Parametric Curves

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Overview

• What is a Spline?

• Specific Examples:
  ◦ Hermite Splines
  ◦ Cardinal Splines
  ◦ Uniform Cubic B-Splines

• Comparing Cardinal and Uniform Cubic B-Splines
What is a Spline in CG?

A spline is a \textit{piecewise polynomial function} whose derivatives satisfy some \textit{continuity constraints} across curve boundaries.

\[ P_i(x) \quad x \in [0,1) \]

\[ P_1(x) \quad x \in [0,1) \]

\[ P_2(x) \quad x \in [0,1) \]

\[ P_3(x) \quad x \in [0,1) \]

\[ P_i(x) = \sum_{j=0}^{n} a_{ij} \cdot x^j \]
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\[
P_i(x) = \sum_{j=0}^{n} a_{ij} \cdot x^j
\]

\[
P_1(1) = P_2(0)
P_1'(1) = P_2'(0)
\]

\[
P_2(1) = P_3(0)
P_2'(1) = P_3'(0)
\]

\[
\ldots
\]
Overview

• What is a Spline?

• Specific Examples:
  ◦ Hermite Splines
  ◦ Cardinal Splines
  ◦ Uniform Cubic B-Splines

• Comparing Cardinal and Uniform Cubic B-Splines
Specific Example: Hermite Splines

- Interpolating piecewise cubic polynomial, each specified by:
  - Start/end positions
  - Start/end tangents

- Iteratively construct the curve between adjacent end points that interpolate positions and tangents.
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Specific Example: Hermite Splines

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  - Start/end positions
  - Start/end tangents

- Iteratively construct the curve between adjacent end points that interpolate positions and tangents.

Because the end-points of adjacent curves share the same position and derivatives, the Hermite spline is $C^1$ by construction.
Specific Example: Hermite Splines

Given the polynomial:

\[ P_k(u) = a \cdot u^3 + b \cdot u^2 + c \cdot u + d \]

we can write its derivative as:

\[ P'_k(u) = 3 \cdot a \cdot u^2 + 2 \cdot b \cdot u + c \]

Using the matrix representations:

\[ P_k(u) = (u^3 \quad u^2 \quad u \quad 1) \cdot \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \quad P'_k(u) = (3 \cdot u^2 \quad 2 \cdot u \quad 1 \quad 0) \cdot \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \]
Specific Example: Hermite Splines

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We can express the values at the end-points as:

\[ p_k = P_k(0) = (0 \quad 0 \quad 0 \quad 1) \cdot \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \quad \hat{t}_k = P'_k(0) = (0 \quad 0 \quad 1 \quad 0) \cdot \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \]

\[ p_{k+1} = P_k(1) = (1 \quad 1 \quad 1 \quad 1) \cdot \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \quad \hat{t}_{k+1} = P'_k(1) = (3 \quad 2 \quad 1 \quad 0) \cdot \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \]
Specific Example: Hermite Splines

\[
p_k = P_k(0) = (0 \ 0 \ 0 \ 1) \cdot \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \quad \ddot{t}_k = P'_k(0) = (0 \ 0 \ 1 \ 0) \cdot \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}
\]

\[
p_{k+1} = P_k(1) = (1 \ 1 \ 1 \ 1) \cdot \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \quad \ddot{t}_{k+1} = P'_k(1) = (3 \ 2 \ 1 \ 0) \cdot \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}
\]

We can combine the equations into a single matrix expression:

\[
\begin{pmatrix} p_k \\ p_{k+1} \\ \ddot{t}_k \\ \ddot{t}_{k+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}
\]
Specific Example: Hermite Splines

\[
\begin{pmatrix}
p_k \\
p_{k+1} \\
\hat{t}_k \\
\hat{t}_{k+1}
\end{pmatrix} =
\begin{pmatrix}
0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 \\
3 & 2 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
c \\
d
\end{pmatrix}
\]

