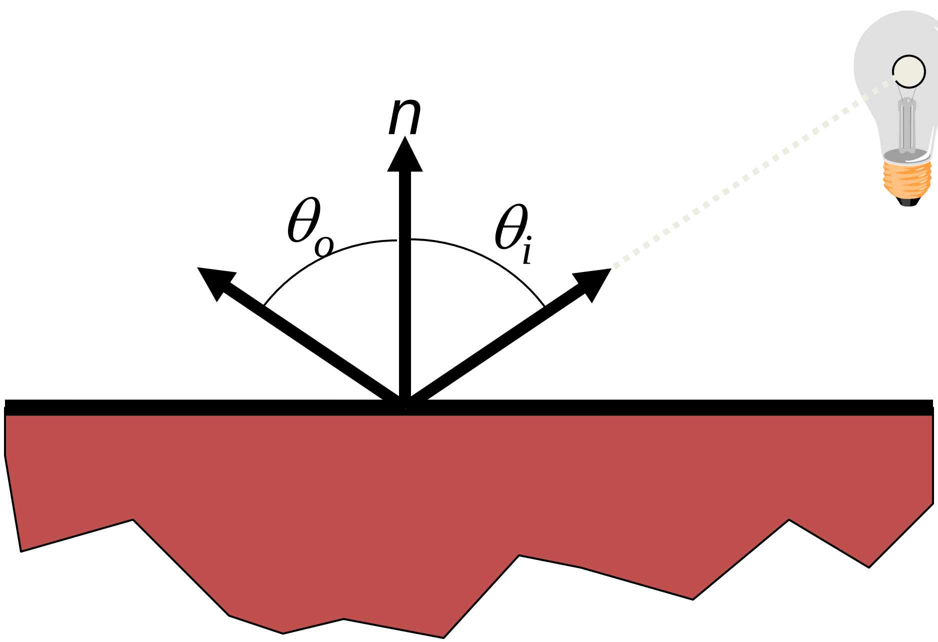
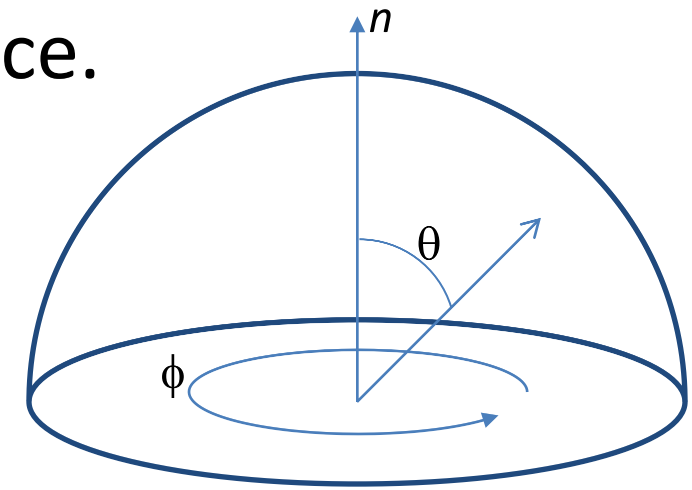
Physically Based Rendering (600.657)

Reflection

Specular Reflection

Light reflects across the surface.



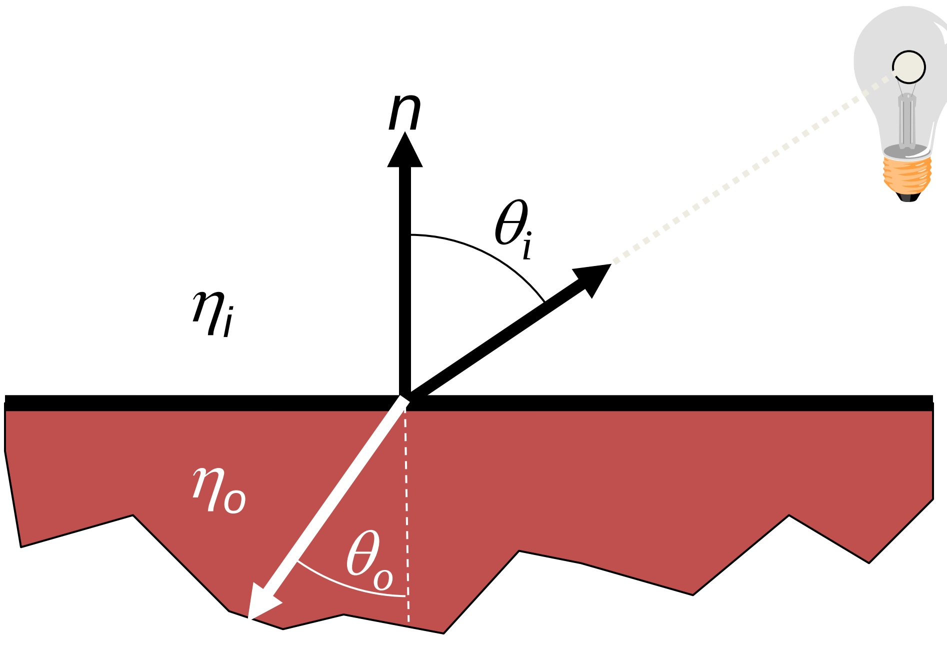


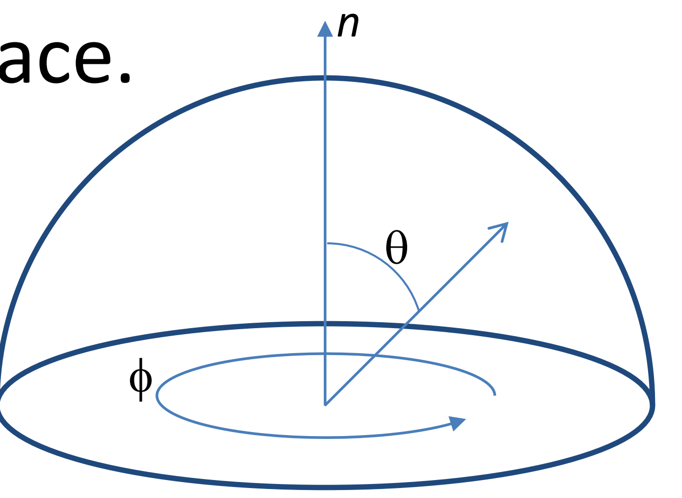
$$\theta_o = \theta_i$$

$$\phi_o = \phi_i \pm \pi$$

Specular Refraction

Light refracts through the surface.





$$\eta_o \sin\theta_o = \eta_i \sin\theta_i$$

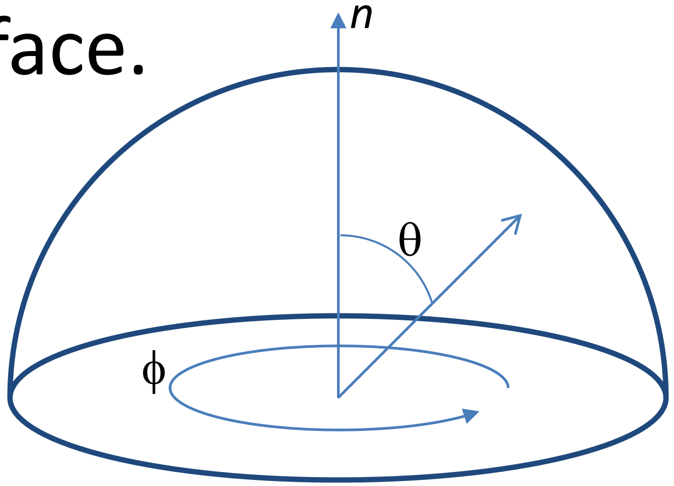
$$\phi_o = \phi_i \pm \pi$$

Specular Refraction

Light refracts through the surface.

