Intersection and Acceleration

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HB Ch. 14.1, 14.2
FvDFH 16.1, 16.2
Ray Casting

• Simple implementation:

Image RayCast( Camera camera, Scene scene, int width, int height)
{
    Image image = new Image( width, height );
    for( int i=0 ; i<width ; i++ ) for( int j=0 ; j<height ; j++ )
    {
        Ray ray = ConstructRayThroughPixel( camera, i, j );
        Intersection hit = FindIntersection( ray, scene );
        image[i][j] = GetColor( hit );
    }
    return image;
}
Ray Casting

• Simple implementation:

```java
Image RayCast( Camera camera, Scene scene, int width, int height)
{
    Image image = new Image( width, height);
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    {
        Ray ray = ConstructRayThroughPixel( camera, i, j );
        Intersection hit = FindIntersection( ray, scene );
        image[i][j] = GetColor( hit );
    }
    return image;
}
```
Ray-Triangle Intersection

1. Intersect ray with plane
2. Check if the point is inside the triangle

$\vec{p}$ $\vec{v}$ $p_0$ $p$
Ray-Plane Intersection

Ray: \( p(t) = p_0 + t \cdot \vec{v}, \quad (0 \leq t < \infty) \)

Plane: \( \Phi(p) = \langle p, \vec{n} \rangle - d = 0 \)

Substituting for \( p \), we get:
\[
\Phi(t) = \langle p_0 + t \cdot \vec{v}, \vec{n} \rangle - d = 0
\]

Solution:
\[
t = -\frac{\langle p_0, \vec{n} \rangle - d}{\langle \vec{v}, \vec{n} \rangle}
\]

Algebraic Method
Ray-Triangle Intersection II

- Check for point-triangle intersection parametrically

In general, given \( p \in \mathbb{R}^3 \) and given three points \( \{ v_1, v_2, v_3 \} \subset \mathbb{R}^3 \) (in general position) we can solve for \( \alpha, \beta, \gamma \in \mathbb{R} \) such that:

\[
p = \alpha v_1 + \beta v_2 + \gamma v_3
\]

\( p \) is in the plane spanned by \( \{ v_1, v_2, v_3 \} \) iff.:

\[
\alpha + \beta + \gamma = 1
\]

\( p \) is inside the triangle with vertices \( \{ v_1, v_2, v_3 \} \) iff.:

\[
\alpha, \beta, \gamma \geq 0
\]
Ray-Triangle Intersection II

• Check for point-triangle intersection parametrically

In general, given \( p \in \mathbb{R}^3 \) and given three points \( \{v_1, v_2, v_3\} \subset \mathbb{R}^3 \) (in general position) we can solve for \( \alpha, \beta, \gamma \in \mathbb{R} \) such that:

\[
p = \alpha v_1 + \beta v_2 + \gamma v_3
\]

To get \( \alpha, \beta, \gamma \), solve the system:

\[
\begin{pmatrix}
v_1^x & v_2^x & v_3^x \\
v_1^y & v_2^y & v_3^y \\
v_1^z & v_2^z & v_3^z
\end{pmatrix}
\begin{pmatrix}
\alpha \\
\beta \\
\gamma
\end{pmatrix}
=
\begin{pmatrix}
p^x \\
p^y \\
p^z
\end{pmatrix}
\]
Ray-Triangle Intersection II

• Check for point-triangle intersection parametrically

In general, given $p \in \mathbb{R}^3$ and given three points $\{v_1, v_2, v_3\} \subset \mathbb{R}^3$ (in general position) we can solve for $\alpha, \beta, \gamma \in \mathbb{R}$ such that:

$$p = \alpha v_1 + \beta v_2 + \gamma v_3$$

To get $\alpha, \beta, \gamma$, solve the system:

$$\begin{pmatrix} v_1^x & v_2^x & v_3^x \\ v_1^y & v_2^y & v_3^y \\ v_1^z & v_2^z & v_3^z \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} p^x \\ p^y \\ p^z \end{pmatrix} \iff \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} v_1^x & v_2^x & v_3^x \\ v_1^y & v_2^y & v_3^y \\ v_1^z & v_2^z & v_3^z \end{pmatrix}^{-1} \begin{pmatrix} p^x \\ p^y \\ p^z \end{pmatrix}$$
Ray-Triangle Intersection II

