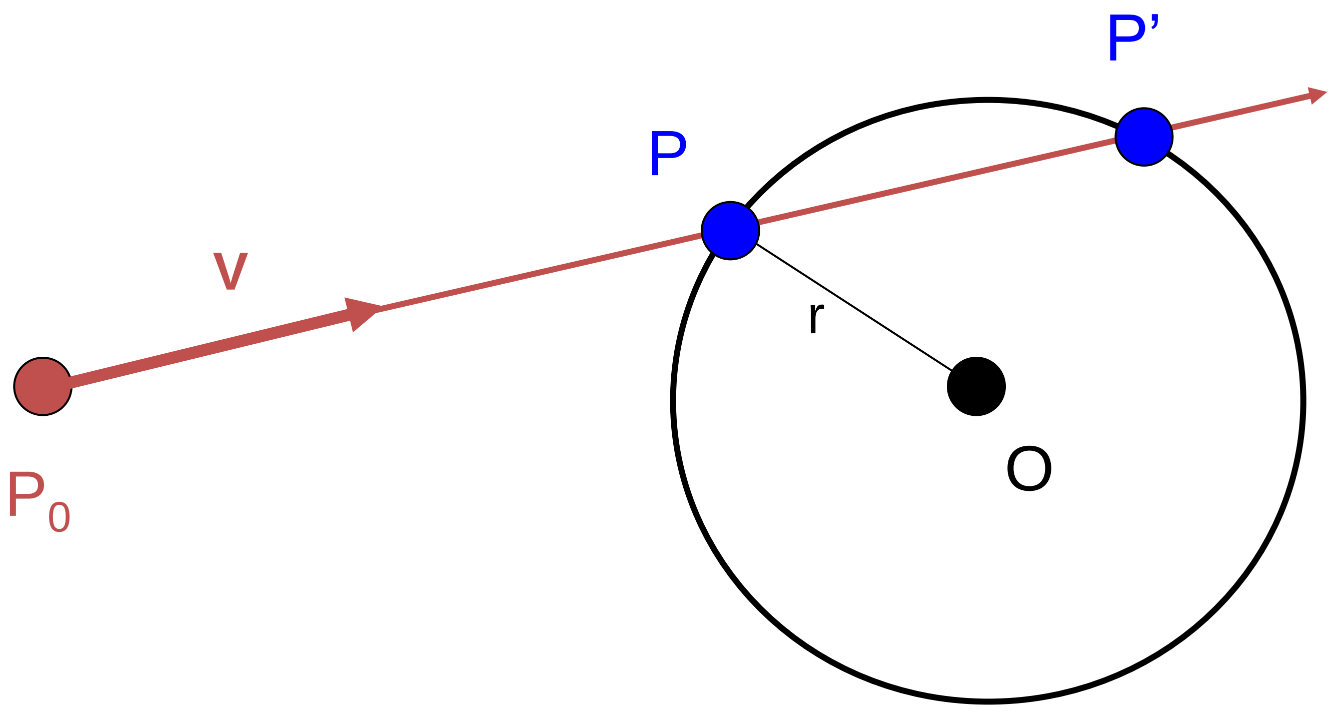
Physically Based Rendering (600.657)

Shapes

Ray-Sphere Intersection

Ray: $P = P_0 + tV$

Sphere: $|P - O|^2 - r^2 = 0$



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Substituting for P, we get:

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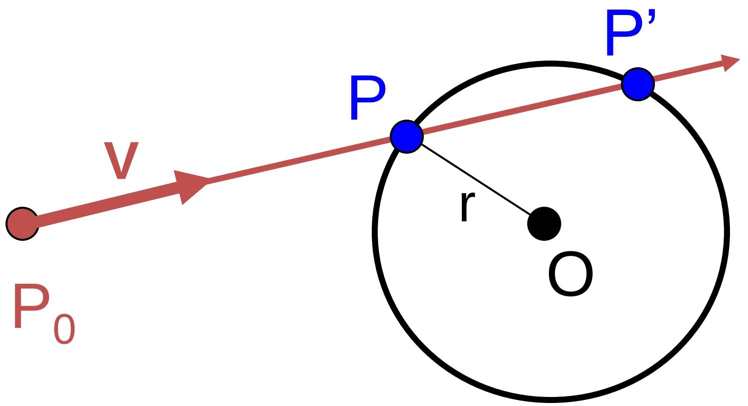

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$$|\mathbf{P_0 + tV} - O|^2 - r^2 = 0$$

Solve quadratic equation:

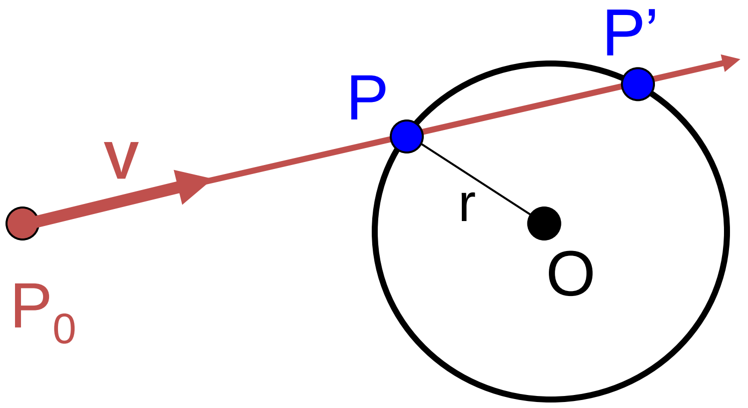
$$at^2 + bt + c = 0$$

where:

$$a = 1$$

$$b = 2\, V \bullet (P_0 - O)$$

$$c = |P_0 - O|^2 - r^2$$



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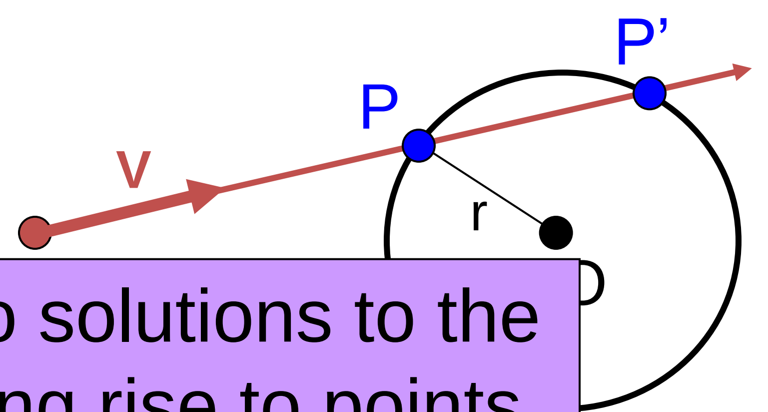

Generally, there are two solutions to the quadratic equation, giving rise to points P and P'.
You want to return the first (positive) hit.

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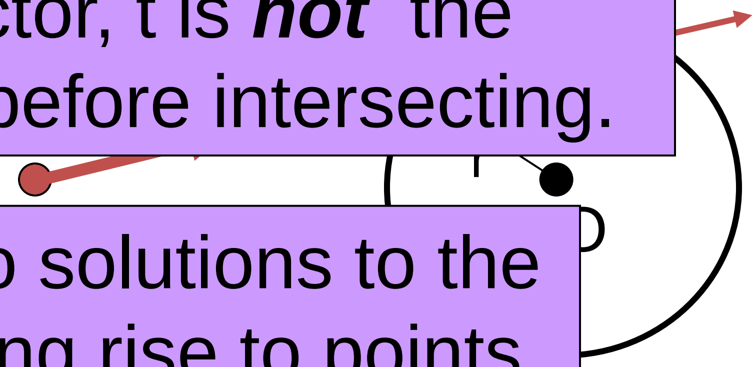
Unless V is a unit-vector, t is ***not*** the distance the ray travels before intersecting.



Generally, there are two solutions to the quadratic equation, giving rise to points P and P'.
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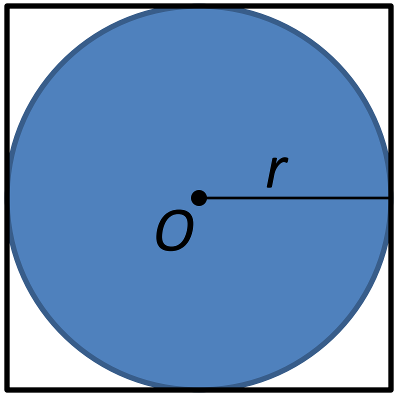
Bounding the Sphere

- Bounding Box:
  - $[O_x\text{-}r, O_y\text{-}r, O_z\text{-}r]$, $[O_x+r, O_y+r, O_z+r]$



Parameterizing the Sphere

- Parametric Equation:

$$p(s,t) = r\left(\sin(\pi t)\cos(2\pi s), \sin(\pi t)\sin(2\pi s), \cos(\pi t)\right) + \left(O_x, O_y, O_z\right)$$

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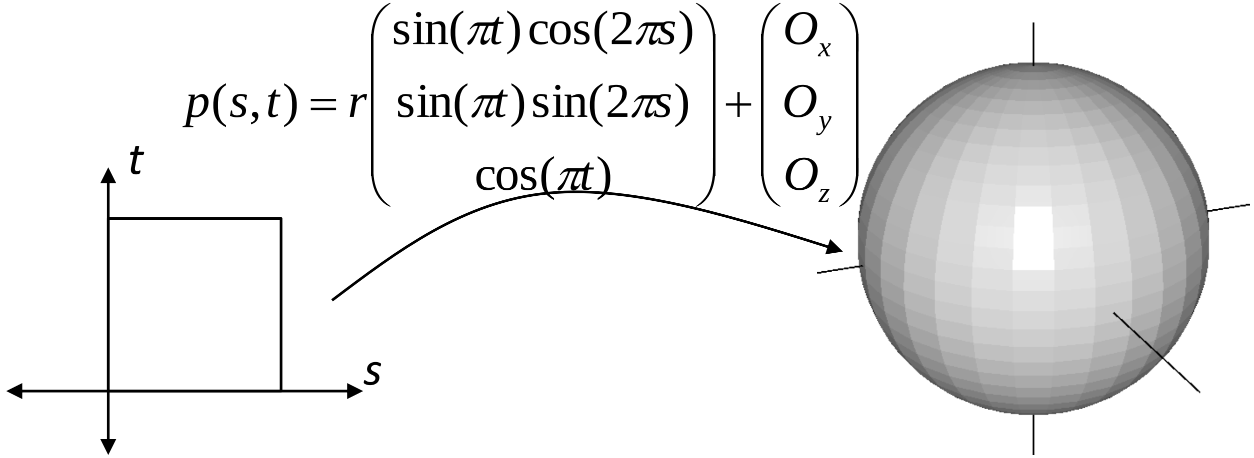

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- Partials:

$$\frac{\partial p}{\partial s}(s,t) = 2\pi r\left(-\sin(\pi t)\sin(2\pi s), \sin(\pi t)\cos(2\pi s), 0\right)$$