Inverting, we get:

\[
\begin{pmatrix}
a \\
b \\
c \\
d
\end{pmatrix} =
\begin{pmatrix}
0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 \\
3 & 2 & 1 & 0
\end{pmatrix}^{-1}
\begin{pmatrix}
p_k \\
p_{k+1} \\
\hat{t}_k \\
\hat{t}_{k+1}
\end{pmatrix} =
\begin{pmatrix}
2 & -2 & 1 & 1 \\
-3 & 3 & -2 & -1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
p_k \\
p_{k+1} \\
\hat{t}_k \\
\hat{t}_{k+1}
\end{pmatrix}
\]
Specific Example: Hermite Splines

\[
\begin{pmatrix}
  a \\
  b \\
  c \\
  d
\end{pmatrix}
= 
\begin{pmatrix}
  2 & -2 & 1 & 1 \\
  -3 & 3 & -2 & -1 \\
  0 & 0 & 1 & 0 \\
  1 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
  p_k \\
  p_{k+1} \\
  \tilde{t}_k \\
  \tilde{t}_{k+1}
\end{pmatrix}
\]

Using the fact that:

\[
P_k(u) = (u^3 \quad u^2 \quad u \quad 1) \cdot 
\begin{pmatrix}
  a \\
  b \\
  c \\
  d
\end{pmatrix}
\]

We get:

\[
P_k(u) = (u^3 \quad u^2 \quad u \quad 1)
\begin{pmatrix}
  2 & -2 & 1 & 1 \\
  -3 & 3 & -2 & -1 \\
  0 & 0 & 1 & 0 \\
  1 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
  p_k \\
  p_{k+1} \\
  \tilde{t}_k \\
  \tilde{t}_{k+1}
\end{pmatrix}
\]

parameters \hspace{1cm} M_{Hermite} \hspace{1cm} boundary info
Specific Example: Hermite Splines

Setting:

- \( H_0(u) = 2u^3 - 3u^2 + 1 \)
- \( H_1(u) = -2u^3 + 3u^2 \)
- \( H_2(u) = u^3 - 2u^2 + u \)
- \( H_3(u) = u^3 - u^2 \)

Blending Functions

\[
P_k(u) = p_k \cdot H_0(u) + p_{k+1} \cdot H_1(u) + \tilde{t}_k \cdot H_2(u) + \tilde{t}_{k+1} \cdot H_3(u)
\]
Specific Example: Hermite Splines

- **Interpolating** piecewise *cubic* polynomial, each specified by:
  - Start/end positions
  - Start/end tangents

- Iteratively construct the curve between adjacent end points that interpolate positions and tangents.

Given the control points, how do we define the value of the tangents/derivatives?
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• Comparing Cardinal and Uniform Cubic B-Splines
Specific Example: Cardinal Splines

- Interpolating piecewise cubic polynomial, each specified by four control points.
- Iteratively construct the curve between middle two points using adjacent points to define tangents.
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Because the end-points of adjacent curves share the same position and derivatives, the Cardinal spline has $C^1$ continuity.
Specific Example: Cardinal Splines

Using Hermite splines, we have:

\[ P_k(u) = \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix} \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_k \\ p_{k+1} \\ \vec{t}_k \\ \vec{t}_{k+1} \end{pmatrix} \]

\[ \vec{t}_k = s(p_{k+1} - p_{k-1}) \]

\[ \vec{t}_{k+1} = s(p_{k+2} - p_k) \]
Specific Example: Cardinal Splines

\[ P_k(u) = (u^3 \quad u^2 \quad u \quad 1) \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_k \\ p_{k+1} \\ \tilde{t}_k \\ \tilde{t}_{k+1} \end{pmatrix} \]

\[
\begin{pmatrix}
    p_k \\
    p_{k+1} \\
    \tilde{t}_k \\
    \tilde{t}_{k+1}
\end{pmatrix} =
\begin{pmatrix}
    p_k \\
    p_{k+1} \\
    s(p_{k+1} - p_k) \\
    s(p_k + 2 - p_k)
\end{pmatrix}
\]