- Check for point-triangle intersection parametrically

In general, given \( p \in \mathbb{R}^3 \) and given three points \( \{v_1, v_2, v_3\} \subset \mathbb{R}^3 \) (in general position) we can solve for \( \alpha, \beta, \gamma \in \mathbb{R} \) such that:

\[
p = \alpha v_1 + \beta v_2 + \gamma v_3
\]

To get \( \alpha, \beta, \gamma \), solve the system:

\[
\begin{pmatrix}
v_1^x & v_2^x & v_3^x \\
v_1^y & v_2^y & v_3^y \\
v_1^z & v_2^z & v_3^z
\end{pmatrix}
\begin{pmatrix}
\alpha \\
\beta \\
\gamma
\end{pmatrix}
= 
\begin{pmatrix}
p^x \\
p^y \\
p^z
\end{pmatrix}
\quad \iff \quad
\begin{pmatrix}
\alpha \\
\beta \\
\gamma
\end{pmatrix}
= 
\begin{pmatrix}
v_1^x & v_2^x & v_3^x \\
v_1^y & v_2^y & v_3^y \\
v_1^z & v_2^z & v_3^z
\end{pmatrix}^{-1}
\begin{pmatrix}
p^x \\
p^y \\
p^z
\end{pmatrix}

This will fail if the vertices \( \{v_1, v_2, v_3\} \) lie in a plane through the origin.
Ray-Triangle Intersection II

• Check for point-triangle intersection parametrically

Embrace the problem case by translating the whole system to the origin:

\[
\begin{pmatrix}
    v_1^x & v_2^x & v_3^x \\
    v_1^y & v_2^y & v_3^y \\
    v_1^z & v_2^z & v_3^z
\end{pmatrix}
\begin{pmatrix}
    \alpha \\
    \beta \\
    \gamma
\end{pmatrix}
= 
\begin{pmatrix}
    p^x \\
    p^y \\
    p^z
\end{pmatrix}

\Rightarrow
\begin{pmatrix}
    0 & v_2^x - v_1^x & v_3^x - v_1^x \\
    0 & v_2^y - v_1^y & v_3^y - v_1^y \\
    0 & v_2^z - v_1^z & v_3^z - v_1^z
\end{pmatrix}
\begin{pmatrix}
    \alpha \\
    \beta \\
    \gamma
\end{pmatrix}
= 
\begin{pmatrix}
    p^x \\
    p^y \\
    p^z
\end{pmatrix}
Ray-Triangle Intersection II

- Check for point-triangle intersection parametrically

Embrace the problem case by translating the whole system to the origin:

\[
\begin{bmatrix}
  v_{1x}^x & v_{2x}^x & v_{3x}^x \\
  v_{1y}^y & v_{2y}^y & v_{3y}^y \\
  v_{1z}^z & v_{2z}^z & v_{3z}^z \\
\end{bmatrix}
\begin{bmatrix}
  \alpha \\
  \beta \\
  \gamma \\
\end{bmatrix} =
\begin{bmatrix}
  p_x \\
  p_y \\
  p_z \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
  (v_{2x}^x - v_{1x}^x) & (v_{3x}^x - v_{1x}^x) \\
  (v_{2y}^y - v_{1y}^y) & (v_{3y}^y - v_{1y}^y) \\
  (v_{2z}^z - v_{1z}^z) & (v_{3z}^z - v_{1z}^z) \\
\end{bmatrix}
\begin{bmatrix}
  \alpha \\
  \beta \\
  \gamma \\
\end{bmatrix} =
\begin{bmatrix}
  p_x - v_{1x}^x \\
  p_y - v_{1y}^y \\
  p_z - v_{1z}^z \\
\end{bmatrix}
\]
Ray-Triangle Intersection II

- Check for point-triangle intersection parametrically

Embrace the problem case by translating the whole system to the origin:

\[
\begin{align*}
\mathbf{p} - \mathbf{v}_1 &= (\mathbf{v}_2 - \mathbf{v}_1)
\end{align*}
\]

This is an over-constrained system!
In general, we can’t express a 3D point as the linear combination of two 3D points.