Solving for η_o we have:

$$\sin\theta_o = \frac{\eta_i}{\eta_o}\sin\theta_i$$



$$\eta_o \sin\theta_o = \eta_i \sin\theta_i$$

$$\phi_o = \phi_i \pm \pi$$

Specular Refraction

Light refracts through the surface.

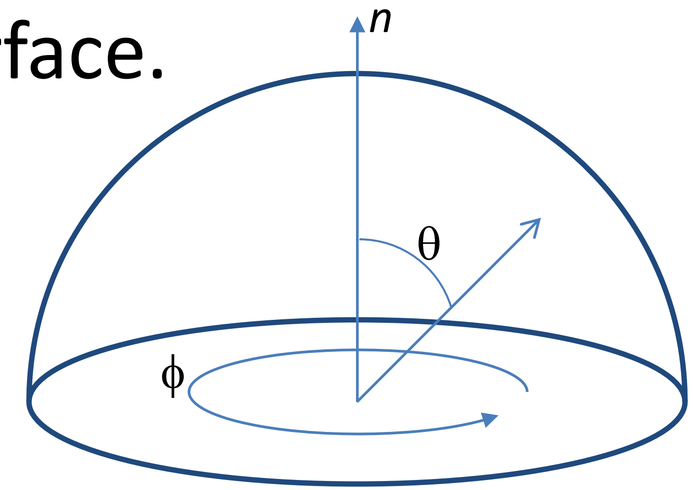
Solving for η_o we have:

$$\sin\theta_o = \frac{\eta_i}{\eta_o}\sin\theta_i$$

When $\eta_i > \eta_o$ and θ_i is big:

$$\theta_i > \sin^{-1}(\eta_o / \eta_i)$$

we have $\sin\theta_o > 1$!!!



$$\eta_o \sin\theta_o = \eta_i \sin\theta_i$$

$$\phi_o = \phi_i \pm \pi$$

Specular Refraction

Light refracts through the surface.

Solving for η_o we have:

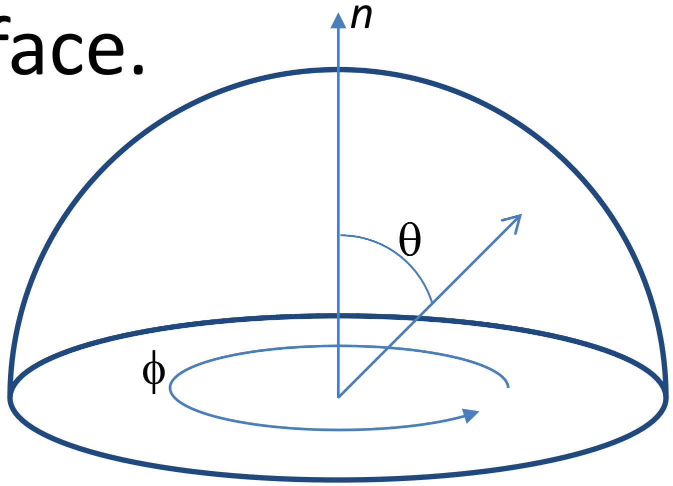
$$\sin\theta_o = \frac{\eta_i}{\eta_o}\sin\theta_i$$

When $\eta_i > \eta_o$ and θ_i is big:

$$\theta_i > \sin^{-1}(\eta_o / \eta_i)$$

we have $\sin\theta_o > 1$!!!

This is the *critical angle*.



$$\eta_o \sin\theta_o = \eta_i \sin\theta_i$$

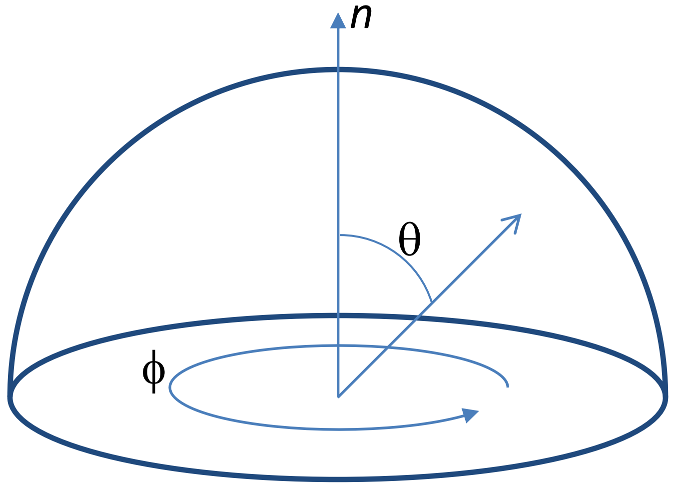
$$\phi_o = \phi_i \pm \pi$$

Specular Refraction

$$\theta_o = \sin^{-1}\left(\frac{\eta_i}{\eta_o}\sin\theta_i\right)$$

Taking the derivative of the outgoing angle as a function of the incoming angle gives:

$$\frac{d\theta_o}{d\theta_i} = \frac{\eta_i \cos\theta_i}{\eta_o \cos\theta_o}$$



$$\eta_o \sin\theta_o = \eta_i \sin\theta_i$$

$$\phi_o = \phi_i \pm \pi$$

Fresnel Reflectance

The fraction of incoming light that is reflected.

Depends on:

- Incoming angle
- Indices of refraction
- Material type

Fresnel Reflectance

The fraction of incoming light that is reflected.

Depends on:

- Incoming angle
- Indices of refraction
- Material type

For reflection we have: $\frac{d\Phi_o}{d\Phi_i} = F_r$

For refraction we have: $\frac{d\Phi_o}{d\Phi_i} = (1 - F_r)$

Fresnel Reflectance

Fresnel Dielectrics:

Don't conduct electricity – can transmit light.

Assuming non-polarized light:

$$F_r = \frac{1}{2}\left(r_{\parallel}{}^2 + r_{\perp}{}^2\right)$$

$$r_{\parallel} = \frac{\eta_o \cos\theta_i - \eta_i \cos\theta_o}{\eta_o \cos\theta_i + \eta_i \cos\theta_o} \quad \text{and} \quad r_{\perp} = \frac{\eta_o \cos\theta_o - \eta_i \cos\theta_i}{\eta_o \cos\theta_o + \eta_i \cos\theta_i}$$

Fresnel Reflectance

Fresnel Dielectrics:

Don't conduct electricity – can transmit light.

Assuming non-polarized light:

$$F_r = \frac{1}{2}\left(r_{\|}^{\,2} + r_{\perp}^{\,2}\right)$$

$$r_{\|} = \frac{\eta_o \cos\theta_i - \eta_i \cos\theta_o}{\eta_o \cos\theta_i + \eta_i \cos\theta_o} \quad \text{and} \quad r_{\perp} = \frac{\eta_o \cos\theta_o - \eta_i \cos\theta_i}{\eta_o \cos\theta_o + \eta_i \cos\theta_i}$$

Note that at the critical angle, $\theta_o=\pi/2$, so that $r_{||}=r_{\perp}=F_r=1$.

