertex
  - Assign pixel colors inside polygon by interpolating colors computed at vertices
Gouraud Shading

- Linearly interpolate colors at vertices down and across scan lines

\[ A = (1 - \alpha) \cdot I_1 + \alpha \cdot I_2 \]

\[ B = (1 - \beta) \cdot I_2 + \beta \cdot I_3 \]

\[ I = (1 - \gamma) \cdot A + \gamma \cdot B \]
Gouraud Shading

- Linearly interpolate colors at vertices down and across scan lines

\[ A = (1 - \alpha) \cdot I_1 + \alpha \cdot I_2 \]

\[ B = (1 - \beta) \cdot I_2 + \beta \cdot I_3 \]

**Note:** The values of \( \alpha \) and \( \beta \) only need to be updated as we move to the next scan-line. The value of \( \gamma \) needs to be updated as we advance along the scan-line.
Gouraud Shading

- Produces smoothly shaded polygonal mesh
  - Continuous shading over adjacent polygons

This is the lighting model implemented on the graphics card as part of the fixed pipeline.
Gouraud Shading

• Produces smoothly shaded polygonal mesh
  ◦ Continuous shading over adjacent polygons

What happens with large polygon & spotlight?
Gouraud Shading

• Produces smoothly shaded polygonal mesh
  ◦ Continuous shading over adjacent polygons

What happens with large polygon & spotlight?
Polygon Shading Algorithms

• Flat Shading
• Gouraud Shading
• **Phong Shading**
**Phong Shading**

- One lighting calculation **per pixel**
  - Approximate surface normals for points inside polygons by linear interpolation of normals from vertices

\[
I = I_E + K_A I_{AL} + \sum_i \left( K_D \langle \vec{N}, \vec{L}_i \rangle I_i + K_S \langle \vec{V}, \vec{R}_i \rangle^n I_i \right)
\]
Phong Shading

- Linearly interpolate surface normals at vertices down and across scan lines

\[ \vec{A} = (1 - \alpha) \cdot \vec{N}_1 + \alpha \cdot \vec{N}_2 \]

\[ \vec{B} = (1 - \beta) \cdot \vec{N}_2 + \beta \cdot \vec{N}_3 \]

\[ \vec{N} = \frac{(1 - \gamma) \cdot \vec{A} + \gamma \cdot \vec{B}}{\| (1 - \gamma) \cdot \vec{A} + \gamma \cdot \vec{B} \|} \]
Phong Shading

• Linearly interpolate surface normals at vertices down and across scan lines

• Compute lighting at each pixel
Phong Shading

- Linearly interpolate surface normals at vertices down and across scan lines
- Compute lighting at each pixel

This was not supported in early generation graphic cards but can now be implemented in the fragment shader of the GPU.
Polygon Shading Algorithms

Wireframe

Flat

Gouraud

Phong
3D Rendering Pipeline (for direct illumination)

3D Primitives

- 3D Modeling Coordinates
  - Modeling Transformation
  - 3D World Coordinates
    - Viewing Transformation
    - 3D Camera Coordinates
      - Lighting
      - 3D Camera Coordinates
        - Projection Transformation
        - 2D Screen Coordinates
          - Clipping
          - 2D Screen Coordinates
            - Viewport Transformation
            - 2D Image Coordinates
              - Scan Conversion
              - Image

3D Model

2D Window

2D Screen
Overview

• Scan conversion
  ○ Figure out which pixels to fill

• Shading
  ○ Determine a color for each filled pixel

• Depth test
  ○ Determine when the color of a pixel comes from the front-most primitive
Hidden Surface Removal (HSR)

• Motivation

• Algorithms for HSR
  ○ Back-face detection
  ○ Depth sort
  ○ Ray casting
  ○ z-buffer
Motivation

In general, we don’t want to draw surfaces that are not visible to the viewer:

• Surfaces may be back-facing
• Surfaces may be covered in 3D
• Surfaces may be covered in the image plane
3D Rendering Pipeline

Somewhere in here we have to decide which objects are visible, and which are hidden.
Visibility algorithms

Figure 29. Characterization of ten opaque-object algorithms & Comparison of the algorithms.
Overview

• Motivation

• Algorithms for HSR
  ◦ Back-face detection
  ◦ BSP-Trees
  ◦ Ray casting
  ◦ z-buffer
Back-face detection

Q: How do we test for back-facing polygons?

A: Dot product of the normal and view directions.