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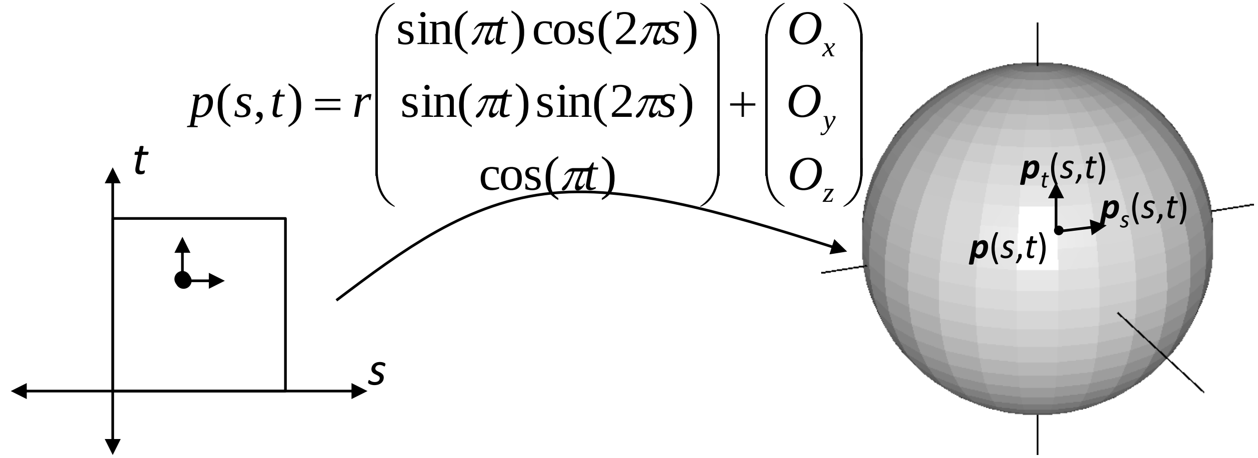

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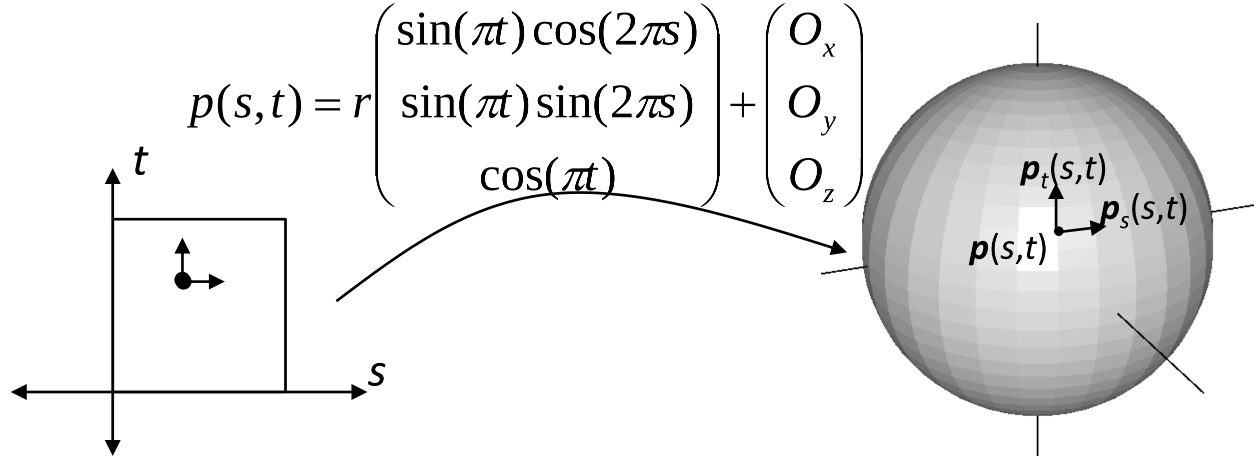

Parameterizing the Sphere

<u>Why do we care?</u>

1. Partial derivatives gives us the normal:

$$\vec{n}(s,t) = \frac{\frac{\partial p}{\partial s} \times \frac{\partial p}{\partial t}}{\left\| \frac{\partial p}{\partial s} \times \frac{\partial p}{\partial t} \right\|}$$

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Parameterizing the Sphere

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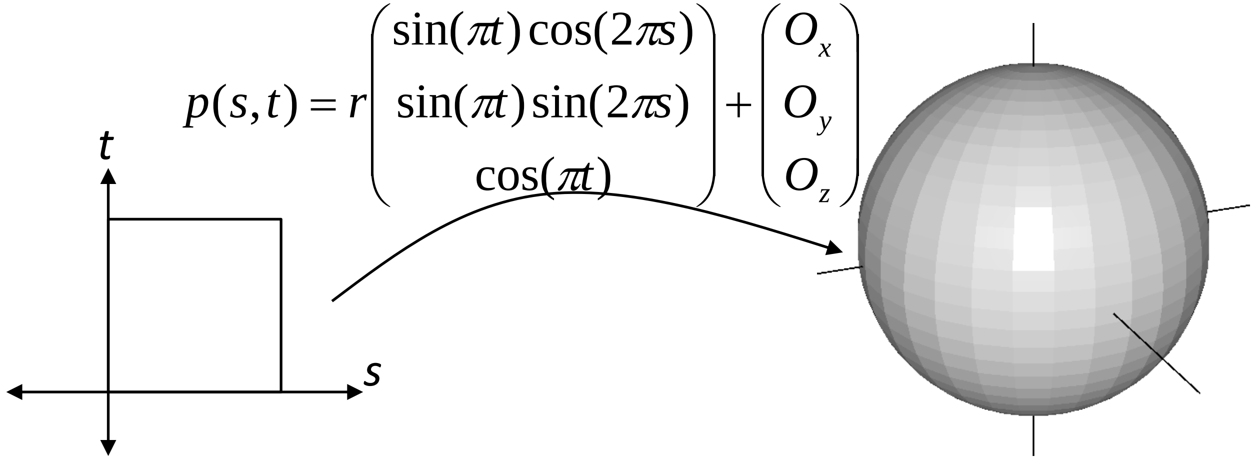
2. Partial derivatives let us compute integrals over the geometry.

$$p(s,t) = r\begin{pmatrix} \sin(\pi t)\cos(2\pi s) \\ \sin(\pi t)\sin(2\pi s) \\ \cos(\pi t) \end{pmatrix} + \begin{pmatrix} O_x \\ O_y \\ O_z \end{pmatrix}$$

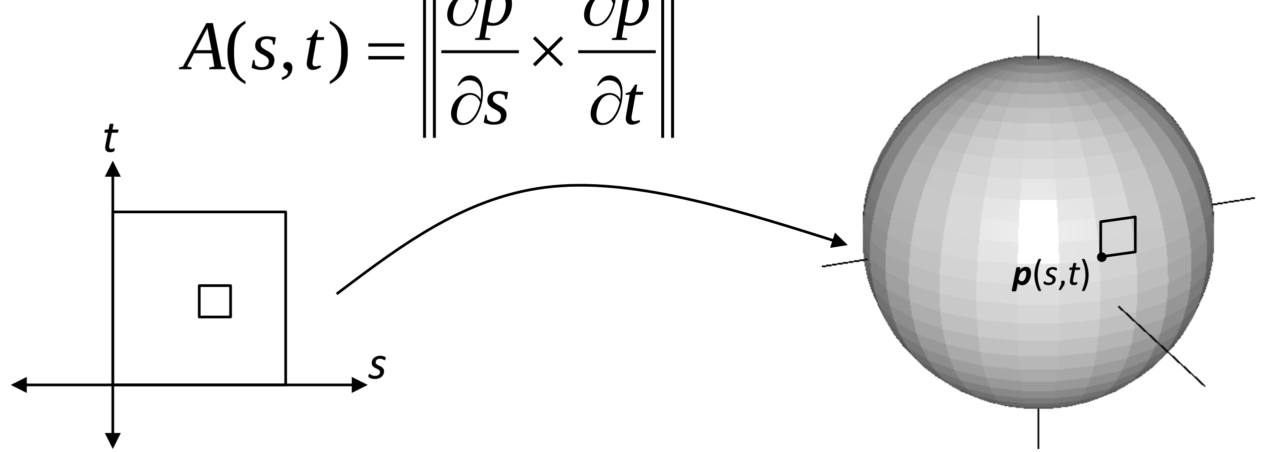


Parameterizing the Sphere

<u>Why do we care?</u>

3. Partial derivatives let us compute integrals over the geometry.
   For a unit square in the parameterization domain, the area of the corresponding patch on the surface is:

$$A(s,t) = \left\| \frac{\partial p}{\partial s} \times \frac{\partial p}{\partial t} \right\|$$



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   $$A(s,t)=\left\|\frac{\partial p}{\partial s}\times\frac{\partial p}{\partial t}\right\|$$
   So integrals become:
   $$\int_{p\in S} f(p)dp=\int_0^1\int_0^1 f(s,t)A(s,t)dtds$$

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⬇

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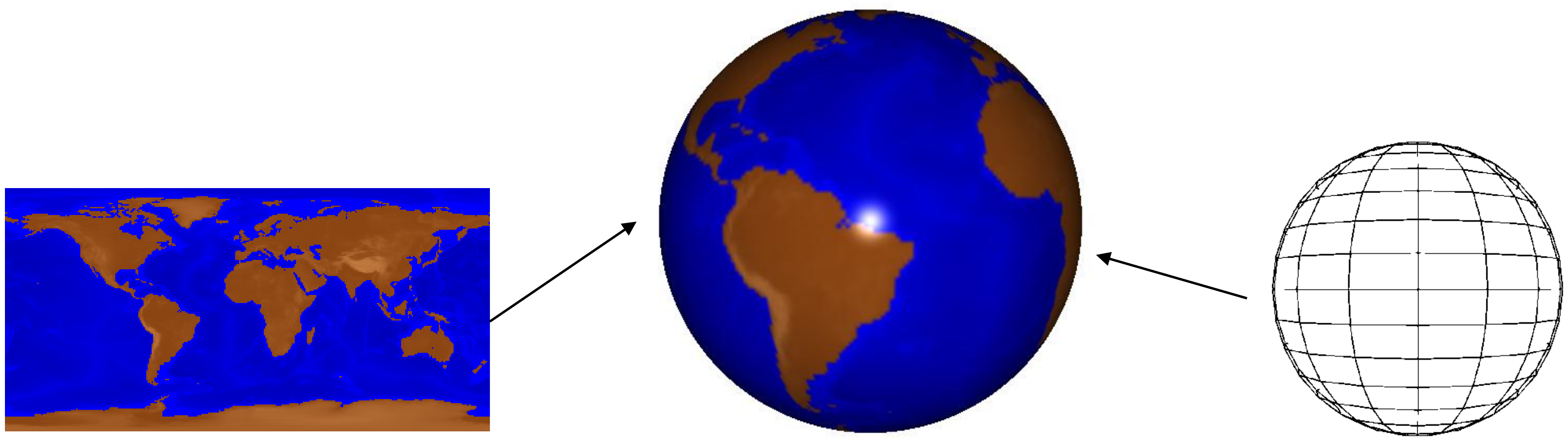
   For example, on the sphere we have:
   $$A(s,t) = 2\pi^2 r^2 \sin(\pi t)$$
   So the area is:
   $$\int_{p\in S^2} 1dp = 2\pi^2 r^2 \int_0^1 \int_0^1 \sin(\pi t)dtds = 4\pi r^2$$

Parameterizing the Sphere

Why do we care?