We can express the boundary conditions as:

\[
\begin{pmatrix}
    p_k \\
    p_{k+1} \\
    \tilde{t}_k \\
    \tilde{t}_{k+1}
\end{pmatrix} =
\begin{pmatrix}
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    -s & 0 & s & 0 \\
    0 & -s & 0 & s
\end{pmatrix} \begin{pmatrix}
    p_{k-1} \\
    p_k \\
    p_{k+1} \\
    p_{k+2}
\end{pmatrix}
\]
Specific Example: Uniform Cubic B-Splines

\[ P_k(u) = (u^3 \quad u^2 \quad u \quad 1) \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_k \\ p_{k+1} \\ \tilde{t}_k \\ \tilde{t}_{k+1} \end{pmatrix} \]

\[ \begin{pmatrix} p_k \\ p_{k+1} \\ \tilde{t}_k \\ \tilde{t}_{k+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s & 0 & s & 0 \\ 0 & -s & 0 & s \end{pmatrix} \begin{pmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{pmatrix} \]

This gives:

\[ P_k(u) = (u^3 \quad u^2 \quad u \quad 1) \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s & 0 & s & 0 \\ 0 & -s & 0 & s \end{pmatrix} \begin{pmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{pmatrix} \]
Specific Example: Cardinal Splines

\[ P_k(u) = (u^3 \quad u^2 \quad u \quad 1) \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s & 0 & s & 0 \\ 0 & -s & 0 & s \end{pmatrix} \begin{pmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{pmatrix} \]

Multiplying, we get the Cardinal matrix representation:

\[ P_k(u) = (u^3 \quad u^2 \quad u \quad 1) \begin{pmatrix} -s & 2 - s & s - 2 & s \\ 2s & s - 3 & 3 - 2s & -s \\ -s & 0 & s & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{pmatrix} \]
Specific Example: Cardinal Splines

Setting:

- \( C_0(u) = -su^3 + 2su^2 - su \)
- \( C_1(u) = (2 - s)u^3 + (s - 3)u^2 + 1 \)
- \( C_2(u) = (s - 2)u^3 + (3 - 2s)u^2 + su \)
- \( C_3(u) = su^3 - su^2 \)

Blending Functions

For \( s = 1/2 \):

\[
P_k(u) = C_0(u) \cdot p_{k-1} + C_1(u) \cdot p_k + C_2(u) \cdot p_{k+1} + C_3(u) \cdot p_{k+2}
\]
Specific Example: Cardinal Splines

- **Interpolating** piecewise *cubic* polynomial, each specified by four control points.

- Iteratively construct the curve between middle two points using adjacent points to define tangents.

At the first and last end-points, you can:
- Not draw the final segments
- Double up end points
- Loop the spline around
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• Comparing Cardinal and Uniform Cubic B-Splines
Specific Example: Uniform Cubic B-Splines

- **Approximating** piecewise cubic polynomial, each specified by four control points.
- Iteratively construct the curve near middle two points using adjacent points to define positions and tangents.

\[ p_0 \quad p_1 \quad p_2 \quad p_3 \quad p_4 \quad p_5 \quad p_6 \quad p_7 \]
Specific Example: Uniform Cubic B-Splines

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- **Approximating** piecewise *cubic* polynomial, each specified by four control points.
- Iteratively construct the curve near middle two points using adjacent points to define positions and tangents.

\[
\begin{align*}
&\mathbf{p_0} \quad \mathbf{p_1} \quad \mathbf{p_2} \quad \mathbf{p_3} \\
&\mathbf{p_4} \quad \mathbf{p_5} \quad \mathbf{p_6} \quad \mathbf{p_7}
\end{align*}
\]
Specific Example: Uniform Cubic B-Splines