If $\langle \vec{V}, \vec{N} \rangle > 0$, then polygon is back-facing
Back-face detection

This method:
× Does not eliminate overlapping primitives
× Does not work for non-solid models and/or models without a well-defined orientation.

In general, back-face expected to remove $\approx$ half of polygon surfaces from removal further visibility tests
A polygon is back-facing if $\langle \vec{V}, \vec{N} \rangle > 0$
A polygon is back-facing if $\langle \vec{V}, \vec{N} \rangle > 0$

Note: When your graphics card does this, it does not use the normals you provide at the vertices for lighting. It uses the cross-product of the triangle edges, so make sure that the ordering of the vertices is consistent.

By default triangles/polygons are back-facing if the vertices are in clockwise order when viewed from the camera.
View-frustrum culling

If the shape is outside the viewing volume, we don’t have to draw it.
View-frustrum culling

If the shape is outside the viewing volume, we don’t have to draw it.
View-frustum culling

3D Primitives

Modeling Transformation

Viewing Transformation

Lighting

Projection Transformation

Clipping

Viewport Transformation

Scan Conversion

Image

Trivial Reject

3D Modeling Coordinates

3D World Coordinates

3D World Coordinates

3D Camera Coordinates

2D Screen Coordinates

2D Screen Coordinates

2D Screen Coordinates

2D Image Coordinates

2D Image Coordinates

2D Image Coordinates
Ideal Solution

Painter’s Algorithm:

- Sort primitives front to back and draw the back ones first, over-writing pixel values with information from the front primitives as they are processed.

Problem:

- In general you can’t sort the primitives.
- ...Unless you are allowed to split them
BSP-Tree Rendering (Object Precision)

- BSP-Trees recursively partition space by planes
  - Given two primitives on either side of a plane, the one on the opposite side from the camera will always be further away.
  - Draw the further side first, and then draw the closer one.
BSP-Tree Rendering (Object Precision)

- Draw further half first, then the closer one.
  - Draw right side of 1
  - Draw left side of 1
BSP-Tree Rendering (Object Precision)

• Draw further half first, then the closer one.
  • Draw right side of 1
    • Draw left side of 3
    • Draw right side of 3
  • Draw left side of 1
BSP-Tree Rendering (Object Precision)

- Draw further half first, then the closer one.
  - Draw right side of 1
    - Draw left side of 3
    - Draw D
    - Draw right side of 3
  - Draw left side of 1
BSP-Tree Rendering (Object Precision)

- Draw further half first, then the closer one.
  - Draw right side of 1
    - Draw left side of 3
      - Draw D
    - Draw right side of 3
      - Draw left side of 5
      - Draw right side of 5
  - Draw left side of 1
BSP-Tree Rendering  (Object Precision)

• Draw further half first, then the closer one.
  • Draw right side of 1
    • Draw left side of 3
    • Draw D
  • Draw right side of 3
    • Draw left side of 5
    • Draw E
    • Draw right side of 5
  • Draw left side of 1
BSP-Tree Rendering (Object Precision)

- Draw further half first, then the closer one.
  - Draw right side of 1
    - Draw left side of 3
      - Draw D
    - Draw right side of 3
      - Draw left side of 5
        - Draw E
      - Draw right side of 5
        - Draw F
  - Draw left side of 1
BSP-Tree Rendering (Object Precision)

• Draw further half first, then the closer one.
  • Draw right side of 1
    • Draw left side of 3
      • Draw D
    • Draw right side of 3
      • Draw left side of 5
        • Draw E
      • Draw right side of 5
        • Draw F
  • Draw left side of 1
    • Draw left side of 2
    • Draw right side of 2
BSP-Tree Rendering (Object Precision)

- Draw further half first, then the closer one.
  - Draw right side of 1
    - Draw left side of 3
      - Draw D
    - Draw right side of 3
      - Draw left side of 5
        - Draw E
      - Draw right side of 5
        - Draw F
  - Draw left side of 1
    - Draw left side of 2
      - Draw left side of 4
      - Draw right side of 4
    - Draw right side of 2
BSP-Tree Rendering (Object Precision)

- Draw further half first, then the closer one.
  - Draw right side of 1
    - Draw left side of 3
      - Draw D
    - Draw right side of 3
      - Draw left side of 5
        - Draw E
      - Draw right side of 5
        - Draw F
  - Draw left side of 1
    - Draw left side of 2
      - Draw left side of 4
        - Draw A
      - Draw right side of 4
    - Draw right side of 2
BSP-Tree Rendering (Object Precision)