Fresnel Reflectance

Fresnel Conductors:

Conduct electricity – don't transmit light.

Assuming non-polarized light:

$$F_r = \frac{1}{2}\left(r_{\parallel}^{\;2} + r_{\perp}^{\;2}\right)$$

$$r_{\parallel} = \frac{\left(\eta^2 + k^2\right)\cos^2\theta_i - 2\eta\cos\theta_i + 1}{\left(\eta^2 + k^2\right)\cos^2\theta_i + 2\eta\cos\theta_i + 1} \quad \text{and} \quad r_{\perp} = \frac{\left(\eta^2 + k^2\right) - 2\eta\cos\theta_i + \cos^2\theta_i}{\left(\eta^2 + k^2\right) + 2\eta\cos\theta_i + \cos^2\theta_i}$$

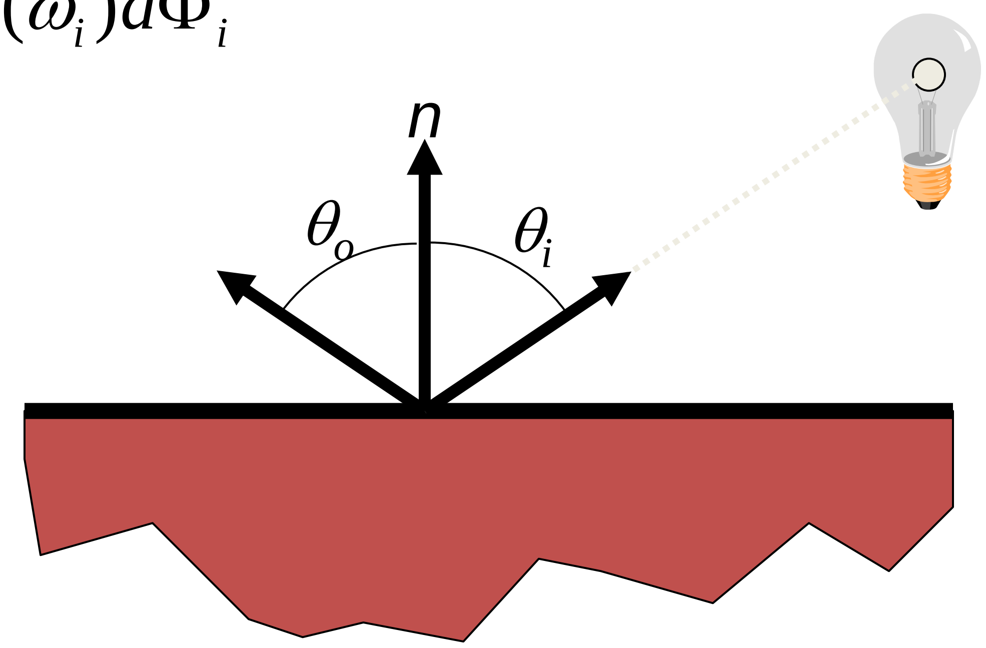
where η is the conductor's index of refraction and *k* is its absorption coefficient.

BRDFs

Specular Reflection

Given incoming direction ω_i and reflected direction ω_o, we want the differential flux to satisfy:

$$d\Phi_o = F_r(\omega_i) d\Phi_i$$



BRDFs

Specular Reflection

Given incoming direction ω_i and reflected direction ω_o, we want the differential flux to satisfy:

$$d\Phi_o = F_r(\omega_i) d\Phi_i$$

$$L_o \cos\theta_o \sin\theta_o dA d\theta_o d\phi_o = F_r(\theta_i) L_i \cos\theta_i \sin\theta_i dA d\theta_i d\phi_i$$

BRDFs

Specular Reflection

Given incoming direction ω_i and reflected direction ω_o, we want the differential flux to satisfy:

$$d\Phi_o = F_r(\omega_i)d\Phi_i$$

$$\Downarrow$$

$$L_o \cos\theta_o \sin\theta_o dA d\theta_o d\phi_o = F_r(\theta_i) L_i \cos\theta_i \sin\theta_i dA d\theta_i d\phi_i$$

Since $\theta_o = \theta_i$ and $\phi_o = \phi_i + \pi$, this gives:

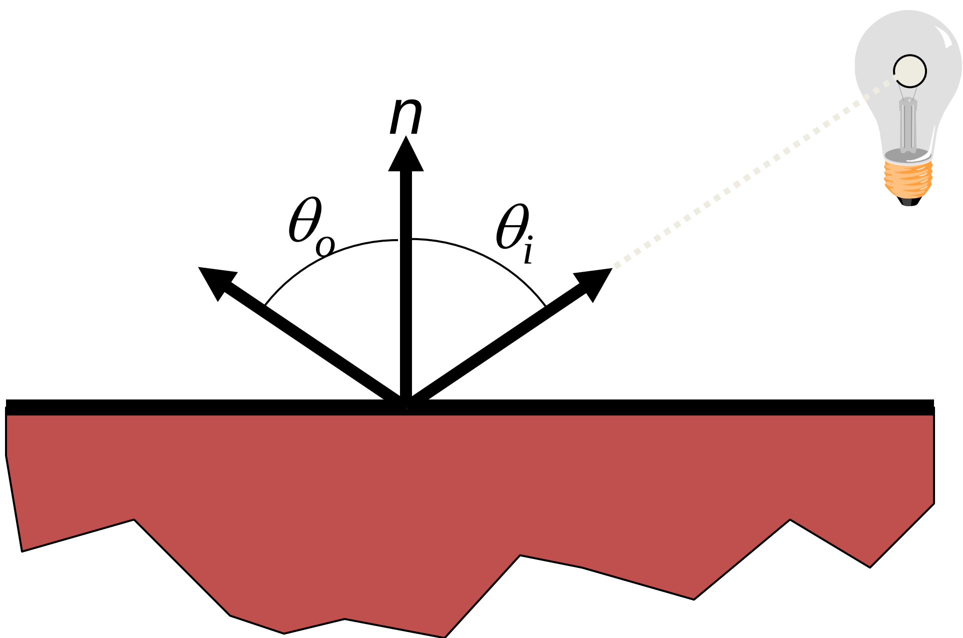
$$L_o = F_r(\theta_i) L_i$$

BRDFs

Specular Reflection

Given incoming direction ω_i and reflected direction ω_o, we want the differential flux to satisfy:

$$d\Phi_o = F_r(\omega_i) d\Phi_i$$



BRDFs

Specular Reflection

Given incoming direction ω_i and reflected direction ω_o, we want the differential flux to satisfy:

$$d\Phi_o = F_r(\omega_i)d\Phi_i$$

This gives:

$$F_r(\theta_i)L_i(\omega_i) = \int_{H^2} f_r(\omega,\omega_o)L_i(\omega)\cos\theta d\omega$$