Using Hermite splines, we have:

\[ P_k(u) = (u^3 \quad u^2 \quad u \quad 1) \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p'_k \\ p'_{k+1} \\ \tilde{t}_k \\ \tilde{t}_{k+1} \end{pmatrix} \]

\[ M_{\text{Hermite}} \]

\[ p'_k = \frac{(p_{k-1} + 4p_k + p_{k+1})}{6} \]

\[ p'_{k+1} = \frac{(p_k + 4p_{k+1} + p_{k+2})}{6} \]

\[ \tilde{t}_k = s(p_{k+1} - p_{k-1}) \]

\[ \tilde{t}_{k+1} = s(p_{k+2} - p_k) \]
Specific Example: Uniform Cubic B-Splines

\[ P_k(u) = (u^3 \quad u^2 \quad u \quad 1) \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p' \k \\ p'_{k+1} \\ \hat{t}_k \\ \hat{t}_{k+1} \end{pmatrix} \]

\[
\begin{pmatrix} p' \k \\ p'_{k+1} \\ \hat{t}_k \\ \hat{t}_{k+1} \end{pmatrix} = \frac{1}{6} \begin{pmatrix} p_{k-1} + 4p_k + p_{k+1} \\ p_k + 4p_{k+1} + p_{k+2} \\ 6s(p_{k+1} - p_{k-1}) \\ 6s(p_{k+2} - p_k) \end{pmatrix}
\]

We can express the boundary conditions as:

\[
\begin{pmatrix} p' \k \\ p'_{k+1} \\ \hat{t}_k \\ \hat{t}_{k+1} \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ -6s & 0 & 6s & 0 \\ 1 & -6s & 0 & 6s \end{pmatrix} \begin{pmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{pmatrix}
\]
Specific Example: Uniform Cubic B-Splines

\[ P_k(u) = (u^3 \quad u^2 \quad u \quad 1) \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p'_k \\ p'_{k+1} \\ \hat{t}_k \\ \hat{t}_{k+1} \end{pmatrix} \]

\[
\begin{pmatrix} p'_k \\ p'_{k+1} \\ \hat{t}_k \\ \hat{t}_{k+1} \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ -6s & 0 & 6s & 0 \\ 1 & -6s & 0 & 6s \end{pmatrix} \begin{pmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{pmatrix}
\]

This gives:

\[ P_k(u) = (u^3 \quad u^2 \quad u \quad 1) \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \frac{1}{6} \begin{pmatrix} 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ -6s & 0 & 6s & 0 \\ 1 & -6s & 0 & 6s \end{pmatrix} \begin{pmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{pmatrix} \]
Specific Example: Uniform Cubic B-Splines

\[ P_k(u) = (u^3 \quad u^2 \quad u \quad 1) \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \frac{1}{6} \begin{pmatrix} 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ -6s & 0 & 6s & 0 \\ 1 & -6s & 0 & 6s \end{pmatrix} \begin{pmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{pmatrix} \]

Multiplying, we get the uniform cubic B-spline matrix representation:

\[ P_k(u) = (u^3 \quad u^2 \quad u \quad 1) \frac{1}{6} \begin{pmatrix} 2 - 6s & 6 - 6s & -6 + 6s & -2 + 6s \\ -3 + 12s & -9 + 6s & 9 - 12s & 3 - 6s \\ -6s & 0 & 6s & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix} \begin{pmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{pmatrix} \]
Specific Example: Uniform Cubic B-Splines

Setting the blending functions to:

- \( B_{0,3}(u) = (\frac{1}{3} - s)u^3 + (-\frac{1}{2} + 2s)u^2 - su + \frac{1}{6} \)
- \( B_{1,3}(u) = (1 - s)u^3 + (-\frac{3}{2} + s)u^2 + \frac{2}{3} \)
- \( B_{2,3}(u) = (-1 + s)u^3 + (\frac{3}{2} - 2s)u^2 + su + \frac{1}{6} \)
- \( B_{3,3}(u) = (-\frac{1}{3} + s)u^3 + (\frac{1}{2} - s)u^2 \)

For \( s = 1/2 \):

\[
P_k(u) = B_{0,3}(u) \cdot p_{k-1} + B_{1,3}(u) \cdot p_k + B_{2,3}(u) \cdot p_{k+1} + B_{3,3}(u) \cdot p_{k+2}
\]
Specific Example: Uniform Cubic B-Splines

- **Approximating** piecewise *cubic* polynomial, each specified by four control points.
- Iteratively construct the curve near middle two points using adjacent points to define positions and tangents.

