Parametric Curves

Michael Kazhdan

(600.457)

HB 10.6 -- 10.9, 10.13
FvDFH 11.2
Overview

• What is a Spline?

• Specific Examples:
  ◦ Hermite Splines
  ◦ Cardinal Splines
  ◦ Uniform Cubic B-Splines

• Comparing Cardinal and Uniform Cubic B-Splines
A spline is a *piecewise polynomial function* whose derivatives satisfy some *continuity constraints* across curve boundaries.

\[ P_i(x) = \sum_{j=0}^{n} a_{ij} \cdot x^j \]
What is a Spline in CG?

A spline is a *piecewise polynomial function* whose derivatives satisfy some *continuity constraints* across curve boundaries.

\[
P_i(x) = \sum_{j=0}^{n} a_{ij} \cdot x^j
\]

\[
P_1(1) = P_2(0) \quad P_1'(1) = P_2'(0)
\]

\[
P_2(1) = P_3(0) \quad P_2'(1) = P_3'(0)
\]

\[
\ldots
\]

\[
\ldots
\]
Overview

• What is a Spline?

• Specific Examples:
  ◦ Hermite Splines
  ◦ Cardinal Splines
  ◦ Uniform Cubic B-Splines

• Comparing Cardinal and Uniform Cubic B-Splines
Specific Example: Hermite Splines

- Interpolating piecewise cubic polynomial, each specified by:
  - Start/end positions
  - Start/end tangents

- Iteratively construct the curve between adjacent end points that interpolate positions and tangents.
Specific Example: Hermite Splines

- Interpolating piecewise *cubic* polynomial, each specified by:
  - Start/end positions
  - Start/end tangents

- Iteratively construct the curve between adjacent end points that interpolate positions and tangents.
Specific Example: Hermite Splines

- Interpolating piecewise cubic polynomial, each specified by:
  - Start/end positions
  - Start/end tangents

- Iteratively construct the curve between adjacent end points that interpolate positions and tangents.
Specific Example: Hermite Splines

- Interpolating piecewise \textit{cubic} polynomial, each specified by:
  - Start/end positions
  - Start/end tangents

- Iteratively construct the curve between adjacent end points that interpolate positions and tangents.
Specific Example: Hermite Splines

- Interpolating piecewise cubic polynomial, each specified by:
  - Start/end positions
  - Start/end tangents

- Iteratively construct the curve between adjacent end points that interpolate positions and tangents.
Specific Example: Hermite Splines

- Interpolating piecewise cubic polynomial, each specified by:
  - Start/end positions
  - Start/end tangents

- Iteratively construct the curve between adjacent end points that interpolate positions and tangents.
Specific Example: Hermite Splines

- Interpolating piecewise *cubic* polynomial, each specified by:
  - Start/end positions
  - Start/end tangents

- Iteratively construct the curve between adjacent end points that interpolate positions and tangents.

Because the end-points of adjacent curves share the same position and derivatives, the Hermite spline is $C^1$ by construction.
Specific Example: Hermite Splines

Given the matrix representations:

\[ P_k(u) = (u^3 \quad u^2 \quad u \quad 1) \cdot \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \quad P'_k(u) = (3 \cdot u^2 \quad 2 \cdot u \quad 1 \quad 0) \cdot \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \]

we can express the values at the end-points as:

\[ p_k = P_k(0) = (0 \quad 0 \quad 0 \quad 1) \cdot \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \quad \tilde{t}_k = P'_k(0) = (0 \quad 0 \quad 1 \quad 0) \cdot \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \]

\[ p_{k+1} = P_k(1) = (1 \quad 1 \quad 1 \quad 1) \cdot \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \quad \tilde{t}_{k+1} = P'_k(1) = (3 \quad 2 \quad 1 \quad 0) \cdot \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \]
Specific Example: Hermite Splines

We can combine the equations:

\[
p_k = P_k(0) = (0 \ 0 \ 0 \ 1) \cdot \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \quad \hat{t}_k = P'_k(0) = (0 \ 0 \ 1 \ 0) \cdot \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}
\]

\[
p_{k+1} = P_k(1) = (1 \ 1 \ 1 \ 1) \cdot \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \quad \hat{t}_{k+1} = P'_k(1) = (3 \ 2 \ 1 \ 0) \cdot \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}
\]

into a single matrix expression:

\[
\begin{pmatrix} p_k \\ p_{k+1} \\ \hat{t}_k \\ \hat{t}_{k+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}
\]
Specific Example: Hermite Splines