3. Partial derivatives tell us how texture gets distorted when mapped to the film plane.



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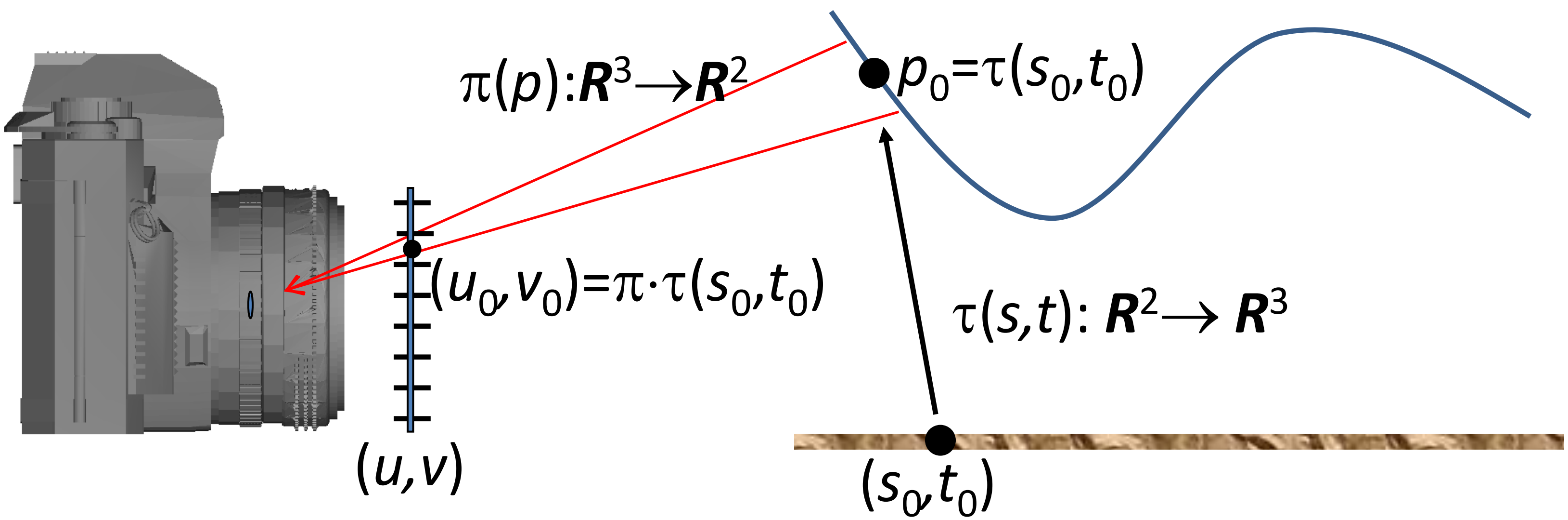

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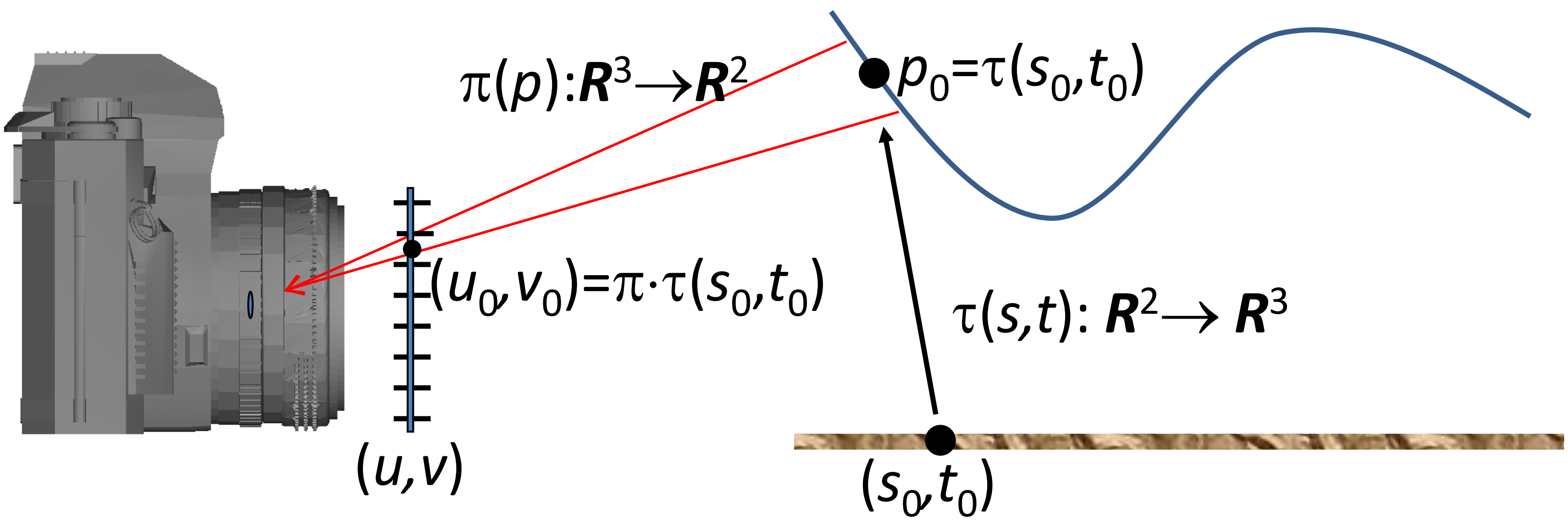
$$(\pi \circ \tau)(s_0+\delta_s, t_0+\delta_t) \approx (\pi \circ \tau)(s_0, t_0) + d(\pi \circ \tau)\begin{pmatrix}\delta_s \\ \delta_t\end{pmatrix}$$



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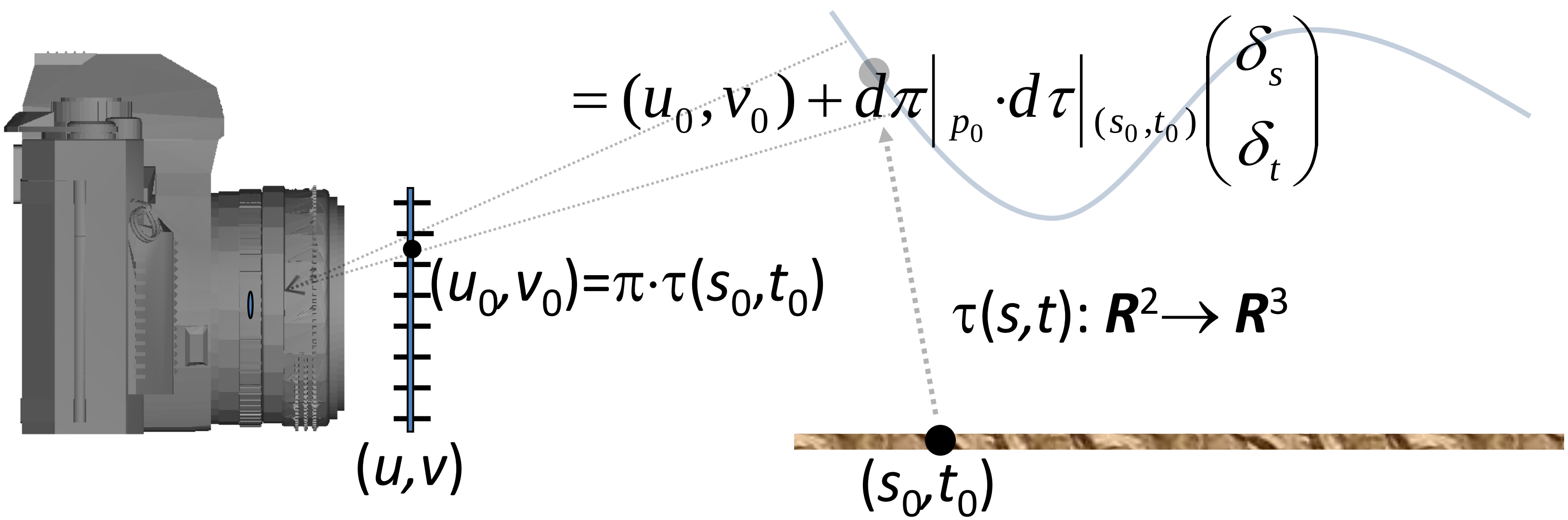

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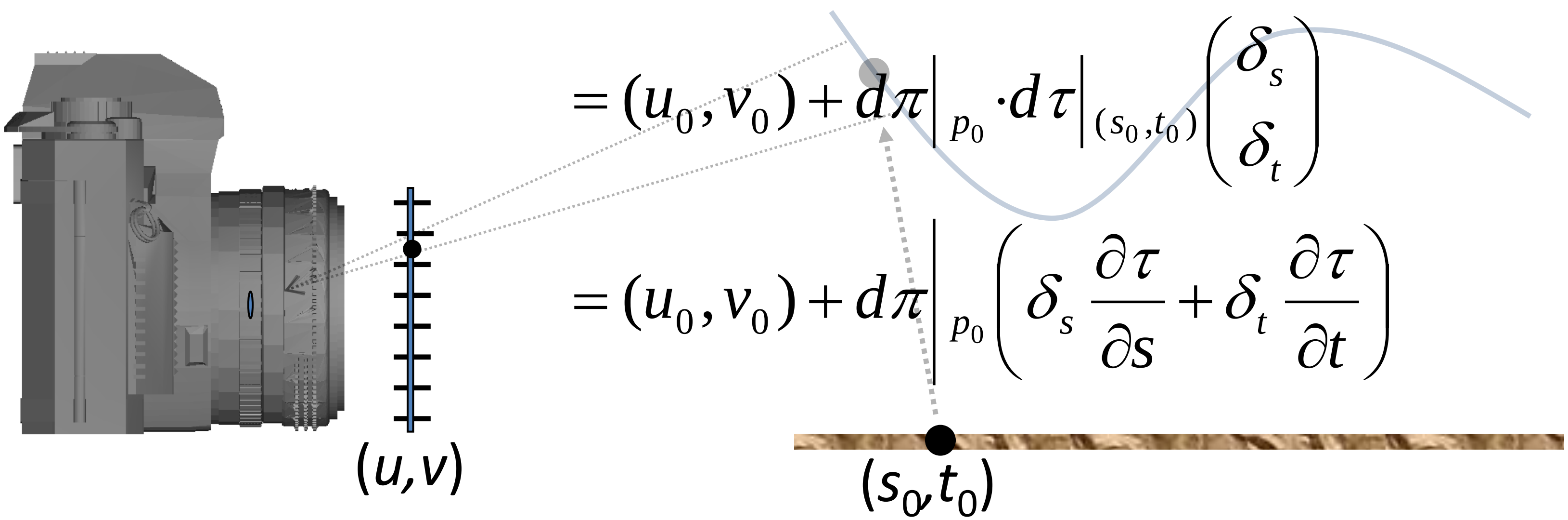

Parameterizing the Sphere

<u>Why do we care?</u>

3. F

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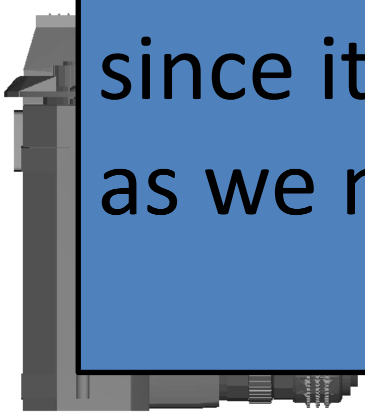
In practice, we are more interested in the equation:

$$(\pi \circ \tau)^{-1}(y_0+\delta_u, v_0+\delta_v) \approx (s_0,t_0) + \left(d(\pi\circ\tau)\big|_{(s_0,t_0)}\right)^{-1}\begin{pmatrix}\delta_u\\ \delta_v\end{pmatrix}$$

since it tells us how texture changes as we move in the film-plane.