At the first and last end-points, you can:
- Not draw the final segments
- Double up end points
- Loop the spline around
Overview

• What is a Spline?

• Specific Examples:
  ◦ Hermite Splines
  ◦ Cardinal Splines
  ◦ Uniform Cubic B-Splines

• Comparing Cardinal and Uniform Cubic B-Splines
Blending Functions

Blending functions provide a way for expressing the functions $P_k(u)$ as a weighted sum of the four control points $p_{k-1}, p_k, p_{k+1},$ and $p_{k+2}$:

$$P_k(u) = BF_0(u) \cdot p_{k-1} + BF_1(u) \cdot p_k + BF_2(u) \cdot p_{k+1} + BF_3(u) \cdot p_{k+2}$$
Blending Functions

Properties:

- **Translation Commutativity:**
  - If we translate all the control points by the same vector \( q \), the position of the new point at the value \( u \) will be the position of the old value at \( u \), translated by \( q \).

\[ P_k(u) = BF_0(u) \cdot p_{k-1} + BF_1(u) \cdot p_k + BF_2(u) \cdot p_{k+1} + BF_3(u) \cdot p_{k+2} \]
Comparison: Cardinal vs. Cubic B

Cardinal Splines ($s = 1/2$)

- $BF_0(u) = -\frac{1}{2}u^3 + u^2 - \frac{1}{2}u$
- $BF_1(u) = \frac{3}{2}u^3 - \frac{5}{2}u^2 + 1$
- $BF_2(u) = -\frac{3}{2}u^3 + 2u^2 + \frac{1}{2}u$
- $BF_3(u) = \frac{1}{2}u^3 - \frac{1}{2}u^2$

$BF_0(u) + BF_1(u) + BF_2(u) + BF_3(u) = 1$

Cubic B-Splines ($s = 1/2$)

- $BF_0(u) = -\frac{1}{6}u^3 + \frac{1}{2}u^2 - \frac{1}{2}u + \frac{1}{6}$
- $BF_1(u) = \frac{1}{2}u^3 - u^2 + \frac{2}{3}$
- $BF_2(u) = -\frac{1}{2}u^3 + \frac{1}{2}u^2 + \frac{1}{2}u + \frac{1}{6}$
- $BF_3(u) = \frac{1}{6}u^3$

$BF_0(u) + BF_1(u) + BF_2(u) + BF_3(u) = 1$

$P_k(u) = BF_0(u) \cdot p_{k-1} + BF_1(u) \cdot p_k + BF_2(u) \cdot p_{k+1} + BF_3(u) \cdot p_{k+2}$
Blending Functions

Properties:

- Translation Commutativity:
  \[ BF_0(u) + BF_1(u) + BF_2(u) + BF_3(u) = 1 \text{ for all } 0 \leq u \leq 1. \]

- Continuity:
  - We need the curve \( P_{k+1}(u) \) to begin where \( P_k(u) \) ended.
  - Taking the difference, we get:
    \[ 0 = P_{k+1}(0) - P_k(1) \]
  - Expanding this out, we get:
    \[
    0 = \left( -BF_0(1) \right) p_{k-1} \\
    + \left( BF_0(0) - BF_1(1) \right) p_k \\
    + \left( BF_1(0) - BF_2(1) \right) p_{k+1} \\
    + \left( BF_2(0) - BF_3(1) \right) p_{k+2} \\
    + \left( BF_3(0) \right) p_{k+3}
    \]
  - Since this is true for all control points \( \{p_{k-1}, p_k, p_{k+1}, p_{k+2}, p_{k+3}\} \), we get:
    \[
    0 = BF_0(1) \\
    BF_0(0) = BF_1(1) \\
    BF_1(0) = BF_2(1) \\
    BF_2(0) = BF_3(1) \\
    BF_3(0) = 0
    \]

\[
P_k(u) = BF_0(u) \cdot p_{k-1} + BF_1(u) \cdot p_k + BF_2(u) \cdot p_{k+1} + BF_3(u) \cdot p_{k+2}
\]
Blending Functions