Inverting the matrix in the equation:

\[
\begin{pmatrix}
   p_k \\
   p_{k+1} \\
   \vec{t}_k \\
   \vec{t}_{k+1}
\end{pmatrix}
= 
\begin{pmatrix}
   0 & 0 & 0 & 1 \\
   1 & 1 & 1 & 1 \\
   0 & 0 & 1 & 0 \\
   3 & 2 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
   a \\
   b \\
   c \\
   d
\end{pmatrix}
\]

we get:

\[
\begin{pmatrix}
   a \\
   b \\
   c \\
   d
\end{pmatrix} = 
\begin{pmatrix}
   0 & 0 & 0 & 1 \\
   1 & 1 & 1 & 1 \\
   0 & 0 & 1 & 0 \\
   3 & 2 & 1 & 0
\end{pmatrix}^{-1}
\begin{pmatrix}
   p_k \\
   p_{k+1} \\
   \vec{t}_k \\
   \vec{t}_{k+1}
\end{pmatrix}
\]
Specific Example: Hermite Splines

Inverting the matrix in the equation:

\[
\begin{pmatrix}
 p_k \\
 p_{k+1} \\
 \vec{t}_k \\
 \vec{t}_{k+1}
\end{pmatrix} = \begin{pmatrix}
 0 & 0 & 0 & 1 \\
 1 & 1 & 1 & 1 \\
 0 & 0 & 1 & 0 \\
 3 & 2 & 1 & 0
\end{pmatrix} \begin{pmatrix}
 a \\
 b \\
 c \\
 d
\end{pmatrix}
\]

we get:

\[
\begin{pmatrix}
 a \\
 b \\
 c \\
 d
\end{pmatrix} = \begin{pmatrix}
 0 & 0 & 0 & 1 \\
 1 & 1 & 1 & 1 \\
 0 & 0 & 1 & 0 \\
 3 & 2 & 1 & 0
\end{pmatrix}^{-1} \begin{pmatrix}
 p_k \\
 p_{k+1} \\
 \vec{t}_k \\
 \vec{t}_{k+1}
\end{pmatrix} = \begin{pmatrix}
 2 & -2 & 1 & 1 \\
 -3 & 3 & -2 & -1 \\
 0 & 0 & 1 & 0 \\
 1 & 0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
 p_k \\
 p_{k+1} \\
 \vec{t}_k \\
 \vec{t}_{k+1}
\end{pmatrix}
\]
Specific Example: Hermite Splines

Using the facts that:
\[
\begin{pmatrix}
  a \\
  b \\
  c \\
  d
\end{pmatrix}
= \begin{pmatrix}
  2 & -2 & 1 & 1 \\
  -3 & 3 & -2 & -1 \\
  0 & 0 & 1 & 0 \\
  1 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
  p_k \\
  p_{k+1} \\
  \vec{t}_k \\
  \vec{t}_{k+1}
\end{pmatrix}
\]

and
\[P_k(u) = (u^3 \ u^2 \ u \ 1) \cdot \begin{pmatrix}
  a \\
  b \\
  c \\
  d
\end{pmatrix}\]

We get:
\[P_k(u) = (u^3 \ u^2 \ u \ 1) \begin{pmatrix}
  2 & -2 & 1 & 1 \\
  -3 & 3 & -2 & -1 \\
  0 & 0 & 1 & 0 \\
  1 & 0 & 0 & 0
\end{pmatrix}\begin{pmatrix}
  p_k \\
  p_{k+1} \\
  \vec{t}_k \\
  \vec{t}_{k+1}
\end{pmatrix}\]
Specific Example: Hermite Splines

\[ P_k(u) = (u^3 \quad u^2 \quad u \quad 1) \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_k \\ p_{k+1} \\ \hat{t}_k \\ \hat{t}_{k+1} \end{pmatrix} \]

Setting:

- \( H_0(u) = 2u^3 - 3u^2 + 1 \)
- \( H_1(u) = -2u^3 + 3u^2 \)
- \( H_2(u) = u^3 - 2u^2 + u \)
- \( H_3(u) = u^3 - u^2 \)

we can re-write the equation as:

\[ P_k(u) = p_k \cdot H_0(u) + p_{k+1} \cdot H_1(u) + \hat{t}_k \cdot H_2(u) + \hat{t}_{k+1} \]
Specific Example: Hermite Splines

Setting:

- $H_0(u) = 2u^3 - 3u^2 + 1$
- $H_1(u) = -2u^3 + 3u^2$
- $H_2(u) = u^3 - 2u^2 + u$
- $H_3(u) = u^3 - u^2$

Blending Functions

$$P_k(u) = p_k \cdot H_0(u) + p_{k+1} \cdot H_1(u) + \tilde{t}_k \cdot H_2(u) + \tilde{t}_{k+1} \cdot H_3(u)$$
Specific Example: Hermite Splines

- Interpolating piecewise \textit{cubic} polynomial, each specified by:
  - Start/end positions
  - Start/end tangents

- Iteratively construct the curve between adjacent end points that interpolate positions and tangents.

Given the control points, how do we define the value of the tangents/derivatives?
Overview

• What is a Spline?