BRDFs

Specular Reflection

Given incoming direction ω_i and reflected direction ω_o, we want the differential flux to satisfy:

$$L_o(\omega_o) = F_r(\theta_i) L_i(\omega_i)$$

This gives:

$$F_r(\theta_i) L_i(\omega_i) = \int_{H^2} f_r(\omega, \omega_o) L_i(\omega) \cos\theta d\omega$$

$$\Downarrow$$

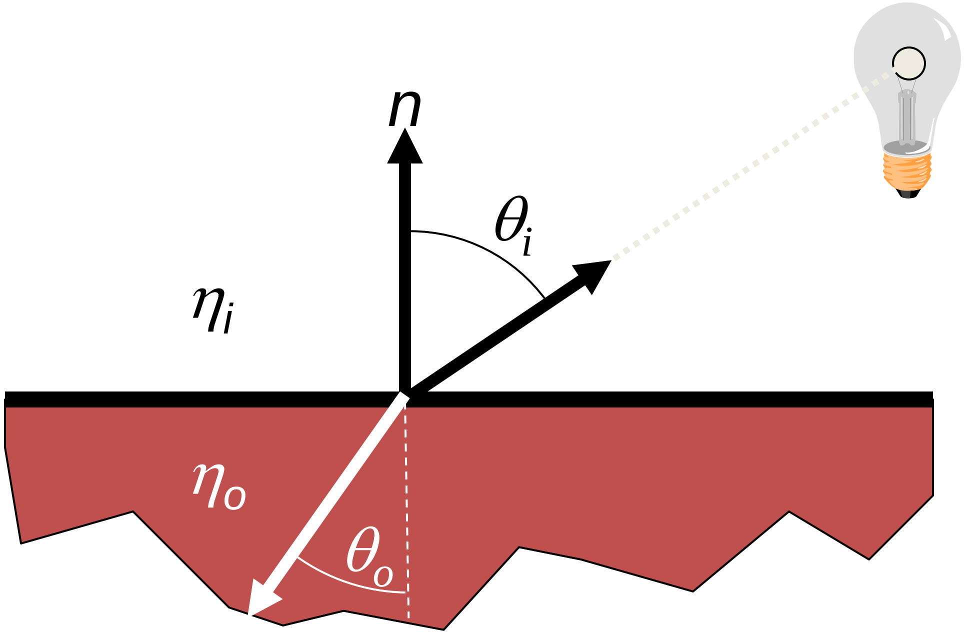
$$f_r(\omega_i, \omega_o) = F_r(\theta_i) \frac{\delta_{\omega_i}(R_n \omega_o)}{\cos\theta_i}$$

BRDFs

Specular Transmission

Given incoming direction ω_i and refracted direction ω_o, we want:

$$d\Phi_o = \left(1 - F_r(\omega_i)\right) d\Phi_i$$



BRDFs

Specular Transmission

Given incoming direction ω_i and refracted direction ω_o, we want:

$$d\Phi_o = \left(1 - F_r(\omega_i)\right) d\Phi_i$$

$$\Downarrow$$

$$L_o \cos\theta_o \sin\theta_o dA d\theta_o d\phi_o = \left(1 - F_r(\theta_i)\right) L_i \cos\theta_i \sin\theta_i dA d\theta_i d\phi_i$$

BRDFs

<u>Specular Transmission</u>

Given incoming direction ω_i and refracted direction ω_o, we want:

$$d\Phi_o = \left(1 - F_r(\omega_i)\right) d\Phi_i$$

$$L_o \cos\theta_o \sin\theta_o \, dA \, d\theta_o \, d\phi_o = \left(1 - F_r(\theta_i)\right) L_i \cos\theta_i \sin\theta_i \, dA \, d\theta_i \, d\phi_i$$

Using the Snell's law and its derivative:

$$L_o \eta_i^2 \, dA \, d\phi_o = \left(1 - F_r(\theta_i)\right) L_i \eta_o^2 \, dA \, d\phi_i$$

BRDFs

Specular Transmission

Given incoming direction ω_i and refracted direction ω_o, we want:

$$d\Phi_o = \left(1 - F_r(\omega_i)\right) d\Phi_i$$

$$L_o \cos\theta_o \sin\theta_o dA d\theta_o d\phi_o = \left(1 - F_r(\theta_i)\right) L_i \cos\theta_i \sin\theta_i dA d\theta_i d\phi_i$$

Using the Snell's law and its derivative:

$$L_o \eta_i^2 dA d\phi_o = \left(1 - F_r(\theta_i)\right) L_i \eta_o^2 dA d\phi_i$$

And since $\phi_i + \pi$, we get:

$$L_o \eta_i^2 = \left(1 - F_r(\theta_i)\right) L_i \eta_o^2$$

BRDFs

<u>Specular Transmission</u>

Given incoming direction ω_i and refracted direction ω_o, we want:

$$L_o = \left(1 - F_r(\theta_i)\right)\frac{\eta_o^2}{\eta_i^2} L_i$$

Which gives:

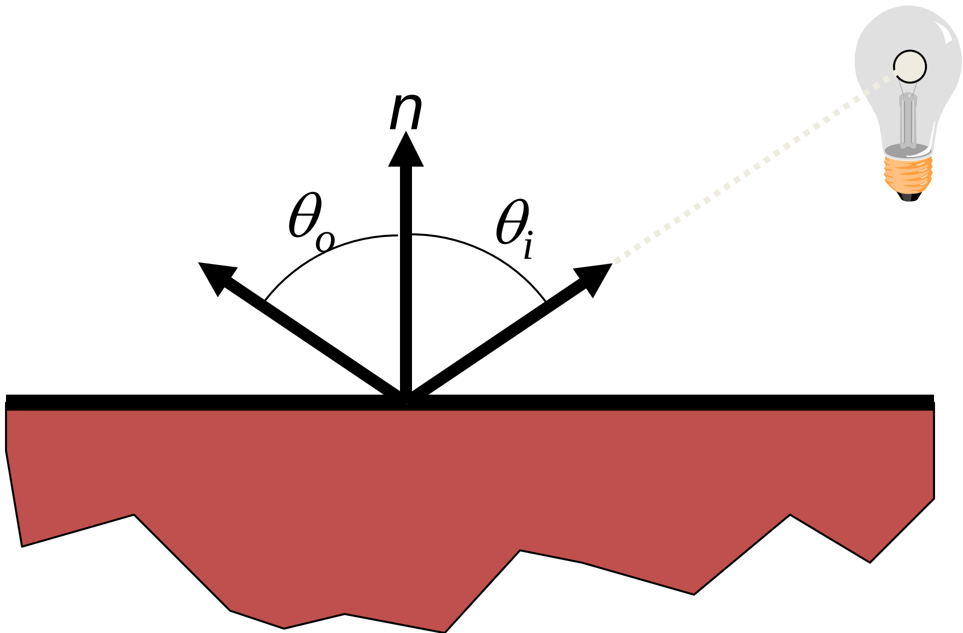
$$f_r(\omega_i, \omega_o) = \left(1 - F_r(\theta_i)\right)\frac{\eta_o^2}{\eta_i^2}\frac{\delta_{\omega_o}(T_n \omega_i)}{\cos\theta_i}$$