This is not the general case!
A solution exists since \( \mathbf{p} \) is in the plane spanned by \( \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} \)

After solving for \( \beta \) and \( \gamma \), we can set:

\[
\alpha = 1 - \beta - \gamma
\]

\[
\begin{align*}
\begin{pmatrix}
v_1^x & v_2^x & v_3^x \\
v_1^y & v_2^y & v_3^y \\
v_1^z & v_2^z & v_3^z
\end{pmatrix}
\begin{pmatrix}
\alpha \\
\beta \\
\gamma
\end{pmatrix}
= 
\begin{pmatrix}
p_x \\
p_y \\
p_z
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{pmatrix}
(v_2^x - v_1^x) & (v_3^x - v_1^x) \\
v_2^y - v_1^y & v_3^y - v_1^y \\
v_2^z - v_1^z & v_3^z - v_1^z
\end{pmatrix}
\begin{pmatrix}
\beta \\
\gamma
\end{pmatrix}
= 
\begin{pmatrix}
p_x \\
p_y \\
p_z
\end{pmatrix}
\end{align*}
\]
Ray-Scene Intersection

- Intersections with geometric primitives
  - Sphere
  - Triangle

- Acceleration techniques
  - Bounding volume hierarchies
  - Spatial partitions
    - Uniform grids
    - Octrees
    - BSP trees
Ray-Scene Intersection

A direct (naïve) approach:

Intersection FindIntersection( Ray ray, Scene scene )
{
    ( min_t , min_shape ) = ( -1 , NULL )
    For each primitive in scene
    {
        t = Intersect( ray , primitive );
        if( t>0 and (t<min_t or min_t<0 ) )
            min_shape = primitive
            min_t = t
    }
}
return Intersection( min_t , min_shape )
Overview

• Acceleration techniques
  ◦ Bounding volume hierarchies
  ◦ Space partitions
    » Uniform (voxel) grids
    » Octrees
    » BSP trees
Intersection Testing

Accelerated techniques try to leverage:

- **Grouping:**
  Discard groups of primitives that are guaranteed to be missed by the ray.

- **Ordering:**
  Test nearer intersections first and allow for early termination if there is a hit.
Bounding Volumes

• Check for intersection with the bounding volume:
  ○ Bounding cubes
  ○ Bounding boxes
  ○ Bounding spheres
  ○ Etc.

Stuff that’s easy to intersect
Bounding Volumes

- Check for intersection with the bounding volume
  - If the ray misses the bounding volume, it can’t intersect its contents

Still need to check for intersections with shape.
Bounding Volume Hierarchies

• Build hierarchy of bounding volumes
  ◦ Bounding volume of a parent node contains all children
Bounding Volume Hierarchies

- Grouping acceleration

```c
FindIntersection( Ray ray , Node node )
{
    ( min_t , min_shape ) = ( -1 , NULL )

    if( !intersect ( node.boundingVolume ) )   // Test Bounding box
        return ( -1 , NULL );

    foreach shape in node                         // Test node’s shape
    {
        t = Intersect( shape )
        if( t>0 && (t<min_t || min_t<0) ) ( min_t , min_shape ) = ( t , shape )
    }

    for each child in node                        // Test node’s children
    {
        ( t , shape ) = FindIntersection( ray , child )
        if( t>0 && (t < min_t || min_t<0) ) ( min_t , min_shape ) = ( t , shape )
    }

    return ( min_t , min_shape );
}
```
Bounding Volume Hierarchies

• Use hierarchy to accelerate ray intersections
  ◦ Intersect node contents only if you hit the bounding volume
Bounding Volume Hierarchies

- Use hierarchy to accelerate ray intersections
  - Intersect node contents only if you hit the bounding volume

Don’t need to test shapes A or B

Need to test groups 1, 2, and 3

Need to test shapes C, D, E, and F
Bounding Volume Hierarchies

• Grouping + Ordering acceleration

```c
FindIntersection( Ray ray , Node node )
{
    // Find intersections with the shapes of the node
    ...
    // Find intersections with child node bounding volumes
    ...
    // Sort child bounding volume intersections front to back
    // and store distances to child bounding boxes in bv_t[]
    ...

    // Process intersections (checking for early termination)
    for each child node whose bounding box is intersected
    {
        if( min_t < bv_t[child] ) break;
        ( t , shape ) = FindIntersection( ray , child );
        if( t>0 && (t < min_t || min_t<0) ) ( min_t , min_shape ) = ( t , shape )
    }
    return ( min_t , min_shape );
}
```
Bounding Volume Hierarchies