• Specific Examples:
  ◦ Hermite Splines
  ◦ Cardinal Splines
  ◦ Uniform Cubic B-Splines

• Comparing Cardinal and Uniform Cubic B-Splines
Specific Example: Cardinal Splines

- Interpolating piecewise cubic polynomial, each specified by four control points.
- Iteratively construct the curve between middle two points using adjacent points to define tangents.
Specific Example: Cardinal Splines

- Interpolating piecewise *cubic* polynomial, each specified by four control points.
- Iteratively construct the curve between middle two points using adjacent points to define tangents.
Specific Example: Cardinal Splines

- Interpolating piecewise cubic polynomial, each specified by four control points.
- Iteratively construct the curve between middle two points using adjacent points to define tangents.
Specific Example: Cardinal Splines

- Interpolating piecewise cubic polynomial, each specified by four control points.
- Iteratively construct the curve between middle two points using adjacent points to define tangents.
Specific Example: Cardinal Splines

- Interpolating piecewise \textit{cubic} polynomial, each specified by four control points.
- Iteratively construct the curve between middle two points using adjacent points to define tangents.
Specific Example: Cardinal Splines

- Interpolating piecewise *cubic* polynomial, each specified by four control points.
- Iteratively construct the curve between middle two points using adjacent points to define tangents.
Specific Example: Cardinal Splines

- Interpolating piecewise cubic polynomial, each specified by four control points.
- Iteratively construct the curve between middle two points using adjacent points to define tangents.
Specific Example: Cardinal Splines

- Interpolating piecewise cubic polynomial, each specified by four control points.
- Iteratively construct the curve between middle two points using adjacent points to define tangents.
Specific Example: Cardinal Splines

• Interpolating piecewise *cubic* polynomial, each specified by four control points.

• Iteratively construct the curve between middle two points using adjacent points to define tangents.
Specific Example: Cardinal Splines

- Interpolating piecewise cubic polynomial, each specified by four control points.
- Iteratively construct the curve between middle two points using adjacent points to define tangents.
Specific Example: Cardinal Splines

- Interpolating piecewise cubic polynomial, each specified by four control points.
- Iteratively construct the curve between middle two points using adjacent points to define tangents.
Specific Example: Cardinal Splines

- Interpolating piecewise cubic polynomial, each specified by four control points.
- Iteratively construct the curve between middle two points using adjacent points to define tangents.
Specific Example: Cardinal Splines

- Interpolating piecewise cubic polynomial, each specified by four control points.
- Iteratively construct the curve between middle two points using adjacent points to define tangents.

Because the end-points of adjacent curves share the same position and derivatives, the Cardinal spline has $C^1$ continuity.
Specific Example: Cardinal Splines

Using Hermite splines, we have:

\[
P_k(u) = (u^3 \ u^2 \ u \ 1) \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_k \\ p_{k+1} \\ \tilde{t}_k \\ \tilde{t}_{k+1} \end{pmatrix}
\]

\[
M_{Hermite}
\]

\[
\tilde{t}_k = s(p_{k+1} - p_{k-1})
\]

\[
\tilde{t}_{k+1} = s(p_{k+2} - p_k)
\]
Specific Example: Cardinal Splines

Using Hermite splines, we have:

\[ P_k(u) = (u^3 \ u^2 \ u \ 1) \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_k \\ p_{k+1} \\ \hat{t}_k \\ \hat{t}_{k+1} \end{pmatrix} \]

\[ = (u^3 \ u^2 \ u \ 1) \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_k \\ p_{k+1} \\ s(p_{k+1} - p_{k-1}) \\ s(p_{k+2} - p_k) \end{pmatrix} \]

\[ = (u^3 \ u^2 \ u \ 1) \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s & 0 & s & 0 \\ 1 & -s & 0 & s \end{pmatrix} \begin{pmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{pmatrix} \]
Specific Example: Cardinal Splines

Combining the matrices:

\[ P_k(u) = (u^3 \quad u^2 \quad u \quad 1) \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s & 0 & s & 0 \\ 1 & -s & 0 & s \end{pmatrix} \begin{pmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{pmatrix} \]

\[ P_k(u) = (u^3 \quad u^2 \quad u \quad 1) \begin{pmatrix} -s & 2 - s & s - 2 & s \\ 2s & s - 3 & 3 - 2s & -s \\ -s & 0 & s & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{pmatrix} \]
Specific Example: Cardinal Splines

Setting:

- $C_0(u) = -su^3 + 2su^2 - su$
- $C_1(u) = (2 - s)u^3 + (s - 3)u^2 + 1$
- $C_2(u) = (s - 2)u^3 + (3 - 2s)u^2 + su$
- $C_3(u) = su^3 - su^2$

For $s = 1$:

$$P_k(u) = C_0(u) \cdot p_{k-1} + C_1(u) \cdot p_k + C_2(u) \cdot p_{k+1} + C_3(u) \cdot p_{k+2}$$
Specific Example: Cardinal Splines

- Interpolating piecewise cubic polynomial, each specified by four control points.
- Iteratively construct the curve between middle two points using adjacent points to define tangents.