BRDFs

Lambertian Reflection

Given incoming direction ω_i, any reflected direction ω_o, and given ρ the fraction of light that is reflected, we want:

$$M = \rho E = \rho \int_{H^2} L_i(\omega_i) \cos\theta_i d\omega_i$$



BRDFs

Lambertian Reflection

Given incoming direction ω_i, any reflected direction ω_o, and given ρ the fraction of light that is reflected, we want:

$$M = \rho E = \rho \int_{H^2} L_i(\omega_i)\cos\theta_i d\omega_i$$

This gives:

$$\rho \int_{H^2} L_i(\omega_i)\cos\theta_i d\omega_i = \int_{H^2}\int_{H^2} f_r(\omega_i,\omega_o) L_i(\omega_i)\cos\theta_i d\omega_i \cos\theta_o d\omega_o$$

$$= f_r \int_{H^2} L_i(\omega_i)\cos\theta_i d\omega_i \int_{H^2}\cos\theta_o d\omega_o$$

$$= \pi f_r \int_{H^2} L_i(\omega_i)\cos\theta_i d\omega_i$$

BRDFs

Lambertian Reflection

Given incoming direction ω_i, any reflected direction ω_o, and given ρ the fraction of light that is reflected, we want:

$$M = \rho E = \rho \int_{H^2} L_i(\omega_i)\cos\theta_i d\omega_i$$

This gives:

$$\rho \int_{H^2} L_i(\omega_i)\cos\theta_i d\omega_i = \int_{H^2}\int_{H^2} f_r(\omega_i,\omega_o) L_i(\omega_i)\cos\theta_i d\omega_i \cos\theta_o d\omega_o$$

$$= f_r \int_{H^2} L_i(\omega_i)\cos\theta_i d\omega_i \int_{H^2}\cos\theta_o d\omega_o$$

$$= \pi f_r \int_{H^2} L_i(\omega_i)\cos\theta_i d\omega_i$$

$$f_r = \frac{\rho}{\pi}$$

Microfacet Models

Assume that the roughness of a surface can be described by local variation of heights/slopes.

Microfacet Models

Assume that the roughness of a surface can be described by local variation of heights/slopes.

Using this model, one can derive different BRDFs depending on whether the underlying facets are assumed to be diffuse or specular.

Microfacet Models

Oren-Nayar:

- The facets are Lambertian
- Microfacet distribution is Gaussian in angle

Taking into account masking, shadowing, and inter-reflection, the BRDF is approximated by:

$$f_r(\omega_i, \omega_o) = \frac{\rho}{\pi}\left(A + B\max(0, \cos(\phi_i - \phi_o)\sin\alpha\tan\beta\right)$$

$$A = 1 - \frac{\sigma^2}{2(\sigma^2 + 0.33)}$$

$$B = \frac{0.45\sigma^2}{\sigma^2 + 0.09}$$

$$\alpha = \max(\theta_i, \theta_o)$$

$$\beta = \min(\theta_i, \theta_o)$$

Microfacet Models

<u>Torrance-Sparrow</u>:

- The facets are specular
- General microfacet distribution *D* (PDF for microfacets with a particular orientation)

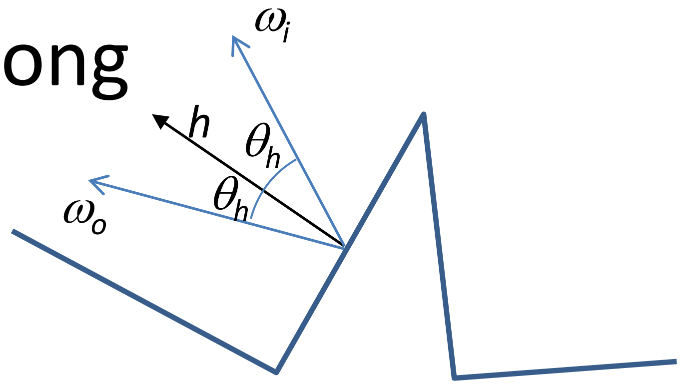
Microfacet Models

Torrance-Sparrow:

- The facets are specular
- General microfacet distribution *D* (PDF for microfacets with a particular orientation)

Since the model is specular, the BRDF value for directions ω_i and ω_o only gets contributions from microfacets oriented along the half-angle direction:

$$h = \frac{\omega_i + \omega_o}{\left|\omega_i + \omega_o\right|}$$



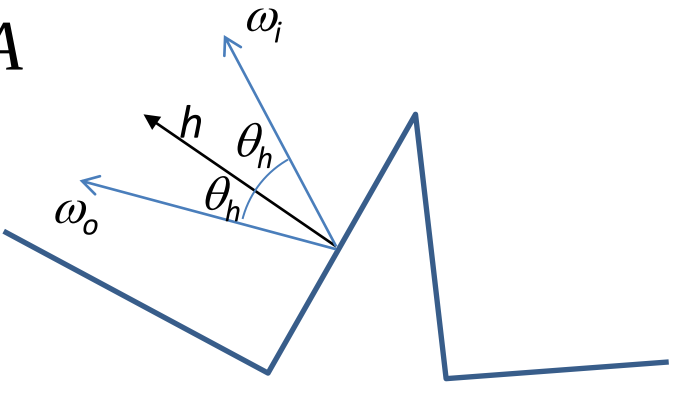
Microfacet Models

Torrance-Sparrow:

- The facets are specular
- General microfacet distribution *D* (PDF for microfacets with a particular orientation)

The change in flux through microfacets with orientation *h* is given by:

$$d\Phi_h = L_i(\omega_i)d\omega_i \cos\theta_h D(\omega_h)dA$$



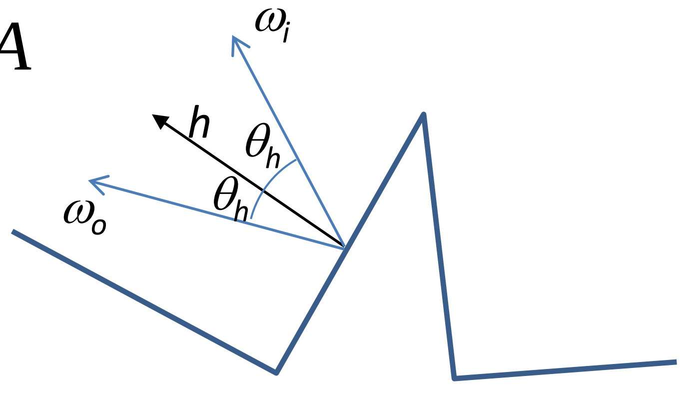
Microfacet Models

Torrance-Sparrow:

- The facets are specular
- General microfacet distribution *D* (PDF for microfacets with a particular orientation)

The change in flux through microfacets with orientation *h* is given by:

$$d\Phi_h = L_i(\omega_i)d\omega_i \cos\theta_h D(\omega_h)dA$$



Assuming Fresnel's law:

$$d\Phi_o = F_r(\omega_i)d\Phi_h$$

$$d\Phi_o = F_r(\omega_i)L_i(\omega_i)d\omega_i \cos\theta_h D(\omega_h)dA$$

Microfacet Models

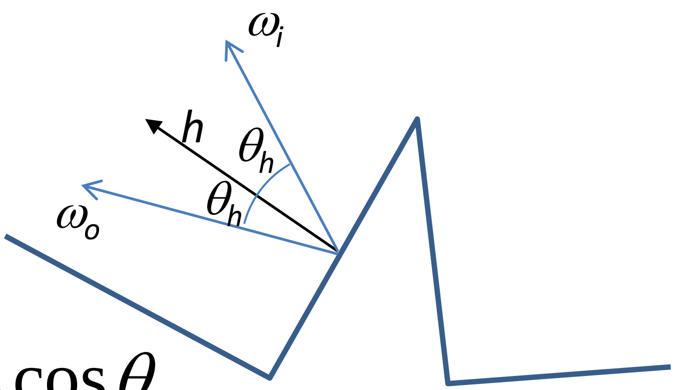
Torrance-Sparrow:

- The facets are specular
- General microfacet distribution *D* (PDF for microfacets with a particular orientation)

$$d\Phi_o = F_r(\omega_i)L_i(\omega_i)d\omega_i \cos\theta_h D(\omega_h)dA$$

Using the fact that the change in flux is:

$$\frac{d\Phi_o}{d\omega_0 dA} = L_o(\omega_o)\cos\theta_o$$



Gives:

$$L(\omega_o) = \frac{F_r(\omega_i)L_i(\omega_i)d\omega_i D(\omega_h)dA\cos\theta_h}{d\omega_0 dA\cos\theta_o}$$

Microfacet Models

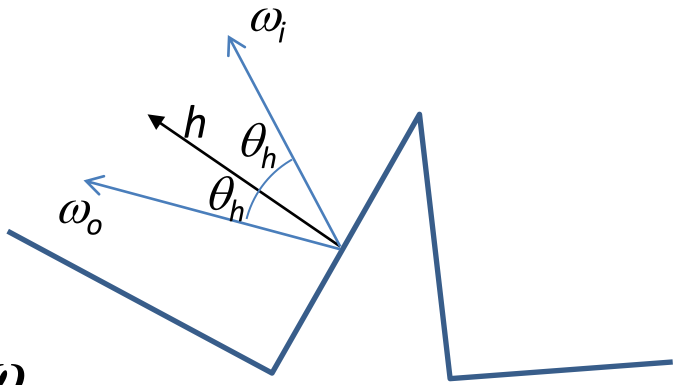
Torrance-Sparrow:

- The facets are specular
- General microfacet distribution D (PDF for microfacets with a particular orientation)

$$L(\omega_o) = \frac{F_r(\omega_i)L_i(\omega_i)d\omega_i D(\omega_h)dA\cos\theta_h}{d\omega_0 dA\cos\theta_o}$$

Finally, using the fact that:

$$d\omega_h = \frac{d\omega_o}{4\cos\theta_h}$$



Gives:

$$L(\omega_o) = \frac{F_r(\omega_i)L_i(\omega_i)D(\omega_h)d\omega_i}{4\cos\theta_o}$$

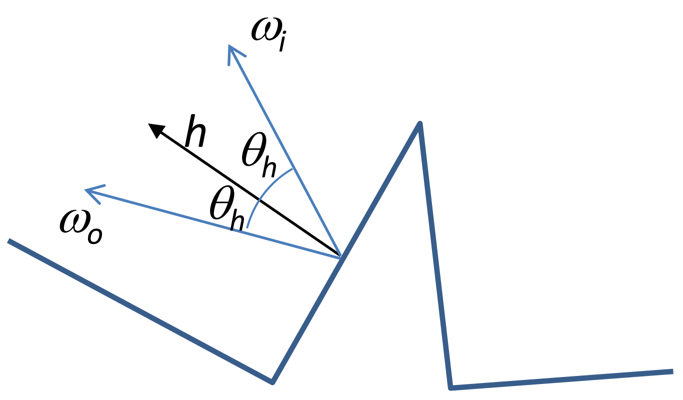
Microfacet Models

Torrance-Sparrow:

- The facets are specular
- General microfacet distribution *D* (PDF for microfacets with a particular orientation)

$$L(\omega_o) = \frac{F_r(\omega_i) L_i(\omega_i) D(\omega_h) d\omega_i}{4\cos\theta_o}$$

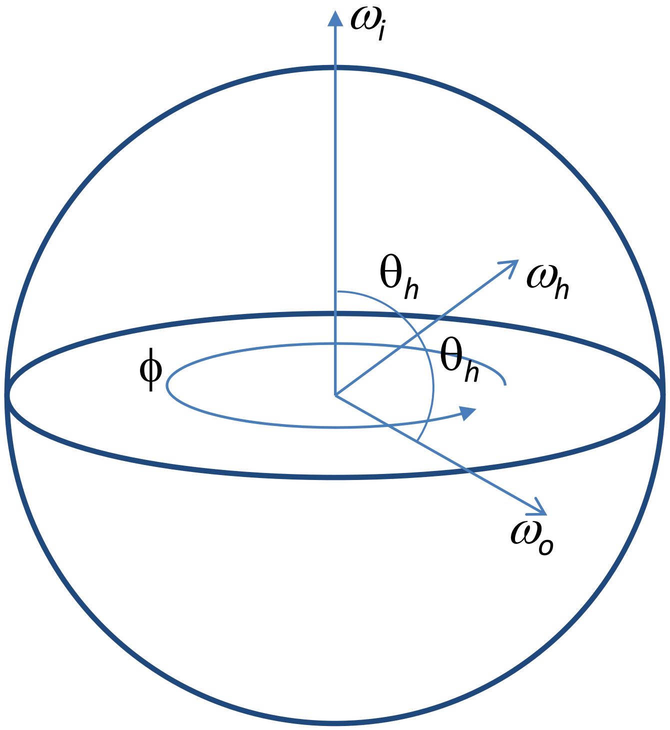
$$f(\omega_o, \omega_i) = \frac{F_r(\omega_i) D(\omega_h)}{4\cos\theta_o \cos\theta_i}$$



Microfacet Models

Relating Half-Angle to Outgoing Change:

$$d\omega_h = \frac{d\omega_o}{4\cos\theta_h}$$



Setting ω_i to the North pole, if (θ_h,ϕ_h) are the spherical angles of the half-angle, then $(2\theta_h,\phi_h)$ are the angles of the outgoing angle:

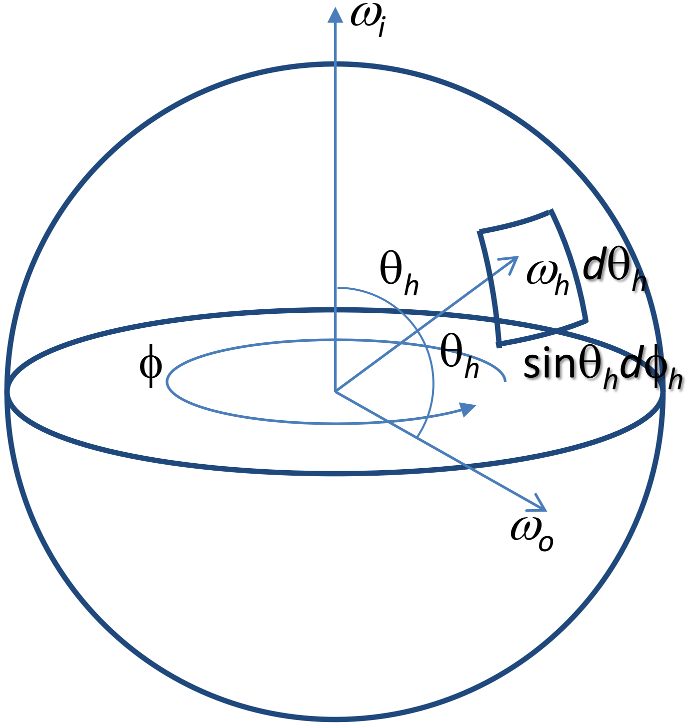
$$\theta_o = 2\theta_h \qquad \phi_o = \phi_h$$