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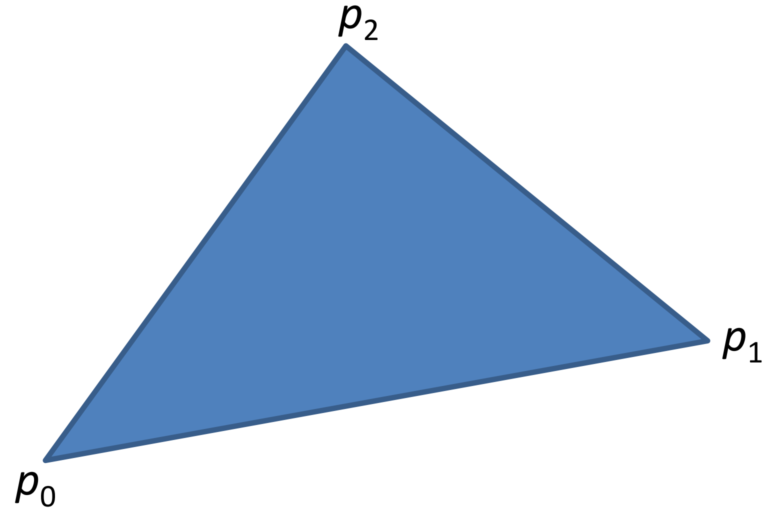
This is good for anti-aliasing textures. To anti-alias textures reflected off the surface, we also need to know how the normals change on the reflecting surface.

Triangles

Described by three vertices:

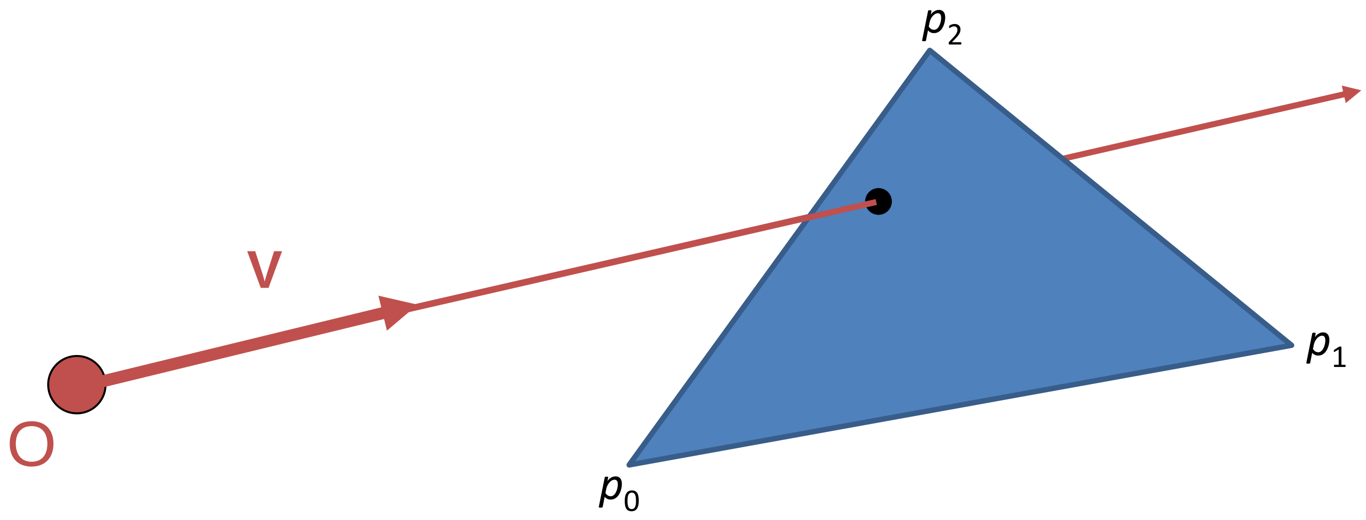
- Position: $\{p_i\}$
- Parameter values: $\{(u_i,v_i)\}$
- Normals: $\{n_i\}$



Triangle Intersection

Ray: $P = O + tV$

Triangle: $(1-b_1-b_2)p_0+b_1p_1+b_2p_2$ with $0 \leq b_1,b_2 \leq 1$



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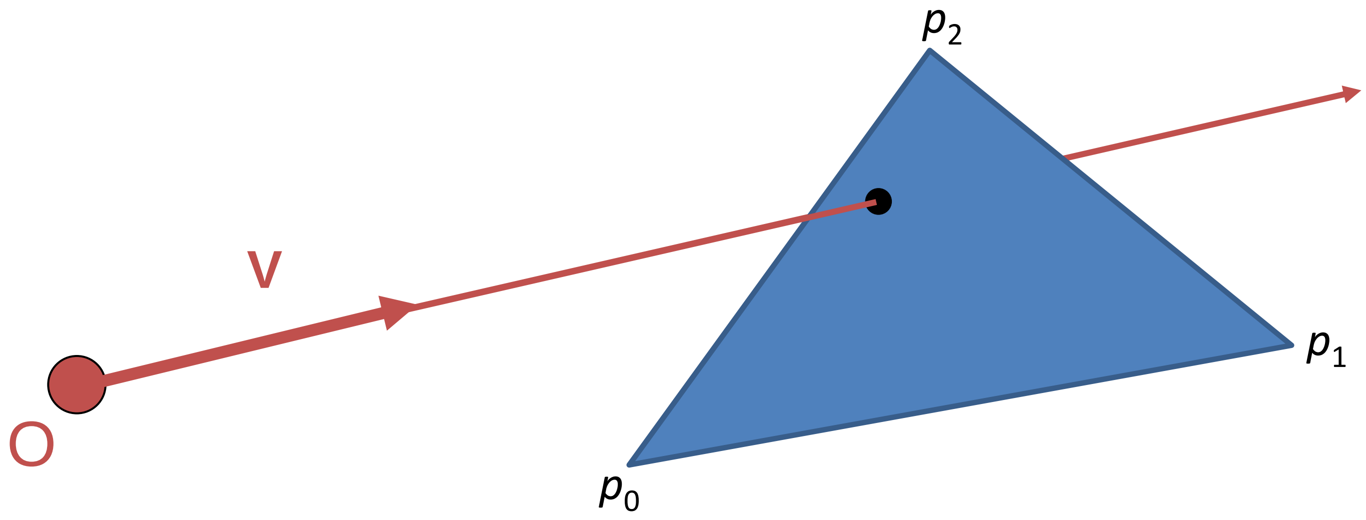
$$\begin{pmatrix} -v_x & p_{1x}-p_{0x} & p_{2x}-p_{0x} \\ -v_y & p_{1y}-p_{0y} & p_{2y}-p_{0y} \\ -v_z & p_{1z}-p_{0z} & p_{2z}-p_{0z} \end{pmatrix} \begin{pmatrix} t \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} O_x - p_{0x} \\ O_y - p_{0y} \\ O_z - p_{0z} \end{pmatrix}$$



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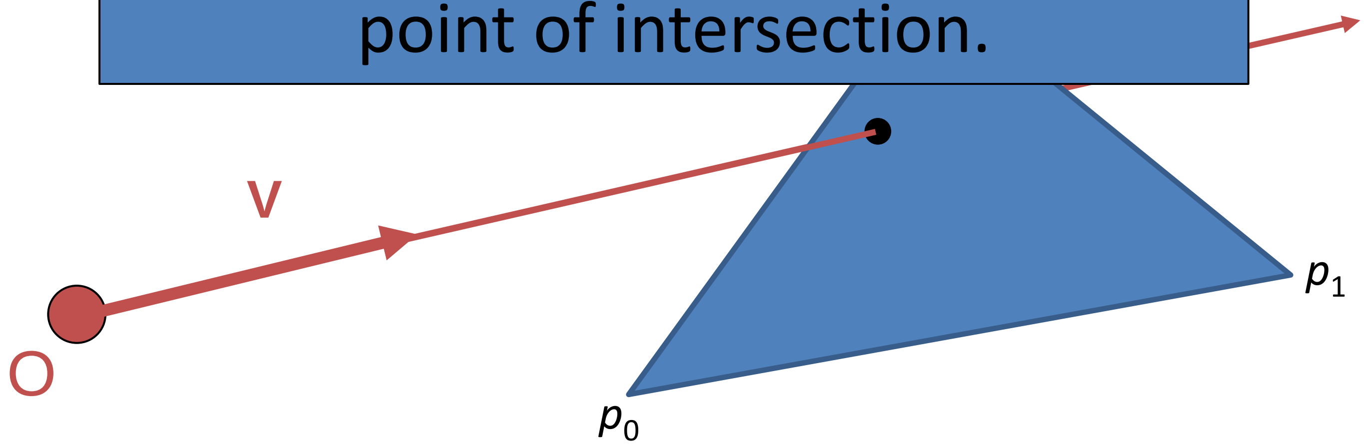
Note that $(1-b_1-b_2)$, b_1, and b_2 are the barycentric coordinates of the point of intersection.



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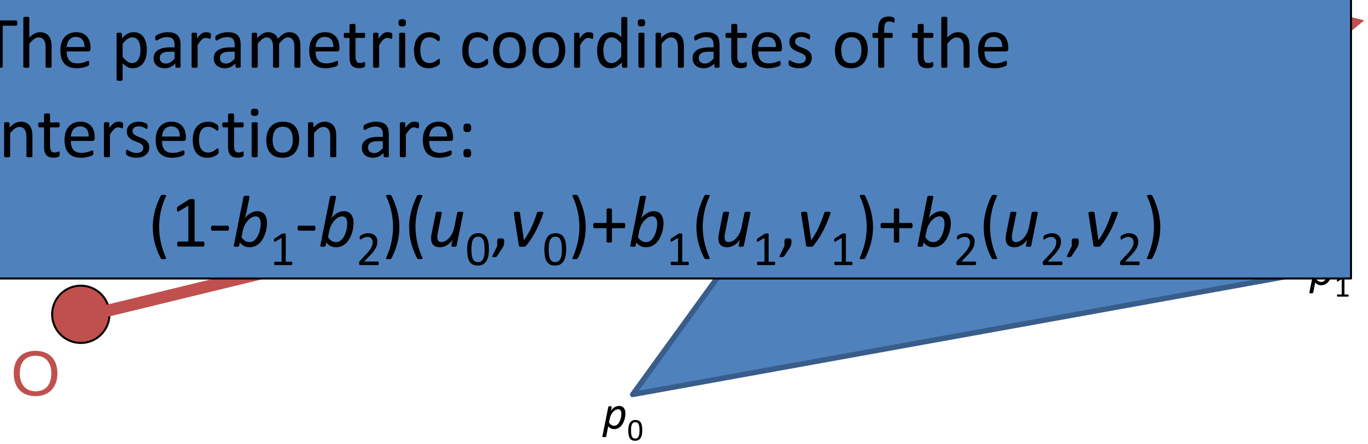
The parametric coordinates of the intersection are:

$$(1\text{-}b_1\text{-}b_2)(u_0,v_0)+b_1(u_1,v_1)+b_2(u_2,v_2)$$

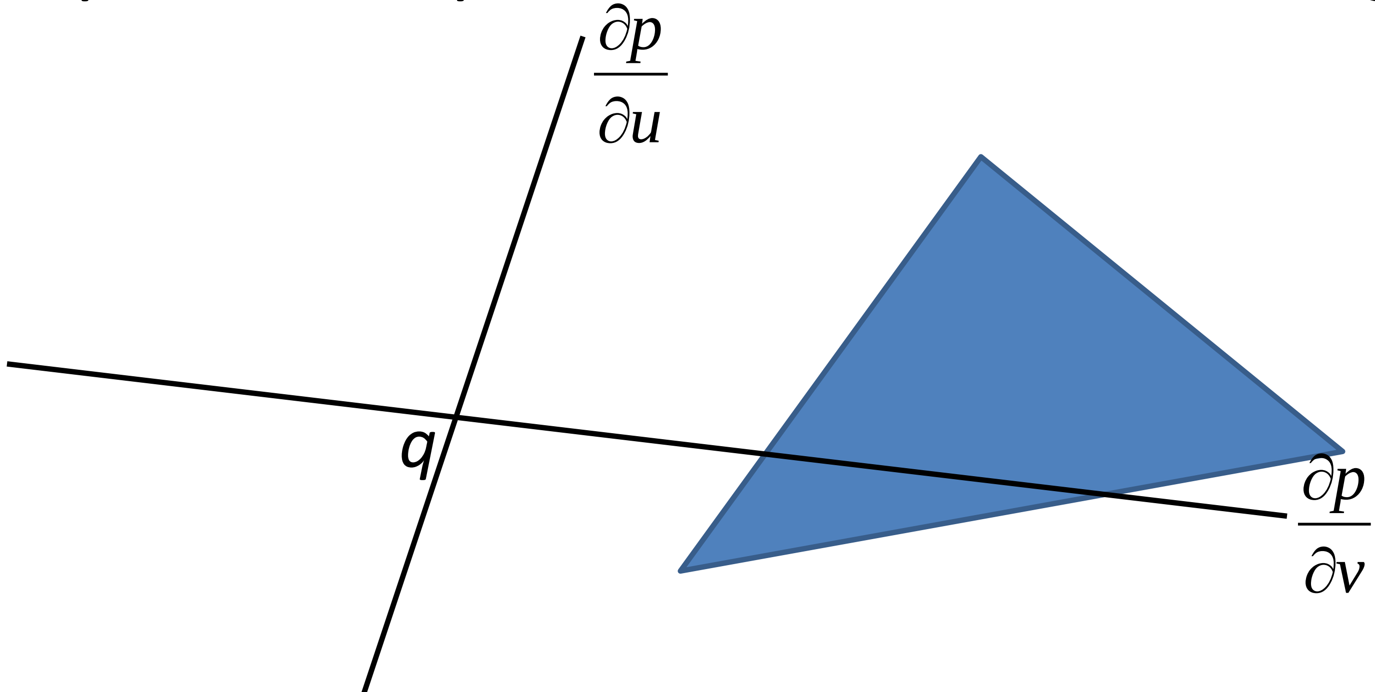


Parameterizing the Triangle

We expect a parameterization of the triangle that is linear:

$$p(u,v) = q + u\frac{\partial p}{\partial u} + v\frac{\partial p}{\partial v}$$

with $\partial p/\partial u$ and $\partial p/\partial v$ constant on the triangle.



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Given the parameter values $\{(u_i,v_i)\}$ at the three corners, q, $\partial p/\partial u$ and $\partial p/\partial v$ satisfy:

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Parameterizing the Triangle

$$p_i - p_1 = (u_i - u_1)\frac{\partial p}{\partial u} + (v_i - v_1)\frac{\partial p}{\partial v}$$

We obtain the partials by solving:

$$\begin{pmatrix} (p_2 - p_1)_{x/y/z} \\ (p_3 - p_1)_{x/y/z} \end{pmatrix} = \begin{pmatrix} u_2 - u_1 & v_2 - v_1 \\ u_3 - u_1 & v_3 - v_1 \end{pmatrix} \begin{pmatrix} \left(\frac{\partial p}{\partial u}\right)_{x/y/z} \\ \left(\frac{\partial p}{\partial v}\right)_{x/y/z} \end{pmatrix}$$

Shading the Triangle

In addition to the true geometry (e.g. constant normal) there is also the shading geometry defined by the normals:

- ~~The shading normal is the normal of the triangle.~~

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- ~~The shading normal is the normal of the triangle.~~
- ~~The (shading) change in normal is zero.~~

Shading the Triangle

In addition to the true geometry (e.g. constant normal) there is also the shading geometry defined by the normals:

– The shading normal is:

$$n = (1 - b_1 - b_2) n_0 + b_1 n_1 + b_2 n_2$$

Shading the Triangle

In addition to the true geometry (e.g. constant normal) there is also the shading geometry defined by the normals:

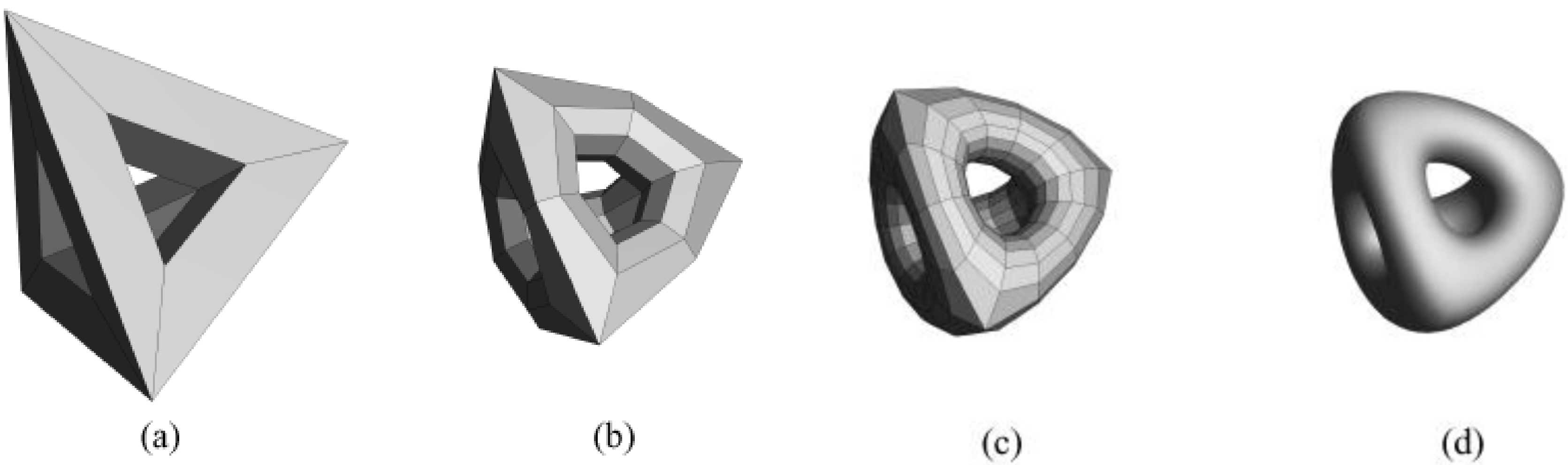
– The shading normal is:

$$n=(1-b_1-b_2)n_0+b_1n_1+b_2n_2$$

– The (shading) change in normal can be obtained by fitting a linear interpolant to the vertex normals and taking the derivative.

Subdivision Surfaces

- Coarse mesh & subdivision rule
  - Define smooth surface as limit of sequence of refinements



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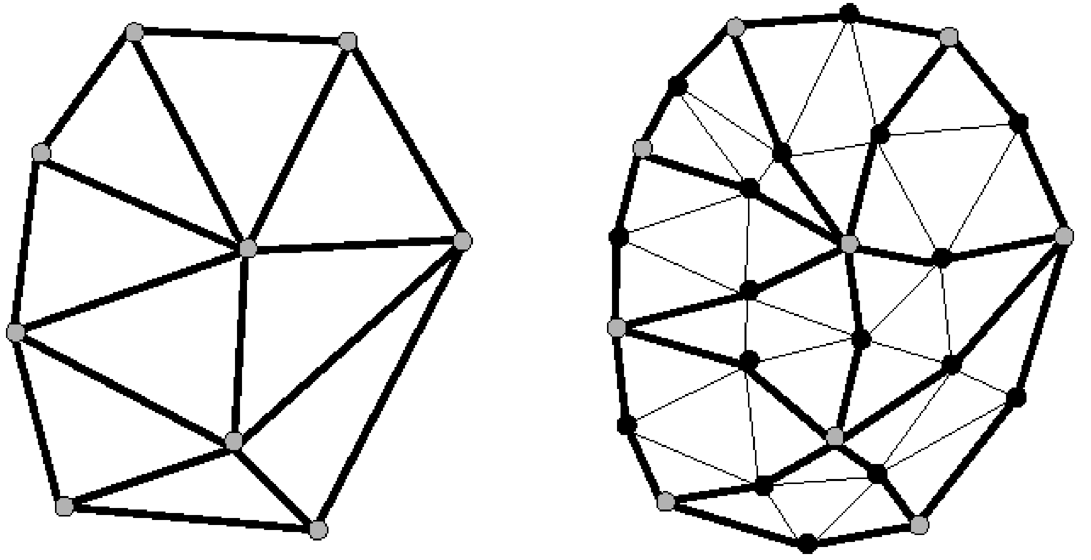
SIGGRAPH 99
Course Notes

Key Questions

- How to subdivide the mesh?
  - Aim for properties like smoothness



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Course Notes

General Subdivision Scheme

- How to subdivide the mesh?

  Two parts:

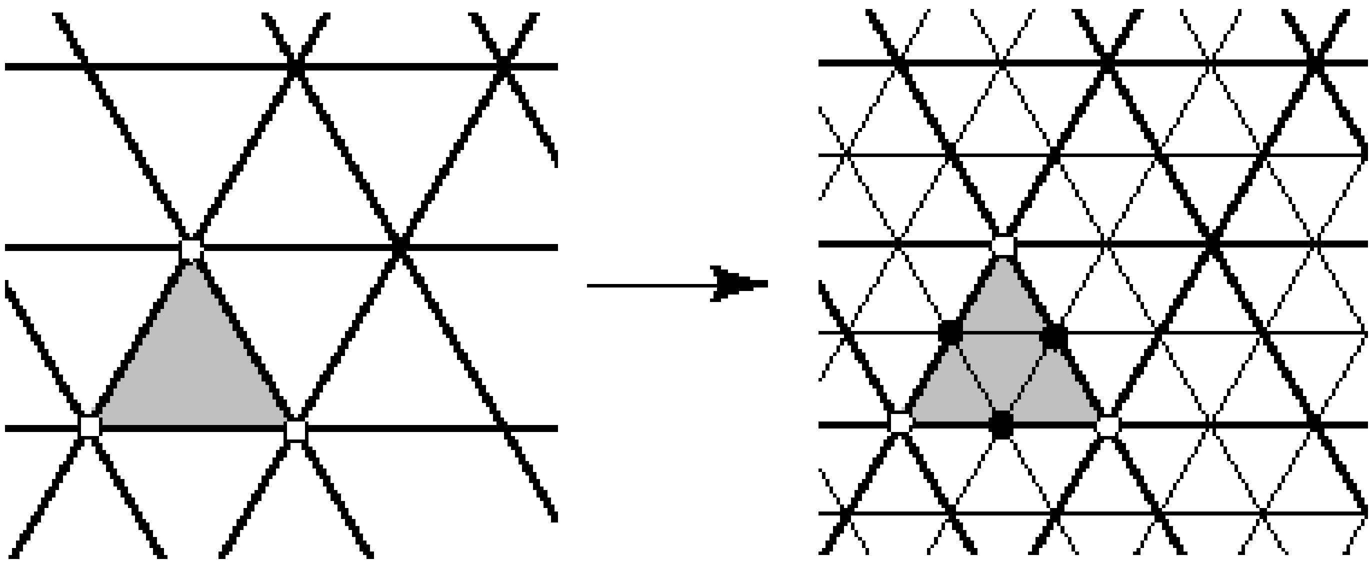
  - Refinement:
    - Add new vertices and connect (topological)
  - Smoothing:
    - Move vertex positions (geometric)

Loop Subdivision Scheme

- How to subdivide the mesh?