At the first and last end-points, you can:
- Not draw the final segments
- Double up end points
- Loop the spline around
Overview

• What is a Spline?

• Specific Examples:
  ◦ Hermite Splines
  ◦ Cardinal Splines
  ◦ Uniform Cubic B-Splines

• Comparing Cardinal and Uniform Cubic B-Splines
Approximating piecewise cubic polynomial, each specified by four control points.

- Iteratively construct the curve between middle two points using adjacent points to define tangents.
Specific Example: Uniform Cubic B-Splines

- Approximating piecewise *cubic* polynomial, each specified by four control points.
- Iteratively construct the curve between middle two points using adjacent points to define tangents.
Specific Example: Uniform Cubic B-Splines

• Approximating piecewise cubic polynomial, each specified by four control points.

• Iteratively construct the curve between middle two points using adjacent points to define tangents.
Specific Example: Uniform Cubic B-Splines

• Approximating piecewise cubic polynomial, each specified by four control points.

• Iteratively construct the curve between middle two points using adjacent points to define tangents.

![Diagram of cubic B-spline curve with control points labeled as $p_0, p_1, p_2, p_3, p_4, p_5, p_6, p_7$.]
Specific Example: Uniform Cubic B-Splines

- Approximating piecewise cubic polynomial, each specified by four control points.
- Iteratively construct the curve between middle two points using adjacent points to define tangents.
Specific Example: Uniform Cubic B-Splines

• Approximating piecewise cubic polynomial, each specified by four control points.

• Iteratively construct the curve between middle two points using adjacent points to define tangents.
Specific Example: Uniform Cubic B-Splines

- Approximating piecewise cubic polynomial, each specified by four control points.
- Iteratively construct the curve between middle two points using adjacent points to define tangents.
Specific Example: Uniform Cubic B-Splines

- Approximating piecewise cubic polynomial, each specified by four control points.
- Iteratively construct the curve between middle two points using adjacent points to define tangents.
Specific Example: Uniform Cubic B-Splines

- Approximating piecewise cubic polynomial, each specified by four control points.
- Iteratively construct the curve between middle two points using adjacent points to define tangents
Specific Example: Uniform Cubic B-Splines

• Approximating piecewise cubic polynomial, each specified by four control points.

• Iteratively construct the curve between middle two points using adjacent points to define tangents.
Specific Example: Uniform Cubic B-Splines

• Approximating piecewise cubic polynomial, each specified by four control points.

• Iteratively construct the curve between middle two points using adjacent points to define tangents.
Specific Example: Uniform Cubic B-Splines

- Approximating piecewise \textit{cubic} polynomial, each specified by four control points.
- Iteratively construct the curve between middle two points using adjacent points to define tangents.
Specific Example: Uniform Cubic B-Splines

- Approximating piecewise cubic polynomial, each specified by four control points.
- Iteratively construct the curve between middle two points using adjacent points to define tangents.
Specific Example: Uniform Cubic B-Splines

- Approximating piecewise cubic polynomial, each specified by four control points.
- Iteratively construct the curve between middle two points using adjacent points to define tangents.
Specific Example: Uniform Cubic B-Splines

- Approximating piecewise *cubic* polynomial, each specified by four control points.
- Iteratively construct the curve between middle two points using adjacent points to define tangents.
Specific Example: Uniform Cubic B-Splines

- Approximating piecewise \textit{cubic} polynomial, each specified by four control points.
- Iteratively construct the curve between middle two points using adjacent points to define tangents.
Specific Example: Uniform Cubic B-Splines

- Approximating piecewise cubic polynomial, each specified by four control points.
- Iteratively construct the curve between middle two points using adjacent points to define tangents.
Specific Example: Uniform Cubic B-Splines

- Approximating piecewise cubic polynomial, each specified by four control points.
- Iteratively construct the curve between middle two points using adjacent points to define tangents.

Specific Example: Uniform Cubic B-Splines

\[ p_0 \rightarrow p_1 \rightarrow p_2 \rightarrow p_3 \rightarrow p_4 \rightarrow p_5 \rightarrow p_6 \rightarrow p_7 \]
Specific Example: Uniform Cubic B-Splines

• Approximating piecewise cubic polynomial, each specified by four control points.