- Draw further half first, then the closer one.
  - Draw right side of 1
    - Draw left side of 3
      - Draw D
    - Draw right side of 3
      - Draw left side of 5
        - Draw E
      - Draw right side of 5
        - Draw F
  - Draw left side of 1
    - Draw left side of 2
      - Draw left side of 4
        - Draw A
      - Draw right side of 4
        - Draw B
    - Draw right side of 2
BSP-Tree Rendering (Object Precision)

- Draw further half first, then the closer one.
  - Draw right side of 1
    - Draw left side of 3
      - Draw D
    - Draw right side of 3
      - Draw left side of 5
        - Draw E
      - Draw right side of 5
        - Draw F
  - Draw left side of 1
    - Draw left side of 2
      - Draw left side of 4
        - Draw A
      - Draw right side of 4
        - Draw B
    - Draw right side of 2
      - Draw C
3D Rendering Pipeline

3D Primitives
- 3D Model Coordinates

Modeling Transformation
- 3D World Coordinates

Viewing Transformation
- 3D World Coordinates

Lighting
- 3D Camera Coordinates

Projection Transformation
- 2D Screen Coordinates

Clipping
- 2D Screen Coordinates

Viewport Transformation
- 2D Image Coordinates

Scan Conversion
- 2D Image Coordinates

Ordered Rendering

Binary Space Partition:
- View-independent
- Linear-time depth sort
Ray Casting

• Fire a ray for every pixel
  ◦ If ray intersects multiple objects, take the closest
Ray Casting Pipeline

Ray casting
- $P(p \log n)$ for $p$ pixels and $n$ shapes
- May (or not) use pixel coherence
- Simple, but generally not used
z-Buffer

- Store color & depth of closest object at each pixel
  - Initialize depth of each pixel to $\infty$
  - Only update pixels whose depth is closer than the depth stored in the buffer
z-Buffer

- Store color & depth of closest object at each pixel
  - Initialize depth of each pixel to $\infty$
  - Only update pixels whose depth is closer than the depth stored in the buffer

Case 1 (Blue before Red):
Blue $\rightarrow (d = 1) < (d = \infty)$:
  Set RGB = (0,0,1), $d = 1$
Red $\rightarrow (d = 2) > (d = 1)$:
  Don’t change pixel
**z-Buffer**

- Store color & depth of closest object at each pixel
  - Initialize depth of each pixel to $\infty$
  - Only update pixels whose depth is closer than the depth stored in the buffer

---

**Case 1 (Blue before Red):**
Blue $\rightarrow (d = 1) < (d = \infty)$:
Set $RGB = (0,0,1), d = 1$
Red $\rightarrow (d = 2) > (d = 1)$:
Don’t change pixel

**Case 2 (Red before Blue):**
Red $\rightarrow (d = 2) > (d = \infty)$:
Set $RGB = (1,0,0), d = 2$
Blue $\rightarrow (d = 1) < (d = 2)$:
Set $RGB = (0,0,1), d = 1$
z-Buffer

- Store color & depth of closest object at each pixel
  - Initialize depth of each pixel to $\infty$
  - Update only pixels whose depth is closer than in buffer
  - Depths are interpolated from vertices, just like colors

\[
A = (1 - \alpha) \cdot d_1 + \alpha \cdot d
\]

\[
B = (1 - \beta) \cdot d_2 + \beta \cdot d_3
\]

\[
d = (1 - \gamma) \cdot A + \gamma \cdot B
\]
**z-Buffer**

Edge antialiasing becomes difficult because you want multiple triangles to write to the same pixel.

- Who sets the $z$-value?
3D Rendering Pipeline

3D Primitives

- Modeling Transformation
- Viewing Transformation
- Lighting
- Projection Transformation
- Clipping
- Viewport Transformation
- Scan Conversion

- 3D Modeling Coordinates
- 3D World Coordinates
- 3D Camera Coordinates
- 2D Screen Coordinates
- 2D Image Coordinates
- Image

- Polygons can be rasterized in any order
- Requires additional memory
  - $z$-buffer size $\approx$ frame buffer
- This is what your graphics card does!

$z$-Buffer
3D Rendering Pipeline (for direct illumination)

3D Primitives

Modeling Transformation

3D Modeling Coordinates

Viewing Transformation

3D World Coordinates

Lighting

3D Camera Coordinates

Projection Transformation

3D Camera Coordinates

Clipping

2D Screen Coordinates

Viewport Transformation

2D Screen Coordinates

Scan Conversion

2D Image Coordinates

Image

2D Window

3D Model

2D Screen
Scan Conversion

How do we average information from the three vertices of a triangle?