  Refinement:

  - Subdivide each triangle into 4 triangles by splitting each edge and connecting new vertices



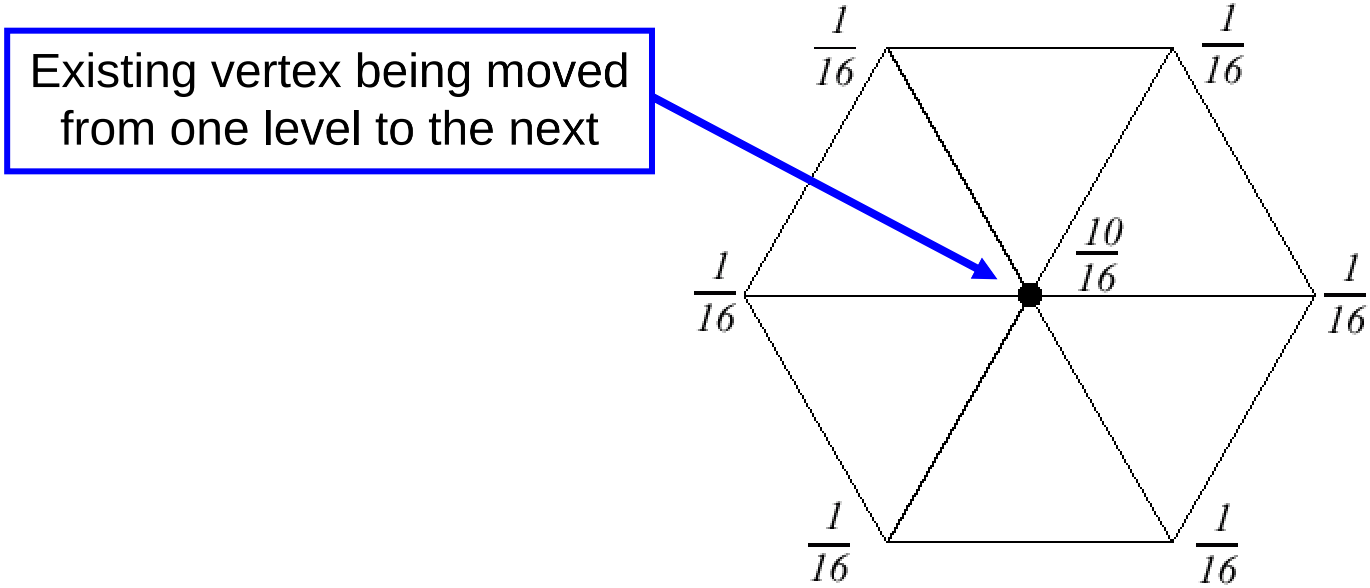
Loop Subdivision Scheme

- How to subdivide the mesh:

  Refinement

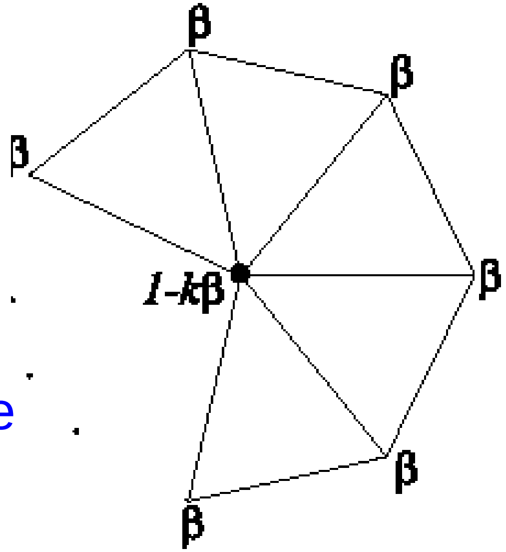
  Smoothing:

  - <u>Existing Vertices</u>: Choose *new* location as weighted average of *original* vertex and its neighbors



Loop Subdivision Scheme

- General rule for moving existing *interior vertices*:

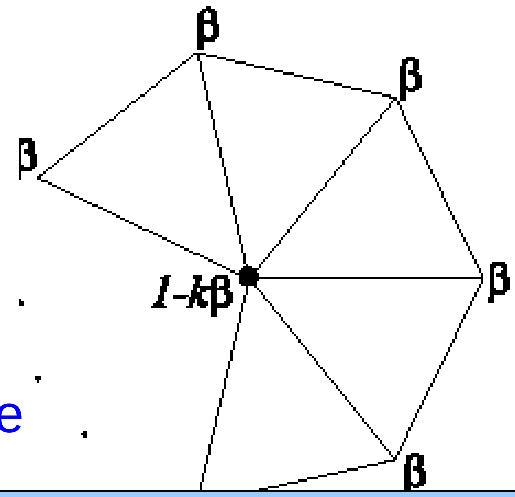


What about vertices that have more
Or less than 6 neighboring faces?

New_position = (1 - $k\beta$)original_position + sum(β * *each_original_vertex*)

Loop Subdivision Scheme

- General rule for moving existing *interior vertices*:



What about vertices that have more
Or less than 6 neighboring faces?

New ... *tex*)

<u>0≤ β ≤1/k</u>:

• As β increases, the contribution from adjacent vertices plays a more important role.

Where do existing vertices move?

- How to choose β?
  - Analyze properties of limit surface
  - Interested in continuity of surface and smoothness
  - Involves calculating eigenvalues of matrices
    - Original Loop

$$\beta = \frac{1}{k}\left(\frac{5}{8} - \left(\frac{3}{8} + \frac{1}{4}\cos\frac{2\pi}{k}\right)^2\right)$$

    - Warren

$$\beta = \begin{cases} \frac{3}{8k} & n > 3 \\ \frac{3}{16} & n = 3 \end{cases}$$

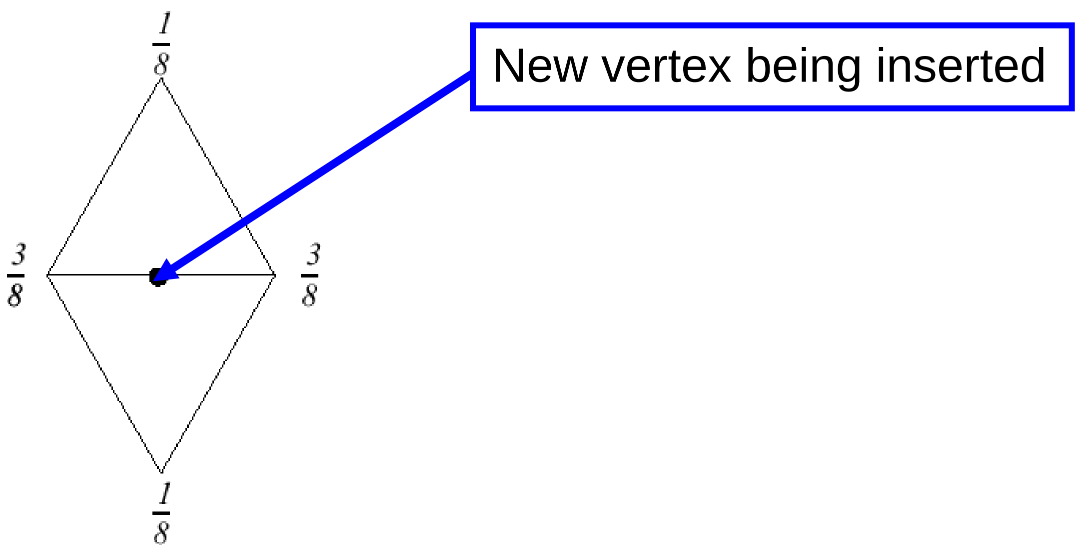
Loop Subdivision Scheme

- How to subdivide the mesh:

  Refinement

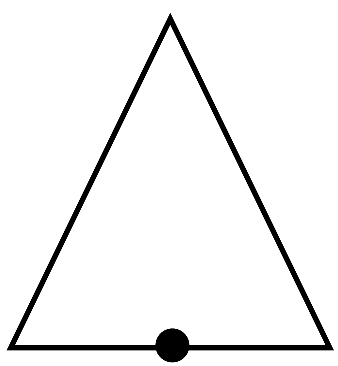
  Smoothing:

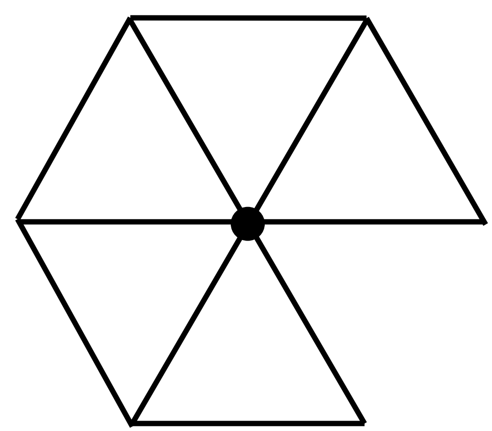
  - <u>Inserted Vertices</u>: Choose location as weighted average of *original* vertices in local neighborhood



Boundary Cases?

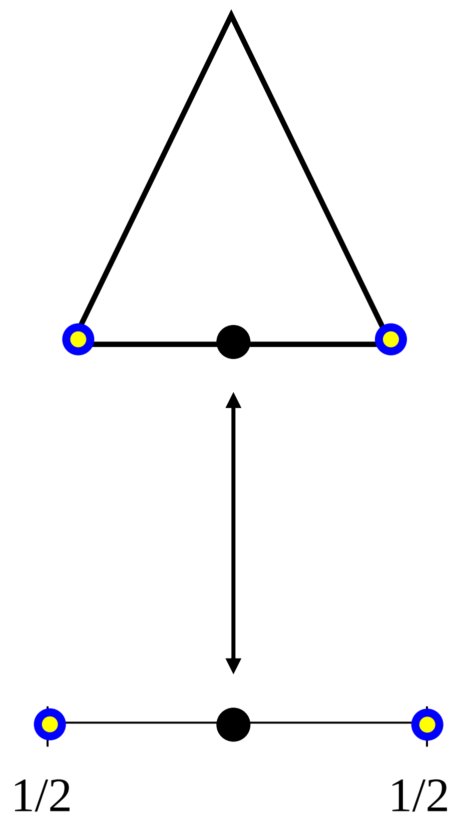
- What about *extraordinary vertices* and *boundary edges?*:
  - Existing vertex adjacent to a missing triangle
  - New vertex bordered by only one triangle