• Iteratively construct the curve between middle two points using adjacent points to define tangents.
Specific Example: Uniform Cubic B-Splines

Using Hermite splines, we have:

\[ P_k(u) = (u^3 \quad u^2 \quad u \quad 1) \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p'_k \\ p'_{k+1} \\ \tilde{t}_k \\ \tilde{t}_{k+1} \end{pmatrix} \]

Specific Example:

Uniform Cubic B-Splines

\[ p'_k = \frac{(p_{k-1} + 4p_k + p_{k+1})}{6} \]

\[ p'_{k+1} = \frac{(p_k + 4p_{k+1} + p_{k+2})}{6} \]

\[ \tilde{t}_k = s(p_{k+1} - p_{k-1}) \]

\[ \tilde{t}_{k+1} = s(p_{k+2} - p_k) \]
Using Hermite splines, we have:

\[ P_k(u) = (u^3 \quad u^2 \quad u \quad 1) \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p'_k \\ p'_{k+1} \\ \tilde{t}_k \\ \tilde{t}_{k+1} \end{pmatrix} \]

Specific Example: Uniform Cubic B-Splines
Specific Example: Uniform Cubic B-Splines

Combining the matrices:

\[ P_k(u) = (u^3 \ u^2 \ u \ 1) \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ -6s & 0 & 6s & 0 \\ 1 & -6s & 0 & 6s \end{pmatrix} \begin{pmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{pmatrix} \]

\[ P_k(u) = (u^3 \ u^2 \ u \ 1) \frac{1}{6} \begin{pmatrix} 2 - 6s & 6 - 6s & -6 + 6s & -2 + 6s \\ -3 + 12s & -9 + 6s & 9 - 12s & 3 - 6s \\ -6s & 0 & 6s & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix} \begin{pmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{pmatrix} \]
Specific Example: Uniform Cubic B-Splines

Setting:

- \( B_{0,3}(u) = \frac{1}{6} (1 - u)^3 \)
- \( B_{1,3}(u) = \frac{1}{6} (3u^3 - 6u^2 + 4) \)
- \( B_{2,3}(u) = \frac{1}{6} (-3u^3 + 3u^2 + 3u + 1) \)
- \( B_{3,3}(u) = \frac{1}{6} u^3 \)

Blending Functions

\[
P_k(u) = B_{0,3}(u) \cdot p_{k-1} + B_{1,3}(u) \cdot p_k + B_{2,3}(u) \cdot p_{k+1} + B_{3,3}(u) \cdot p_{k+2}
\]
Specific Example: Uniform Cubic B-Splines

- Approximating piecewise cubic polynomial, each specified by four control points.
- Iteratively construct the curve between middle two points using adjacent points to define tangents.

At the first and last end-points, you can:
- Not draw the final segments
- Double up end points
- Loop the spline around
Overview

• What is a Spline?

• Specific Examples:
  ◦ Hermite Splines
  ◦ Cardinal Splines
  ◦ Uniform Cubic B-Splines

• Comparing Cardinal and Uniform Cubic B-Splines
Blending Functions

Blending functions provide a way for expressing the functions $P_k(u)$ as a weighted sum of the four control points $p_{k-1}, p_k, p_{k+1},$ and $p_{k+2}$:

$$P_k(u) = BF_0(u) \cdot p_{k-1} + BF_1(u) \cdot p_k + BF_2(u) \cdot p_{k+1} + BF_3(u) \cdot p_{k+2}$$
Blending Functions

Properties:

- **Translation Commutativity:**
  \[ BF_0(u) + BF_1(u) + BF_2(u) + BF_3(u) = 1 \text{ for all } 0 \leq u \leq 1. \]

If we translate all the control points by the same vector \( q \), the position of the new point at the value \( u \) will just be the position of the old value at \( u \), translated by \( q \):

\[
Q_k(u) = BF_0(u)(q + p_{k-1}) + BF_1(u)(q + p_k) + BF_2(u)(q + p_{k+1}) + BF_3(u)(q + p_{k+2})
= (BF_0(u) + BF_1(u) + BF_2(u) + BF_3(u))q + P_k(u)
= q + P_k(u)
\]

- **Interpolation:**
  - \( BF_0(0) = BF_2(0) = BF_3(0) = 0 \)
  - \( BF_0(1) = BF_1(1) = BF_3(1) = 0 \)
  - \( BF_2(0) = 1 \)
  - \( BF_2(1) = 1 \)

\[
P_k(u) = BF_0(u) \cdot p_{k-1} + BF_1(u) \cdot p_k + BF_2(u) \cdot p_{k+1} + BF_3(u) \cdot p_{k+2}
\]
Comparison: Cardinal vs. Cubic B

Cardinal Splines ($s = 1/2$)

- $BF_0(u) = -\frac{1}{2}u^3 + u^2 - \frac{1}{2}u$
- $BF_1(u) = \frac{3}{2}u^3 - \frac{5}{2}u^2 + 1$
- $BF_2(u) = -\frac{3}{2}u^3 + 2u^2 + \frac{1}{2}u$
- $BF_3(u) = \frac{1}{2}u^3 - \frac{1}{2}u^2$

$BF_0(u) + BF_1(u) + BF_2(u) + BF_3(u) = 1$

Cubic B-Splines

- $BF_0(u) = -\frac{1}{6}u^3 + \frac{1}{2}u^2 - \frac{1}{2}u + \frac{1}{6}$
- $BF_1(u) = \frac{1}{2}u^3 - u^2 + \frac{2}{3}$
- $BF_2(u) = -\frac{1}{2}u^3 + \frac{1}{2}u^2 + \frac{1}{2}u + \frac{1}{6}$
- $BF_3(u) = \frac{1}{6}u^3$

$BF_0(u) + BF_1(u) + BF_2(u) + BF_3(u) = 1$
Blending Functions

Properties:

• Translation Commutativity:
  \[ BF_0(u) + BF_1(u) + BF_2(u) + BF_3(u) = 1 \] for all \( 0 \leq u \leq 1 \).