Properties:

• Translation Commutativity:

More Generally, for the spline to have continuous $n$-th order derivatives, the blending functions need to satisfy:

\[ 0 = BF_0^{(n)}(1) \]
\[ BF_0^{(n)}(0) = BF_1^{(n)}(1) \]
\[ BF_1^{(n)}(0) = BF_2^{(n)}(1) \]
\[ BF_2^{(n)}(0) = BF_3^{(n)}(1) \]
\[ BF_3^{(n)}(0) = 0 \]

More Generally, for the spline to have continuous $n$-th order derivatives, the blending functions need to satisfy:

\[ P_k(u) = BF_0(u) \cdot p_{k-1} + BF_1(u) \cdot p_k + BF_2(u) \cdot p_{k+1} + BF_3(u) \cdot p_{k+2} \]
Comparison: Cardinal vs. Cubic B

Cardinal Splines \((s = \frac{1}{2})\)

\[
BF_0(u) = -\frac{1}{2} u^3 + u^2 - \frac{1}{2} u \\
BF_1(u) = \frac{3}{2} u^3 - \frac{5}{2} u^2 + 1 \\
BF_2(u) = -\frac{3}{2} u^3 + 2u^2 + \frac{1}{2} u \\
BF_3(u) = \frac{1}{2} u^3 - \frac{1}{2} u^2
\]

\[
BF_0(0) = 0 \quad BF_0(1) = 0 \\
BF_1(0) = 1 \quad BF_1(1) = 0 \\
BF_2(0) = 0 \quad BF_2(1) = 1 \\
BF_3(0) = 0 \quad BF_3(1) = 0
\]

Cubic B-Splines \((s = \frac{1}{2})\)

\[
BF_0(u) = -\frac{1}{6} u^3 + \frac{1}{2} u^2 - \frac{1}{2} u + \frac{1}{6} \\
BF_1(u) = \frac{1}{2} u^3 - u^2 + \frac{2}{3} \\
BF_2(u) = -\frac{1}{2} u^3 + \frac{1}{2} u^2 + \frac{1}{2} u + \frac{1}{6} \\
BF_3(u) = \frac{1}{6} u^3
\]

\[
BF_0(0) = \frac{1}{6} \quad BF_0(1) = 0 \\
BF_1(0) = \frac{2}{3} \quad BF_1(1) = \frac{1}{6} \\
BF_2(0) = \frac{1}{6} \quad BF_2(1) = \frac{2}{3} \\
BF_3(0) = 0 \quad BF_3(1) = \frac{1}{6}
\]

\[ P_k(u) = BF_0(u) \cdot p_{k-1} + BF_1(u) \cdot p_k + BF_2(u) \cdot p_{k+1} + BF_3(u) \cdot p_{k+2} \]
Comparison: Cardinal vs. Cubic B

Cardinal Splines ($s = 1/2$)

- $BF_0(u) = -\frac{1}{2}u^3 + u^2 - \frac{1}{2}u$
- $BF_1(u) = \frac{3}{2}u^3 - \frac{5}{2}u^2 + 1$
- $BF_2(u) = -\frac{3}{2}u^3 + 2u^2 + \frac{1}{2}u$
- $BF_3(u) = \frac{1}{2}u^3 - \frac{1}{2}u^2$

- $BF_0'(0) = -\frac{1}{2}$
- $BF_1'(0) = 0$
- $BF_2'(0) = \frac{1}{2}$
- $BF_3'(0) = 0$

- $BF_0'(1) = 0$
- $BF_1'(1) = -\frac{1}{2}$
- $BF_2'(1) = 0$
- $BF_3'(1) = \frac{1}{2}$