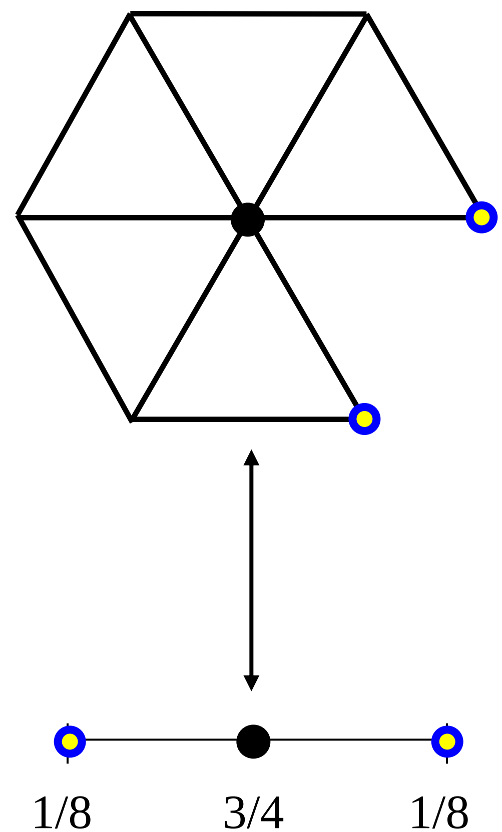




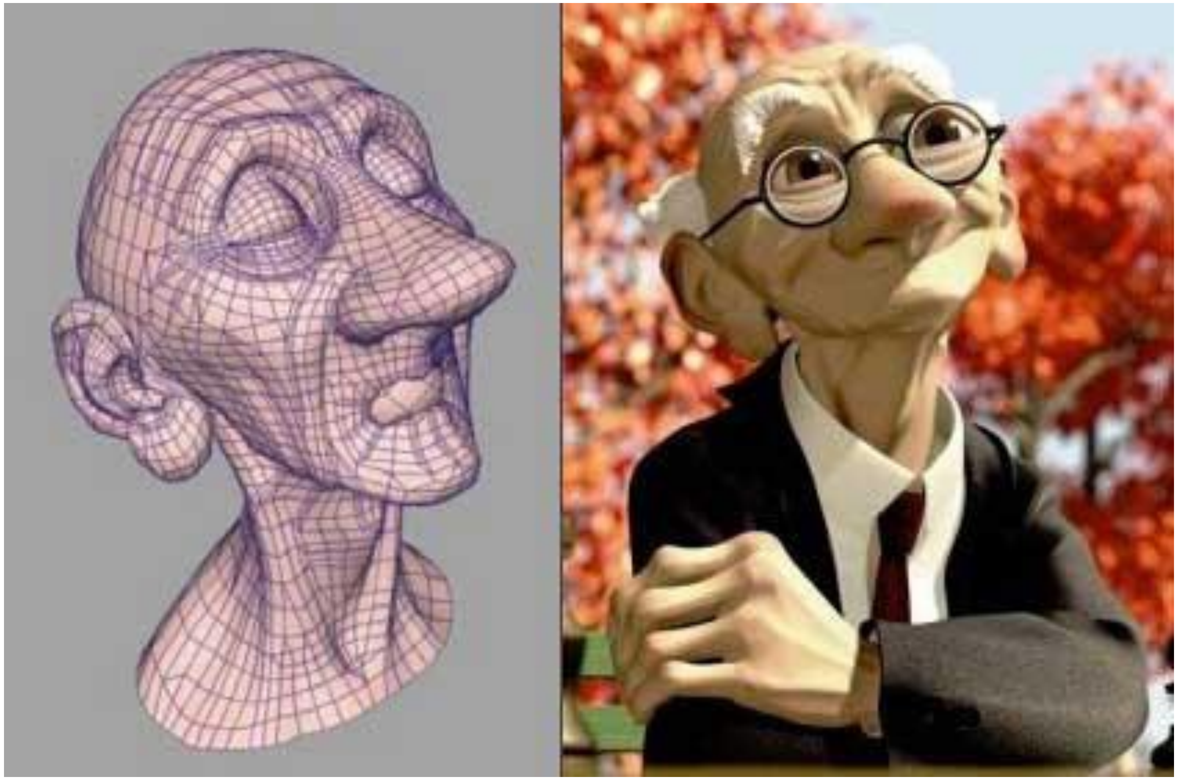
Boundary Cases?

- Rules for *extraordinary vertices* and *boundaries*:



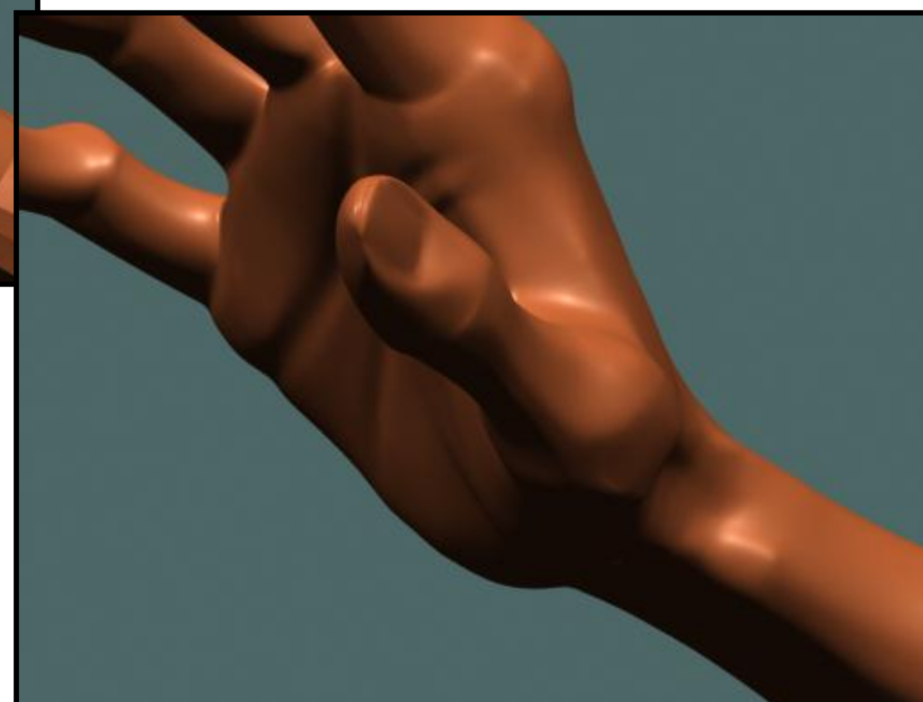


Loop Subdivision Scheme



Pixar

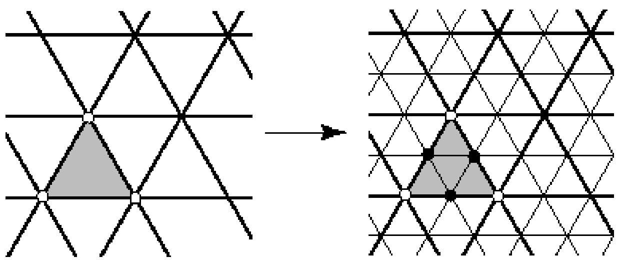
Loop Subdivision Scheme



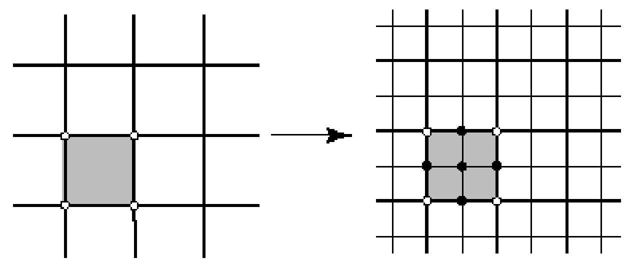
Zorin & Schroeder
SIGGRAPH 99
Course Notes

Subdivision Schemes

- There are different subdivision schemes
  - Different methods for refining topology
  - Different rules for positioning vertices



Face split for triangles



Face split for quads

Face split		
	Triangular meshes	*Quad. meshes*
Approximating	Loop (C^2)	Catmull-Clark (C^2)
Interpolating	Mod. Butterfly (C^1)	Kobbelt (C^1)

Vertex split
Doo-Sabin, Midedge (C^1)
Biquartic (C^2)

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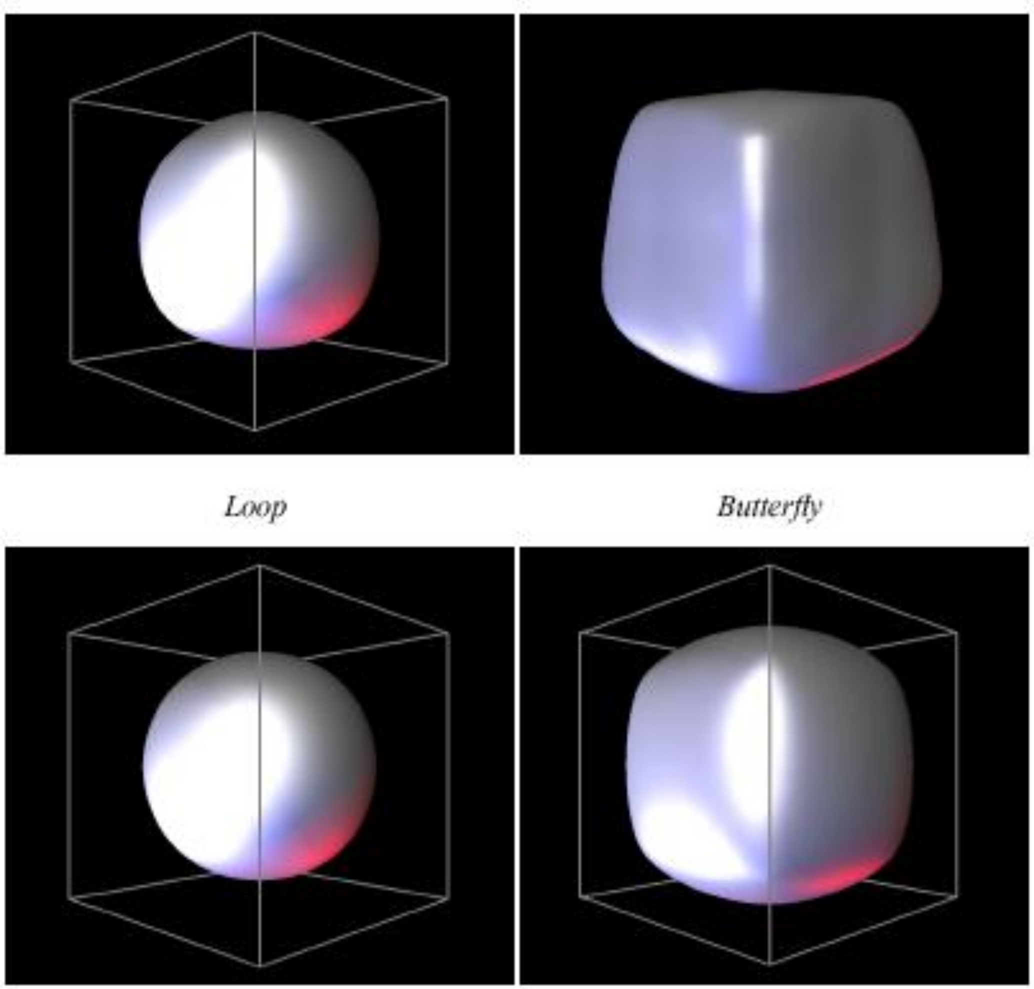
Subdivision Schemes



Loop *Butterfly*

Catmull-Clark *Doo-Sabin*

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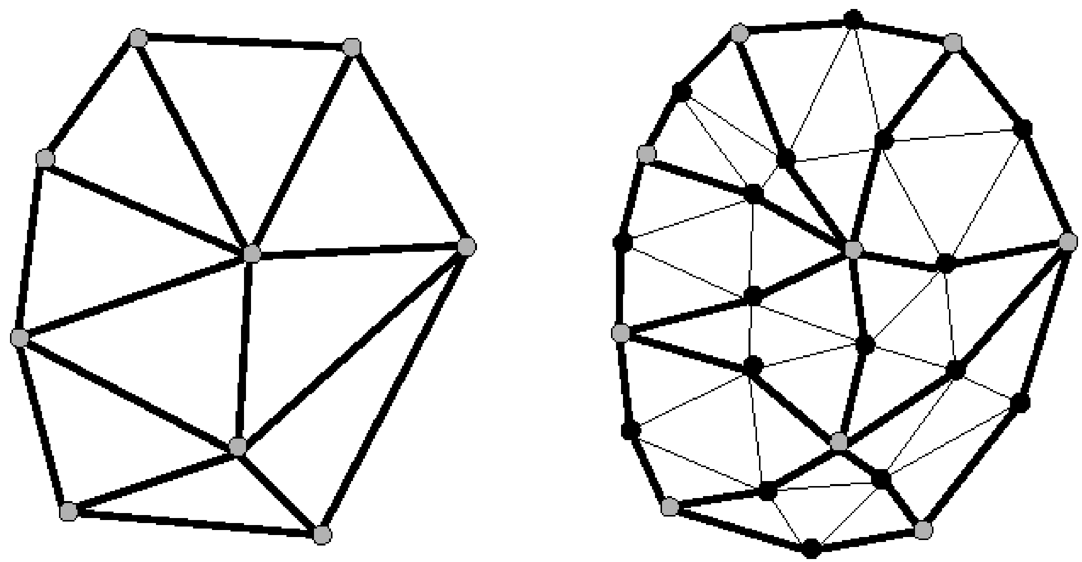
Key Questions