• Continuity:
  \[ BF_0(1) = BF_3(0) = 0 \]
  \[ BF_1(1) = BF_0(0) \]
  \[ BF_2(1) = BF_1(0) \]
  \[ BF_3(1) = BF_2(0) \]

We need the curve \( P_{k+1}(u) \) begin where \( P_k(u) \) ended:

\[
0 = P_{k+1}(0) - P_k(1)
\]

Since this equation has to hold true regardless of the values of \( p_k \), the conditions on the left have to be true.

\[
P_k(u) = BF_0(u) \cdot p_{k-1} + BF_1(u) \cdot p_k + BF_2(u) \cdot p_{k+1} + BF_3(u) \cdot p_{k+2}
\]
Blending Functions

Properties:

• Translation Commutativity:
  \[ BF_0(u) + BF_1(u) + BF_2(u) + BF_3(u) = 1 \text{ for all } 0 \leq u \leq 1. \]

• Continuity:
  \[ BF_0(1) = BF_3(0) = 0 \]
  \[ BF_1(1) = BF_0(0) \]
  \[ BF_2(1) = BF_1(0) \]
  \[ BF_3(1) = BF_2(0) \]

• Convex Hull Containment:
  \[ BF_0(u), BF_1(u), BF_2(u), BF_3(u) \geq 1, \text{ for all } 0 \leq u \leq 1. \]

• Interpolation:
  \[ BF_0(0) = BF_2(0) = BF_3(0) = 0 \]
  \[ BF_0(1) = BF_1(1) = BF_3(1) = 0 \]
  \[ BF_1(0) = 1 \]
  \[ BF_2(1) = 1 \]

We need to have the curve \( P_{k+1}(u) = P_k(u) + \sum \) begin where the curve \( P_k(u) \) ended:

Since this equation has to hold true regardless of the values of \( p_k \), the conditions on the left have to be true.

More Generally, if we want the spline to have continuous \( n \)-th order derivatives, the blending functions need to satisfy:

\[ BF_0^{(n)}(1) = BF_3^{(n)}(0) = 0 \]
\[ BF_1^{(n)}(1) = BF_0^{(n)}(0) \]
\[ BF_2^{(n)}(1) = BF_1^{(n)}(0) \]
\[ BF_3^{(n)}(1) = BF_2^{(n)}(0) \]

\[ P_k(u) = BF_0(u) \cdot p_{k-1} + BF_1(u) \cdot p_k + BF_2(u) \cdot p_{k+1} + BF_3(u) \cdot p_{k+2} \]
Comparison: Cardinal vs. Cubic B

Cardinal Splines \((s = 1/2)\)

- \(BF_0(u) = -\frac{1}{2}u^3 + u^2 - \frac{1}{2}u\)
- \(BF_1(u) = \frac{3}{2}u^3 - \frac{5}{2}u^2 + 1\)
- \(BF_2(u) = -\frac{3}{2}u^3 + 2u^2 + \frac{1}{2}u\)
- \(BF_3(u) = \frac{1}{2}u^3 - \frac{1}{2}u^2\)

- \(BF_0(0) = 0\)
- \(BF_1(0) = 1\)
- \(BF_2(0) = 0\)
- \(BF_3(0) = 0\)
- \(BF_0(1) = 0\)
- \(BF_1(1) = 0\)
- \(BF_2(1) = 1\)
- \(BF_3(1) = 0\)

Cubic B-Splines

- \(BF_0(u) = -\frac{1}{6}u^3 + \frac{1}{2}u^2 - \frac{1}{2}u + \frac{1}{6}\)
- \(BF_1(u) = \frac{1}{2}u^3 - u^2 + \frac{2}{3}\)
- \(BF_2(u) = -\frac{1}{2}u^3 + \frac{1}{2}u^2 + \frac{1}{2}u + \frac{1}{6}\)
- \(BF_3(u) = \frac{1}{6}u^3\)

- \(BF_0(0) = \frac{1}{6}\)
- \(BF_1(0) = \frac{2}{3}\)
- \(BF_2(0) = \frac{1}{6}\)
- \(BF_3(0) = 0\)
- \(BF_0(1) = 0\)
- \(BF_1(1) = \frac{1}{6}\)
- \(BF_2(1) = \frac{2}{3}\)
- \(BF_3(1) = \frac{1}{6}\)

\[P_k(u) = BF_0(u) \cdot p_{k-1} + BF_1(u) \cdot p_k + BF_2(u) \cdot p_{k+1} + BF_3(u) \cdot p_{k+2}\]
Comparison: Cardinal vs. Cubic B