• Use hierarchy to accelerate ray intersections
  ◦ Intersect nodes only if you haven’t hit anything closer
Bounding Volume Hierarchies

- Use hierarchy to accelerate ray intersections
  - Intersect nodes only if you haven’t hit anything closer

- Don’t need to test shapes A, B, D, E, or F
- Need to test groups 1, 2, and 3
- Need to test shape C
Ray-Scene Intersection

- Intersections with geometric primitives
  - Sphere
  - Triangle

- Acceleration techniques
  - Bounding volume hierarchies
  - Spatial partitions
    - Uniform (Voxel) grids
    - Octrees
    - BSP trees
Uniform (Voxel) Grid

• Construct uniform grid over the scene
  ◦ Index primitives according to overlaps with grid cells

• A primitive may belong to multiple cells
• A cell may have multiple primitives
Uniform (Voxel) Grid

- Trace rays through grid cells
  - Fast
  - Incremental

Only check primitives in intersected grid cells
Uniform (Voxel) Grid

- Potential problem:
  - How choose suitable grid resolution?

Too much cost if grid is too fine

Too little benefit if grid is too coarse
Ray-Scene Intersection

- Intersections with geometric primitives
  - Sphere
  - Triangle

  » Acceleration techniques
    - Bounding volume hierarchies
    - Spatial partitions
      » Uniform (Voxel) grids
      » Octrees
      » BSP trees
Octrees

- We can think of a voxel grid as a tree.
  - The root node is the entire region
  - Each node has eight children obtained by subdividing the parent into eight equal regions
Octrees

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  - The root node is the entire region
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Octrees

- In an octree, we only subdivide regions that contain more than one shape.
Octrees
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Octrees

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Octrees

• In an octree, we only subdivide regions that contain more than one shape.

• Adaptively determines grid resolution.
Octrees

• In an octree, we only subdivide regions that contain more than one shape.

• Adaptively determines grid resolution.

Efficiently tracing a ray through an adaptive octree is trickier than tracing a ray through a regular grid!
Overview

• Acceleration techniques
  ◦ Bounding volume hierarchies
  ◦ Spatial partitions
    » Uniform (Voxel) grids
    » Octrees
    » BSP trees
      – $k$-D trees
**k-D Trees**

- Alternate between splitting along the $x$-axis, $y$-axis, and $z$-axis.
**$k$-D Trees**

- Alternate between splitting along the $x$-axis, $y$-axis, and $z$-axis.
**$k$-D Trees**

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**k-D Trees**

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**$k$-D Trees**

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**k-D Trees**

- Alternate between splitting along the $x$-axis, $y$-axis, and $z$-axis.

**Note:**
- Either primitives need to be split, or they belong to multiple nodes.

**Limitation:**
- The splitting planes still have to be axis-aligned
Binary Space Partition (BSP) Tree

• Recursively partition space by planes
Binary Space Partition (BSP) Tree

• Recursively partition space by planes
  ◦ Generate a tree structure where the leaves store the shapes.
Binary Space Partition (BSP) Tree

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Binary Space Partition (BSP) Tree

- Recursively partition space by planes
  - Generate a tree structure where the leaves store the shapes.
Binary Space Partition (BSP) Tree

- Example: Point Intersection
Binary Space Partition (BSP) Tree

- Example: Point Intersection
  - Recursively test what side we are on
Binary Space Partition (BSP) Tree

- Example: Point Intersection
  - Recursively test what side we are on
    - Left of 1 (root) → 2
Binary Space Partition (BSP) Tree

- Example: Point Intersection
  - Recursively test what side we are on
    » Left of 2 → 4
Binary Space Partition (BSP) Tree

• Example: Point Intersection
  ◦ Recursively test what side we are on
    » Right of 4 → Test B
Binary Space Partition (BSP) Tree

- Example: Point Intersection
  - Recursively test what side we are on
    » Missed B. No intersection!
Binary Space Partition (BSP) Tree

- Example: Point Intersection
  - Recursively test what side we are on
    - Missed B. No intersection!