Cubic B-Splines ($s = 1/2$)

- $BF_0(u) = -\frac{1}{6}u^3 + \frac{1}{2}u^2 - \frac{1}{2}u + \frac{1}{6}$
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- $BF_0'(0) = -\frac{1}{2}$
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- $BF_0'(1) = 0$
- $BF_1'(1) = -\frac{1}{2}$
- $BF_2'(1) = 0$
- $BF_3'(1) = \frac{1}{2}$

$p_k(u) = BF_0(u) \cdot p_{k-1} + BF_1(u) \cdot p_k + BF_2(u) \cdot p_{k+1} + BF_3(u) \cdot p_{k+2}$
Comparison: Cardinal vs. Cubic B

Cardinal Splines (\(s = 1/2\))

\[
\begin{align*}
BF_0(u) &= -\frac{1}{2}u^3 + u^2 - \frac{1}{2}u \\
BF_1(u) &= \frac{3}{2}u^3 - \frac{5}{2}u^2 + 1 \\
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BF_3(u) &= \frac{1}{2}u^3 - \frac{1}{2}u^2
\end{align*}
\]

\[
\begin{align*}
BF_0''(0) &= 2 & BF_0''(1) &= 5 \\
BF_1''(0) &= -5 & BF_1''(1) &= 4 \\
BF_2''(0) &= 4 & BF_2''(1) &= -5 \\
BF_3''(0) &= -1 & BF_3''(1) &= 2
\end{align*}
\]

Cubic B-Splines (\(s = 1/2\))

\[
\begin{align*}
BF_0(u) &= -\frac{1}{6}u^3 + \frac{1}{2}u^2 - \frac{1}{2}u + \frac{1}{6} \\
BF_1(u) &= \frac{1}{2}u^3 - u^2 + \frac{2}{3} \\
BF_2(u) &= -\frac{1}{2}u^3 + \frac{1}{2}u^2 + \frac{1}{2}u + \frac{1}{6} \\
BF_3(u) &= \frac{1}{6}u^3
\end{align*}
\]

\[
\begin{align*}
BF_0''(0) &= 1 & BF_0''(1) &= 0 \\
BF_1''(0) &= -2 & BF_1''(1) &= 1 \\
BF_2''(0) &= 1 & BF_2''(1) &= -2 \\
BF_3''(0) &= 0 & BF_3''(1) &= 1
\end{align*}
\]

\[P_k(u) = BF_0(u) \cdot p_{k-1} + BF_1(u) \cdot p_k + BF_2(u) \cdot p_{k+1} + BF_3(u) \cdot p_{k+2}\]
Blending Functions

Properties:

• Translation Commutativity:
  \[ BF_0(u) + BF_1(u) + BF_2(u) + BF_3(u) = 1 \text{ for all } 0 \leq u \leq 1. \]

• Continuity:
  \[ 0 = BF_0(1), \quad BF_0(0) = BF_1(1), \quad BF_1(0) = BF_2(1), \quad BF_2(0) = BF_3(1), \quad BF_3(0) = 0 \]

• Convex Hull Containment:
  ◦ A point is inside the convex hull of a collection of points if and only if it can be expressed as the weighted average of the points, where all the weights are non-negative.
  \[ \Rightarrow BF_0(u), BF_1(u), BF_2(u), BF_3(u) \geq 0, \text{ for all } 0 \leq u \leq 1. \]

\[ P_k(u) = BF_0(u) \cdot p_{k-1} + BF_1(u) \cdot p_k + BF_2(u) \cdot p_{k+1} + BF_3(u) \cdot p_{k+2} \]
Comparison: Cardinal vs. Cubic B

Cardinal Splines ($s = 1/2$)

Cubic B-Splines ($s = 1/2$)

$P_k(u) = BF_0(u) \cdot p_{k-1} + BF_3(u)$

Note:
The weights need not be positive for every choice of $s$. 
Comparison: Cardinal vs. Cubic B

Cardinal Splines ($s = 1/2$)  

\[ P_k(u) = BF_0(u) \cdot p_{k-1} + BF_1(u) \cdot p_k + BF_2(u) \cdot p_{k+1} + BF_3(u) \cdot p_{k+2} \]

Cubic B-Splines ($s = 1/2$)
Blending Functions

Properties:

• Translation Commutativity:
  \[ BF_0(u) + BF_1(u) + BF_2(u) + BF_3(u) = 1 \] for all \( 0 \leq u \leq 1 \).