Cardinal Splines ($s = 1/2$)

- $BF_0(u) = -\frac{1}{2}u^3 + u^2 - \frac{1}{2}u$
- $BF_1(u) = \frac{3}{2}u^3 - \frac{5}{2}u^2 + 1$
- $BF_2(u) = -\frac{3}{2}u^3 + 2u^2 + \frac{1}{2}u$
- $BF_3(u) = \frac{1}{2}u^3 - \frac{1}{2}u^2$

<table>
<thead>
<tr>
<th>Derivative Condition</th>
<th>Cardinal Splines</th>
<th>Cubic B-Splines</th>
</tr>
</thead>
<tbody>
<tr>
<td>$BF_0'(0) = -\frac{1}{2}$</td>
<td>$BF_0'(1) = 0$</td>
<td>$BF_0'(0) = -\frac{1}{6}u^3 + \frac{1}{2}u^2 - \frac{1}{2}u + \frac{1}{6}$</td>
</tr>
<tr>
<td>$BF_1'(0) = 0$</td>
<td>$BF_1'(1) = -\frac{1}{2}$</td>
<td>$BF_1'(0) = 0$</td>
</tr>
<tr>
<td>$BF_2'(0) = \frac{1}{2}$</td>
<td>$BF_2'(1) = 0$</td>
<td>$BF_2'(0) = \frac{1}{2}$</td>
</tr>
<tr>
<td>$BF_3'(0) = 0$</td>
<td>$BF_3'(1) = \frac{1}{2}$</td>
<td>$BF_3'(0) = 0$</td>
</tr>
</tbody>
</table>

$P_k(u) = BF_0(u) \cdot p_{k-1} + BF_1(u) \cdot p_k + BF_2(u) \cdot p_{k+1} + BF_3(u) \cdot p_{k+2}$
### Comparison: Cardinal vs. Cubic B

<table>
<thead>
<tr>
<th>Cardinal Splines ($s = 1/2$)</th>
<th>Cubic B-Splines</th>
</tr>
</thead>
<tbody>
<tr>
<td>$BF_0(u) = -\frac{1}{2}u^3 + u^2 - \frac{1}{2}u$</td>
<td>$BF_0(u) = -\frac{1}{6}u^3 + \frac{1}{2}u^2 - \frac{1}{2}u + \frac{1}{6}$</td>
</tr>
<tr>
<td>$BF_1(u) = \frac{3}{2}u^3 - \frac{5}{2}u^2 + 1$</td>
<td>$BF_1(u) = \frac{1}{2}u^3 - u^2 + \frac{2}{3}$</td>
</tr>
<tr>
<td>$BF_2(u) = -\frac{3}{2}u^3 + 2u^2 + \frac{1}{2}u$</td>
<td>$BF_2(u) = -\frac{1}{2}u^3 + \frac{1}{2}u^2 + \frac{1}{2}u + \frac{1}{6}$</td>
</tr>
<tr>
<td>$BF_3(u) = \frac{1}{2}u^3 - \frac{1}{2}u^2$</td>
<td>$BF_3(u) = \frac{1}{6}u^3$</td>
</tr>
</tbody>
</table>

- $BF_0''(0) = 2$  
- $BF_0''(1) = 5$  
- $BF_1''(0) = -5$  
- $BF_1''(1) = 4$  
- $BF_2''(0) = 4$  
- $BF_2''(1) = -5$  
- $BF_3''(0) = -1$  
- $BF_3''(1) = 2$  

- $BF_0''(0) = 1$  
- $BF_0''(1) = 0$  
- $BF_1''(0) = -2$  
- $BF_1''(1) = 1$  
- $BF_2''(0) = 1$  
- $BF_2''(1) = -2$  
- $BF_3''(0) = 0$  
- $BF_3''(1) = 1$

\[ P_k(u) = BF_0(u) \cdot p_{k-1} + BF_1(u) \cdot p_k + BF_2(u) \cdot p_{k+1} + BF_3(u) \cdot p_{k+2} \]
Blending Functions

Properties:

• Translation Commutativity:
  \( BF_0(u) + BF_1(u) + BF_2(u) + BF_3(u) = 1 \) for all \( 0 \leq u \leq 1 \).