- Interpolate using weights determined by the 2D screen space projection.
- Interpolate using weights determined by the 3D locations.

It’s easier to do the interpolation in 2D.

Is there a difference?
Scan Conversion

• Projective transformations (recall)

\[
\begin{bmatrix}
x' \\
y' \\
z' \\
w'
\end{bmatrix} = \begin{bmatrix}
a & b & c & d \\
e & f & g & h \\
i & j & k & l \\
m & n & o & p
\end{bmatrix} \begin{bmatrix}
x \\
y \\
z \\
w
\end{bmatrix}
\]

Properties of projective transformations:

○ Origin does not necessarily map to origin
○ Lines map to lines
○ *(Weighted) average is not necessarily preserved*
○ Parallel lines do not necessarily remain parallel
○ Closed under composition
Scan Conversion Example

A line segment in 2D projected onto a 1D screen.

How should we interpolate the information from vertices $p_1$ and $p_2$ at the pixel corresponding to ray $R$?
Scan Conversion Example

A line segment in 2D projected onto a 1D screen.

1. $R$ intersects the projected line segment in the middle:
   - We should use equal contributions from $p_1$ and $p_2$. 

$$z = 0 \quad z = 1$$
Scan Conversion Example

A line segment in 2D projected onto a 1D screen.

1. $R$ intersects the projected line segment in the middle:
   - We should use equal contributions from $p_1$ and $p_2$.

2. $R$ intersects the 2D line segment closer to $p_1$:
   - We should use more information from $p_1$ than from $p_2$. 
Scan Conversion Example

A line segment in 2D projected onto a 1D screen.

• How do we interpolate correctly?

\[ z = 0 \quad z = 1 \]
Scan Conversion Example

A line segment in 2D projected onto a 1D screen.

- How do we interpolate correctly?

**Recall:** The 2D point \((x, z)\) maps to the point \((x/z)\) in 1D.

If \(p_1 = (x_1, z_1)\) and \(p_2 = (x_2, z_2)\), to find the blending value \(\alpha\) for a pixel falling at position \(x\) in the screen we need to solve:

\[
(1 - \alpha)(x_1, z_1) + \alpha(x_2, z_2) \rightarrow (x, 1)
\]

\[
((1 - \alpha)x_1 + \alpha x_2, (1 - \alpha)z_1 + \alpha z_2) \rightarrow (x, 1)
\]

\[
\frac{(1 - \alpha)x_1 + \alpha x_2}{(1 - \alpha)z_1 + \alpha z_2} = \frac{x}{1}
\]
Scan Conversion Example

A line segment in 2D projected onto a 1D screen.

• How do we interpolate correctly?

Recall: The 2D point \((x, z)\) maps to the point \((x/z)\) in 1D.

If \(p_1 = (x_1, z_1)\) and \(p_2 = (x_2, z_2)\), to find the blending value \(\alpha\) for a pixel falling at position \(x\) in the screen we need to solve:

\[
\begin{align*}
\frac{(1 - \alpha)x_1 + \alpha x_2}{(1 - \alpha)z_1 + \alpha z_2} &= x \\
\frac{1 - \alpha}{1 - \alpha} x_1 + \alpha x_2 &= x \\
(1 - \alpha)z_1 + \alpha z_2 &= 1
\end{align*}
\]

To compute the interpolation weights correctly, we need to perform a perspective divide:

\[
\begin{align*}
\frac{(1 - \alpha)x_1 + \alpha x_2}{(1 - \alpha)z_1 + \alpha z_2} &= x \\
\frac{1 - \alpha}{1 - \alpha} x_1 + \alpha x_2 &= x \\
(1 - \alpha)z_1 + \alpha z_2 &= 1
\end{align*}
\]

This is not the same as solving for the blending value in the image plane:

\[
\begin{align*}
(1 - \alpha) \frac{x_1}{z_1} + \alpha \frac{x_2}{z_2} &= x \\
\frac{1 - \alpha}{1 - \alpha} \frac{x_1}{z_1} + \alpha \frac{x_2}{z_2} &= x \\
(1 - \alpha)z_1 + \alpha z_2 &= 1
\end{align*}
\]
Scan Conversion Example

Without perspective divide

With perspective divide

courtesy of H. Pfister