• Continuity:
  \[ 0 = BF_0(1), BF_0(0) = BF_1(1), BF_1(0) = BF_2(1), BF_2(0) = BF_3(1), BF_3(0) = 0 \]

• Convex Hull Containment:
  \( BF_0(u), BF_1(u), BF_2(u), BF_3(u) \geq 0 \), for all \( 0 \leq u \leq 1 \).

• Interpolation:
  
  We want the spline segments to satisfy:
  \[ P_k(0) = p_k \quad \text{and} \quad P_k(1) = p_{k+1} \]

  \[ BF_0(0) = 0 \quad BF_0(1) = 0 \]
  \[ BF_1(0) = 1 \quad BF_1(1) = 0 \]
  \[ BF_2(0) = 0 \quad BF_2(1) = 1 \]
  \[ BF_3(0) = 0 \quad BF_3(1) = 0 \]

\[ P_k(u) = BF_0(u) \cdot p_{k-1} + BF_1(u) \cdot p_k + BF_2(u) \cdot p_{k+1} + BF_3(u) \cdot p_{k+2} \]
Comparison: Cardinal vs. Cubic B

Cardinal Splines ($s = 1/2$)

$$BF_0(u) = -\frac{1}{2}u^3 + u^2 - \frac{1}{2}u$$
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<tbody>
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<td>$BF_0(0)$</td>
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Cubic B-Splines ($s = 1/2$)

$$BF_0(u) = -\frac{1}{6}u^3 + \frac{1}{2}u^2 - \frac{1}{2}u + \frac{1}{6}$$
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$$P_k(u) = BF_0(u) \cdot p_{k-1} + BF_1(u) \cdot p_k + BF_2(u) \cdot p_{k+1} + BF_3(u) \cdot p_{k+2}$$
Blending Functions

Properties:

- Translation Commutativity:
  \[ BF_0(u) + BF_1(u) + BF_2(u) + BF_3(u) = 1 \] for all \( 0 \leq u \leq 1 \).

- Continuity:
  \[
  
  \begin{align*}
  0 &= BF_0(1) \\
  BF_0(0) &= BF_1(1) \\
  BF_1(0) &= BF_2(1) \\
  BF_2(0) &= BF_3(1) \\
  BF_3(0) &= 0
  \end{align*}
  \]

- Convex Hull Containment:
  \[ BF_0(u), BF_1(u), BF_2(u), BF_3(u) \geq 0, \text{ for all } 0 \leq u \leq 1. \]

- Interpolation:
  \[
  
  \begin{array}{ccc}
  BF_0(0) & 0 & BF_0(1) & 0 \\
  BF_1(0) & = & 1 & BF_1(1) & = & 0 \\
  BF_2(0) & 0 & BF_2(1) & = & 1 \\
  BF_3(0) & 0 & BF_3(1) & 0 \\
  \end{array}
  \]

Required Conditions

Desirable Conditions

\[
  P_k(u) = BF_0(u) \cdot p_{k-1} + BF_1(u) \cdot p_k + BF_2(u) \cdot p_{k+1} + BF_3(u) \cdot p_{k+2}
  \]
Summary

• A spline is a \textit{piecewise polynomial function} whose derivatives satisfy some \textit{continuity constraints} across curve junctions.

• Looked at specification for 3 splines:
  ◦ Hermite \text{\{Interpolating, cubic, $C^1$\}}
  ◦ Cardinal \text{\{Approximating, convex-hull containment, cubic, $C^2$\}}
  ◦ Uniform Cubic B-Spline

Spline Demo