• Continuity:
  \( BF_0(1) = BF_3(0) = 0 \)
  \( BF_1(1) = BF_0(0) \)
  \( BF_2(1) = BF_1(0) \)
  \( BF_3(1) = BF_2(0) \)

• Convex Hull Containment:
  \( BF_0(u), BF_1(u), BF_2(u), BF_3(u) \geq 0 \), for all \( 0 \leq u \leq 1 \).
  A point is inside the convex hull of a collection of points if and only if it can be expressed as the weighted average of the points, where all the weights are non-negative.

\[
P_k(u) = BF_0(u) \cdot p_{k-1} + BF_1(u) \cdot p_k + BF_2(u) \cdot p_{k+1} + BF_3(u) \cdot p_{k+2}
\]
Comparison: Cardinal vs. Cubic B

Cardinal Splines ($s = 1/2$)

Cubic B-Splines

\[ P_k(u) = BF_0(u) \cdot p_{k-1} + BF_1(u) \cdot p_k + BF_2(u) \cdot p_{k+1} + BF_3(u) \cdot p_{k+2} \]
Comparison: Cardinal vs. Cubic B

Cardinal Splines \((s = 1/2)\)  

\[
P_k(u) = BF_0(u) \cdot p_{k-1} + BF_1(u) \cdot p_k + BF_2(u) \cdot p_{k+1} + BF_3(u) \cdot p_{k+2}
\]
Blending Functions

Properties:

• Translation Commutativity:
  \[ BF_0(u) + BF_1(u) + BF_2(u) + BF_3(u) = 1 \text{ for all } 0 \leq u \leq 1. \]

• Continuity:
  \[ BF_0(1) = BF_3(0) = 0 \]
  \[ BF_1(1) = BF_0(0) \]
  \[ BF_2(1) = BF_1(0) \]
  \[ BF_3(1) = BF_2(0) \]

• Convex Hull Containment:
  \[ BF_0(u), BF_1(u), BF_2(u), BF_3(u) \geq 0, \text{ for all } 0 \leq u \leq 1. \]

• Interpolation:
  \[ BF_0(0) = BF_2(0) = BF_3(0) = 0 \]
  \[ BF_0(1) = BF_1(1) = BF_3(1) = 0 \]
  \[ BF_1(0) = 1 \]
  \[ BF_2(1) = 1 \]

Because we want the spline segments to satisfy:

\[ P_k(0) = p_k \]
\[ P_k(1) = p_{k+1} \]

\[ P_k(u) = BF_0(u) \cdot p_{k-1} + BF_1(u) \cdot p_k + BF_2(u) \cdot p_{k+1} + BF_3(u) \cdot p_{k+2} \]
Comparison: Cardinal vs. Cubic B

Cardinal Splines \((s = 1/2)\)

Cardinal Splines

\[
BF_0(u) = -\frac{1}{2} u^3 + u^2 - \frac{1}{2} u \\
BF_1(u) = \frac{3}{2} u^3 - \frac{5}{2} u^2 + u \\
BF_2(u) = -\frac{3}{2} u^3 + 2u^2 + \frac{1}{2} u \\
BF_3(u) = \frac{1}{2} u^3 - \frac{1}{2} u^2
\]

Cubic B-Splines

\[
BF_0(u) = -\frac{1}{6} u^3 + \frac{1}{2} u^2 - \frac{1}{2} u + \frac{1}{6} \\
BF_1(u) = \frac{1}{2} u^3 - u^2 + \frac{2}{3} u + \frac{1}{6} \\
BF_2(u) = -\frac{1}{2} u^3 + \frac{1}{2} u^2 + \frac{1}{2} u + \frac{1}{6} \\
BF_3(u) = \frac{1}{6} u^3
\]

\[
P_k(u) = BF_0(u) \cdot p_{k-1} + BF_1(u) \cdot p_k + BF_2(u) \cdot p_{k+1} + BF_3(u) \cdot p_{k+2}
\]
Blending Functions

Properties:

- **Translation Commutativity:**
  \[ BF_0(u) + BF_1(u) + BF_2(u) + BF_3(u) = 1 \] for all \( 0 \leq u \leq 1 \).

- **Continuity:**
  \[ BF_0(1) = BF_3(0) = 0 \]
  \[ BF_1(1) = BF_0(0) \]
  \[ BF_2(1) = BF_1(0) \]
  \[ BF_3(1) = BF_2(0) \]

- **Convex Hull Containment:**
  \[ BF_0(u), BF_1(u), BF_2(u), BF_3(u) \geq 0, \text{ for all } 0 \leq u \leq 1. \]

- **Interpolation:**
  \[ BF_0(0) = BF_2(0) = BF_3(0) = 0 \]
  \[ BF_0(1) = BF_1(1) = BF_3(1) = 0 \]
  \[ BF_1(0) = 1 \]
  \[ BF_2(1) = 1 \]

\[ P_k(u) = BF_0(u) \cdot p_{k-1} + BF_1(u) \cdot p_k + BF_2(u) \cdot p_{k+1} + BF_3(u) \cdot p_{k+2} \]
Summary

• A spline is a *piecewise polynomial function* whose derivatives satisfy some *continuity constraints* across curve junctions.

• Looked at specification for 3 splines:
  ◦ Hermite \( \{ \text{Interpolating, cubic, } C^1 \} \)
  ◦ Cardinal \( \{ \text{Approximating, convex-hull containment, cubic, } C^2 \} \)
  ◦ Uniform Cubic B-Spline