Worst-case / Expected complexity: proportional to the depth of the tree
Binary Space Partition (BSP) Tree

- Example: Ray Intersection 1
  - Recursively split the ray and test nearer and farther halves, nearest first. Stop once you hit something:

```plaintext
Recursively split the ray and test nearer and farther halves, nearest first. Stop once you hit something:
```
Binary Space Partition (BSP) Tree

- Example: Ray Intersection 1
  - Recursively split the ray and test nearer and farther halves, nearest first. Stop once you hit something:
    » Test half to the left of 1
Binary Space Partition (BSP) Tree

• Example: Ray Intersection 1
  ◦ Recursively split the ray and test nearer and farther halves, nearest first. Stop once you hit something:
    » Test half to the right of 2
Binary Space Partition (BSP) Tree

• Example: Ray Intersection 1
  ◦ Recursively split the ray and test nearer and farther halves, nearest first. Stop once you hit something:
    » Intersection with C. Done!
Binary Space Partition (BSP) Tree

• Example: Ray Intersection 2
  ◦ Recursively split the ray and test nearer and farther halves, nearest first. Stop once you hit something:
Binary Space Partition (BSP) Tree

• Example: Ray Intersection 2
  ◦ Recursively split the ray and test nearer and farther halves, nearest first. Stop once you hit something:
    » Test half to the left of 1
Binary Space Partition (BSP) Tree

• Example: Ray Intersection 2
  ◦ Recursively split the ray and test nearer and farther halves, nearest first. Stop once you hit something:
    » Test half to the right of 2
Binary Space Partition (BSP) Tree

- **Example: Ray Intersection 2**
  - Recursively split the ray and test nearer and farther halves, nearest first. Stop once you hit something:
    » Missed C. Recurse!
Binary Space Partition (BSP) Tree

• Example: Ray Intersection 2
  ◦ Recursively split the ray and test nearer and farther halves, nearest first. Stop once you hit something:
    » Test half to left of 2
Binary Space Partition (BSP) Tree

- Example: Ray Intersection 2
  - Recursively split the ray and test nearer and farther halves, nearest first. Stop once you hit something:
    - Test half to left of 4
Binary Space Partition (BSP) Tree

- Example: Ray Intersection 2
  - Recursively split the ray and test nearer and farther halves, nearest first. Stop once you hit something:
    » Missed A. Recurse!
Binary Space Partition (BSP) Tree

- Example: Ray Intersection 2
  - Recursively split the ray and test nearer and farther halves, nearest first. Stop once you hit something:
    » No half to right of 4.
Binary Space Partition (BSP) Tree

- Example: Ray Intersection 2
  - Recursively split the ray and test nearer and farther halves, nearest first. Stop once you hit something:
    - Test half to right of 1
Binary Space Partition (BSP) Tree

- Example: Ray Intersection 2
  - Recursively split the ray and test nearer and farther halves, nearest first. Stop once you hit something:
    » Test half to left of 3
Binary Space Partition (BSP) Tree

- Example: Ray Intersection 2
  - Recursively split the ray and test nearer and farther halves, nearest first. Stop once you hit something:
    » Intersection with D. Done!
Binary Space Partition (BSP) Tree

- Example: Ray Intersection 2
  - Recursively split the ray and test nearer and farther halves, nearest first. Stop once you hit something:
    - Intersection with D. Done!

Worst-case: Proportional to the number of nodes in the tree
Expected: substantially faster

How should we choose the splitting planes?
RayTreeIntersect(Ray ray, Node node)
{
    if (Node is a leaf) return intersection of closest primitive in cell, or NULL if none
    else
    {
        // Find near and far children
        near_child = child of node that contains the origin of Ray
        far_child = other child of node

        // Recurse down near child first
        isect = RayTreeIntersect(ray, near_child)
        if( isect ) return isect  // If there’s a hit, we are done

        // If there’s no hit, test the far child
        return RayTreeIntersect(ray, far_child)
    }
}