Microfacet Models

Relating Half-Angle to Outgoing Change:

$$d\omega_h = \frac{d\omega_o}{4\cos\theta_h}$$

$$\theta_o = 2\theta_h \qquad \phi_o = \phi_h$$



The area of a small rectangle about ω_h is:

$$d\omega_h = d\theta_h \sin\theta_h d\phi_h$$

Microfacet Models

Relating Half-Angle to Outgoing Change:

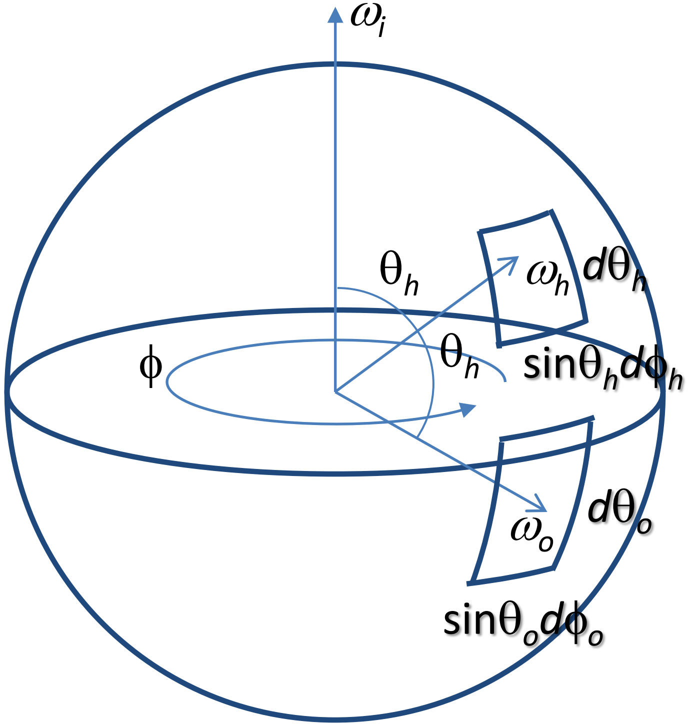
$$d\omega_h = \frac{d\omega_o}{4\cos\theta_h}$$

$$\theta_o = 2\theta_h \qquad \phi_o = \phi_h$$

$$d\omega_h = d\theta_h \sin\theta_h d\phi_h$$

The area of the corresponding rectangle about ω_o is:

$$d\omega_o = d\theta_o \sin\theta_o d\phi_o$$



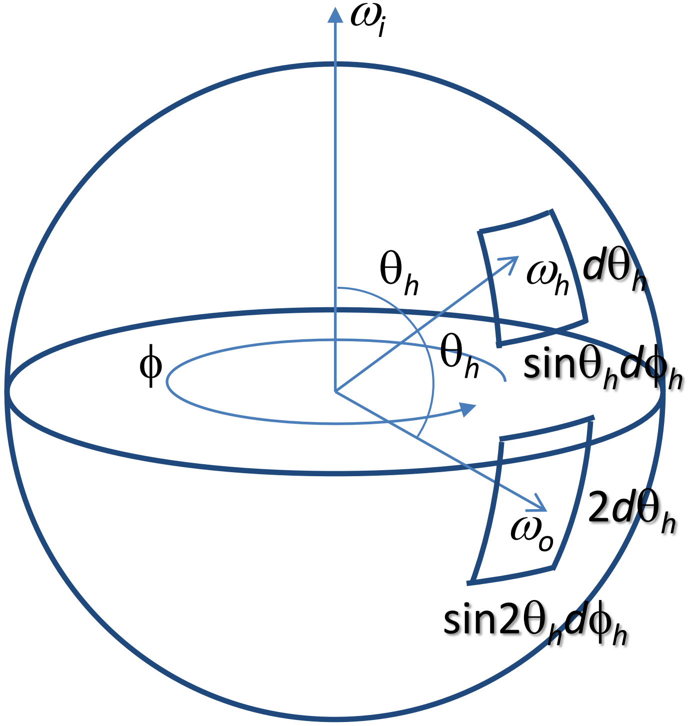
Microfacet Models

Relating Half-Angle to Outgoing Change:

$$d\omega_h = \frac{d\omega_o}{4\cos\theta_h}$$

$$\theta_o = 2\theta_h \qquad \phi_o = \phi_h$$

$$d\omega_h = d\theta_h \sin\theta_h d\phi_h$$



The area of the corresponding rectangle about ω_o is:

$$\begin{aligned} d\omega_o &= d\theta_o \sin\theta_o d\phi_o \\ &= 2d\theta_h \sin 2\theta_h d\phi_h \\ &= 4\sin\theta_h \cos\theta_h d\theta_h d\phi_h \end{aligned}$$

Microfacet Models

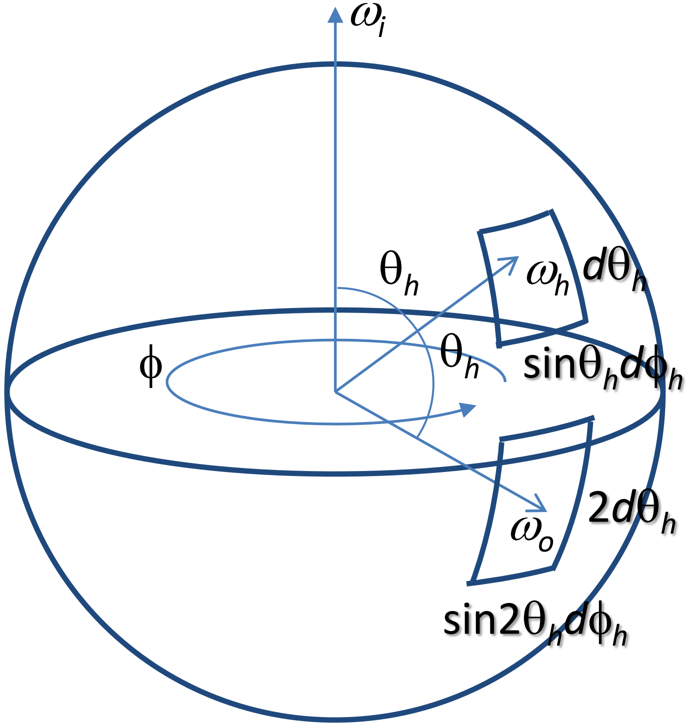
Relating Half-Angle to Outgoing Change:

$$d\omega_h = \frac{d\omega_o}{4\cos\theta_h}$$

$$\theta_o = 2\theta_h \qquad \phi_o = \phi_h$$

$$d\omega_h = d\theta_h \sin\theta_h d\phi_h$$

$$d\omega_o = 4\sin\theta_h \cos\theta_h d\theta_h d\phi_h$$



The ratio of areas is:

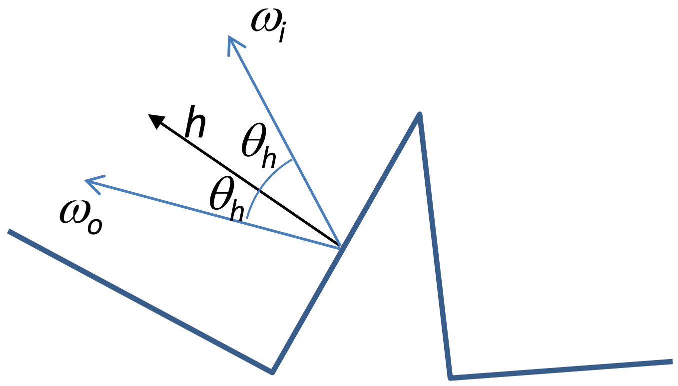
$$\frac{d\omega_h}{d\omega_o} = \frac{1}{4\cos\theta_h}$$

Microfacet Models

Microfacet Distribution:

– Blinn: Cosine of microfacet normals with surface normals falls off exponentially:

$$D(\omega_h) \propto \langle \omega_h, n \rangle^e$$



Microfacet Models

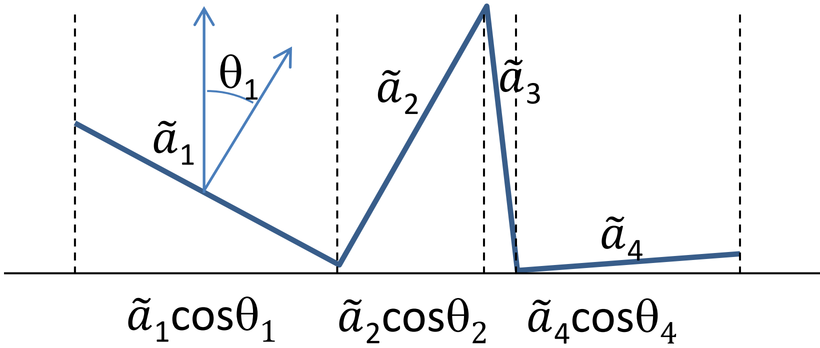
Microfacet Distribution:

– Blinn: Cosine of microfacet normals with surface normals falls off exponentially:

$$D(\omega_h) \propto \langle \omega_h, n \rangle^e$$

Since the projection of the microfacets over a patch onto the plane has to cover the patch:

$$\int_{H^2} D(\omega_h) \cos\theta_h d\omega_h = 1$$



Microfacet Models

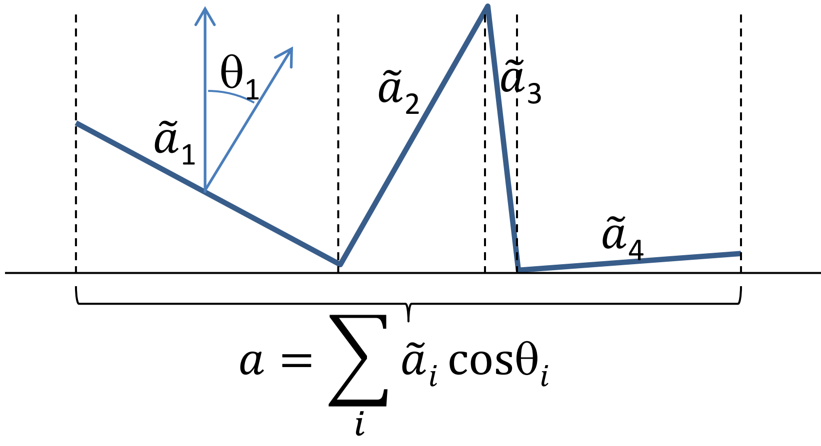
Microfacet Distribution:

– Blinn: Cosine of microfacet normals with surface normals falls off exponentially:

$$D(\omega_h) \propto \langle \omega_h, n \rangle^e$$

Since the projection of the microfacets over a patch onto the plane has to cover the patch:

$$\int_{H^2} D(\omega_h) \cos\theta_h d\omega_h = 1$$



Microfacet Models

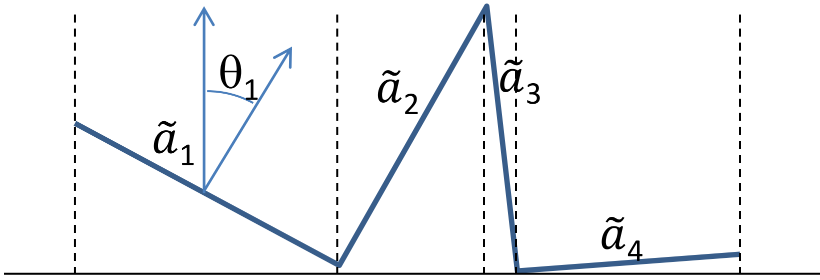
Microfacet Distribution:

– Blinn: Cosine of microfacet normals with surface normals falls off exponentially:

$$D(\omega_h) \propto \langle \omega_h, n \rangle^e$$

Since the projection of the microfacets over a patch onto the plane has to cover the patch:

$$\int_{H^2} D(\omega_h) \cos\theta_h d\omega_h = 1$$



$$1 = c \int_0^{2\pi} \int_0^{\pi/2} (\cos\theta)^e \cos\theta \sin\theta d\theta d\phi$$

$$= c2\pi \int_0^{\pi/2} (\cos\theta)^{e+1} \sin\theta d\theta = -c2\pi \int_1^0 x^{e+1} dx = \frac{c2\pi}{e+2}$$

Microfacet Models

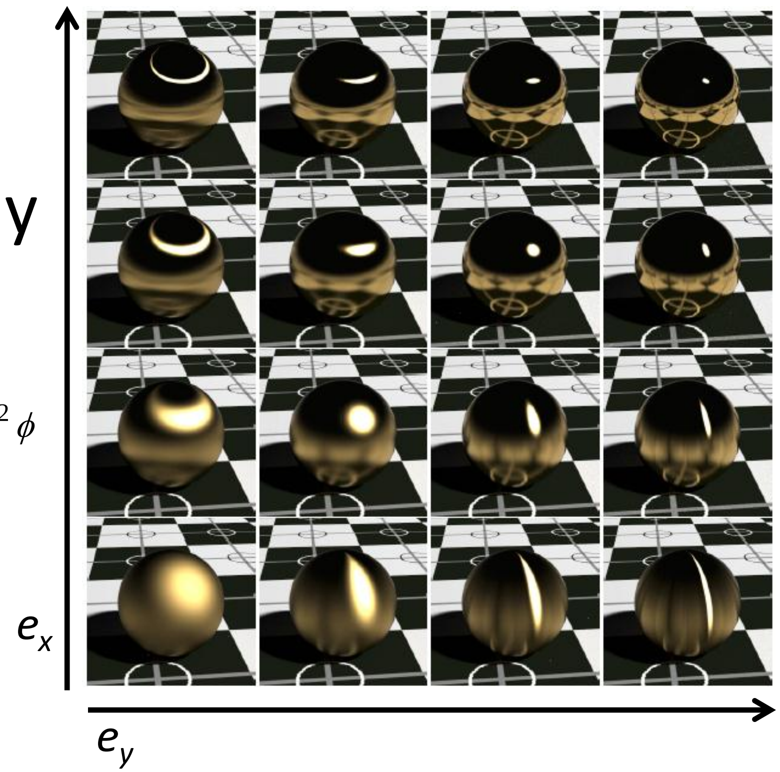
Microfacet Distribution:

- Blinn: Cosine of microfacet normals with surface normals falls off exponentially:

$$D(\omega_h) \propto \langle \omega_h, n \rangle^e$$

- Ashikhmin and Shirley: the normals fall off exponentially but not isotropically:

$$D(\omega_h) = \frac{\sqrt{(e_x+2)(e_y+2)}}{2\pi} \langle \omega_h, n \rangle^{e_x \cos^2\phi + e_y \sin^2\phi}$$



[Ashikhmin and Shirley, 2000]