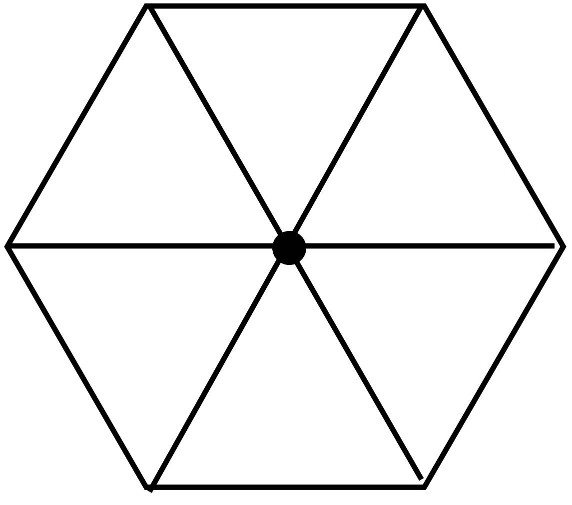
- How to refine the mesh?
  - **Aim for properties like smoothness**



Subdivision Smoothness

To determine the smoothness of the subdivision:

- Repeatedly apply the subdivision scheme
- Look at the neighborhood in the limit.



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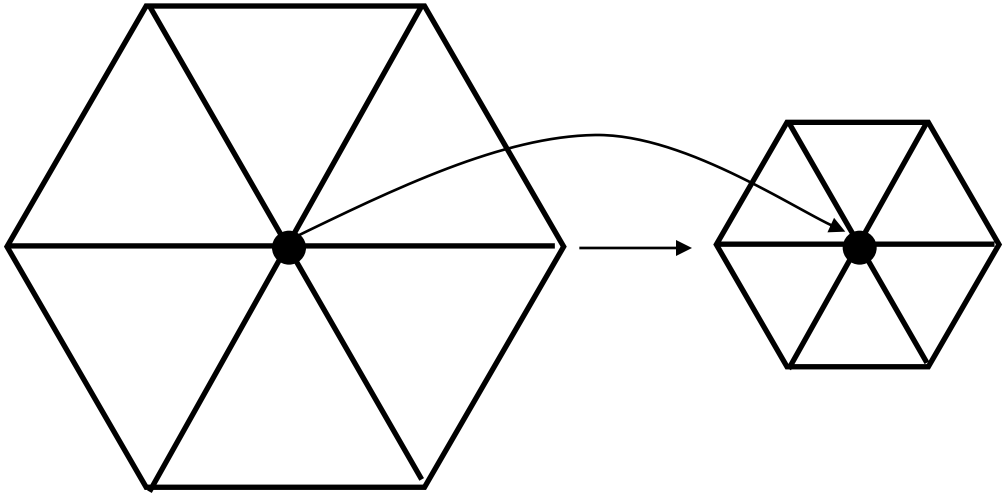

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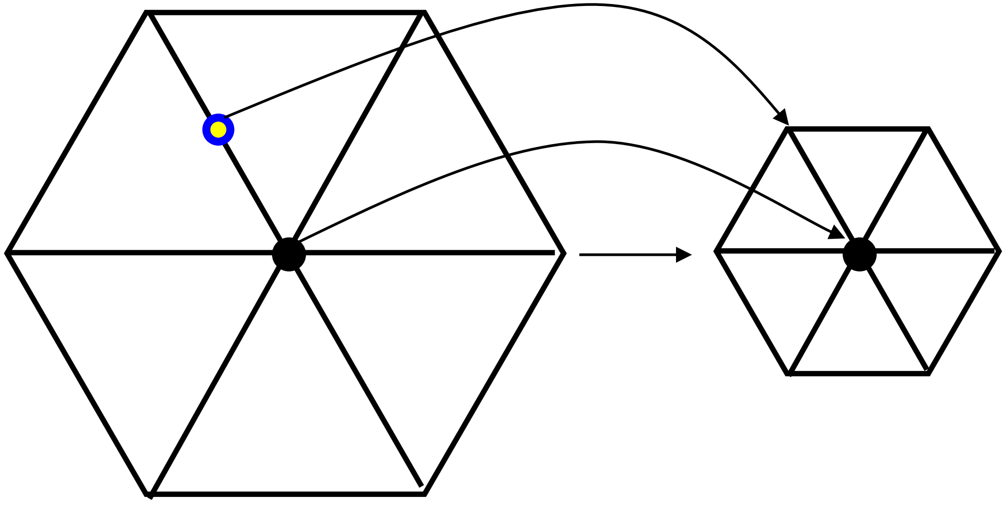

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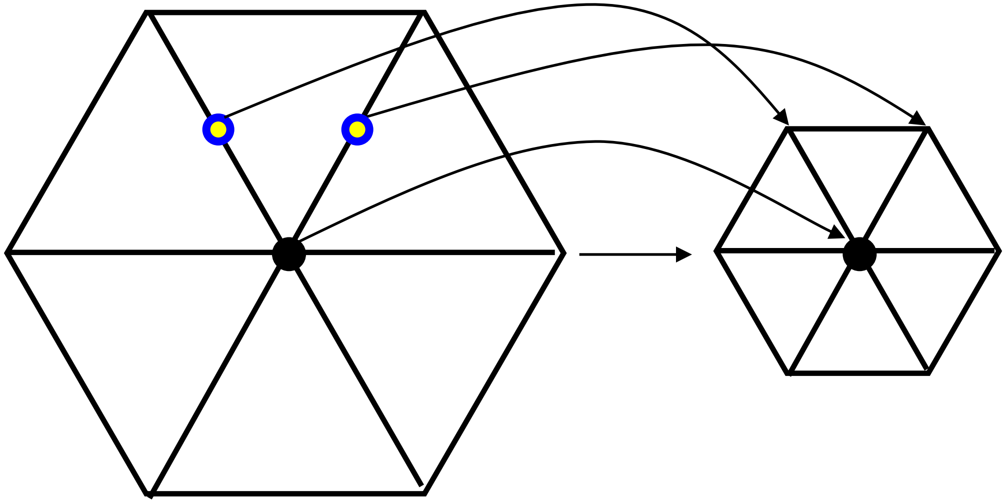

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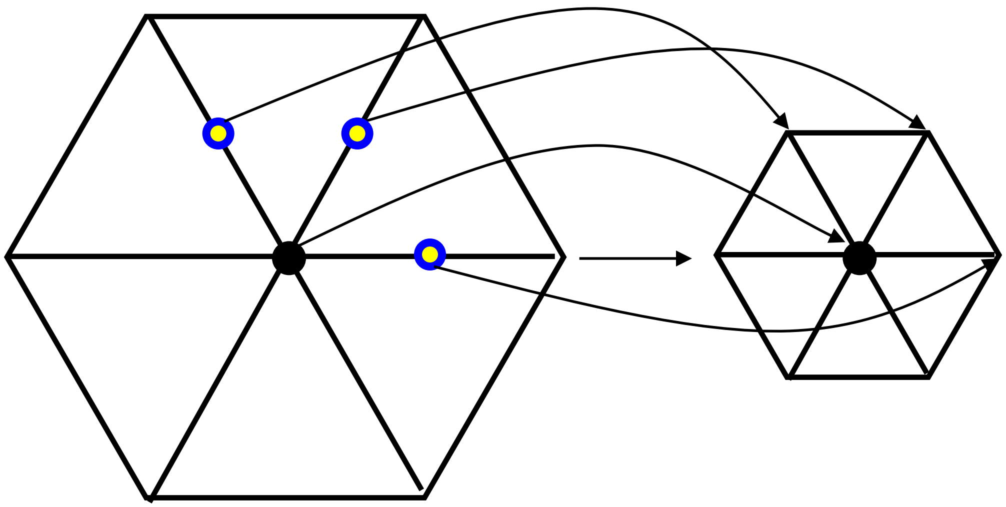

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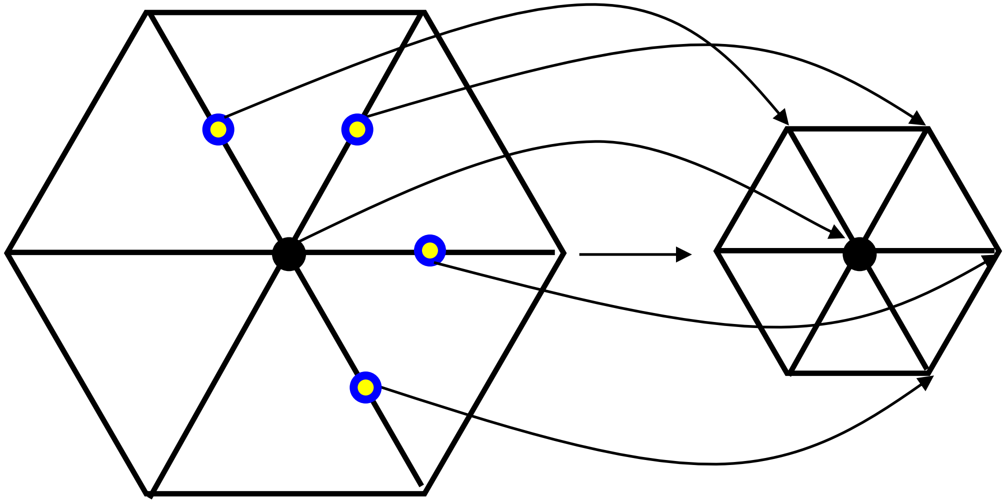

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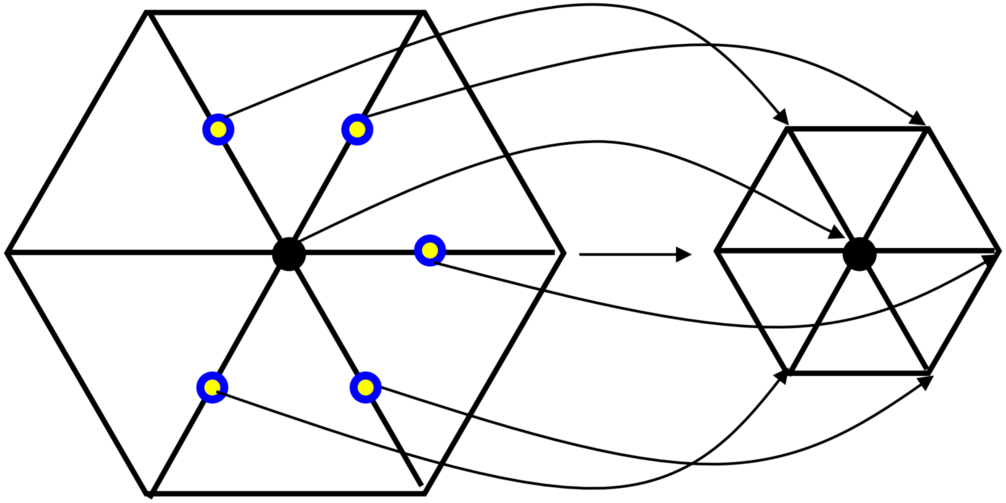

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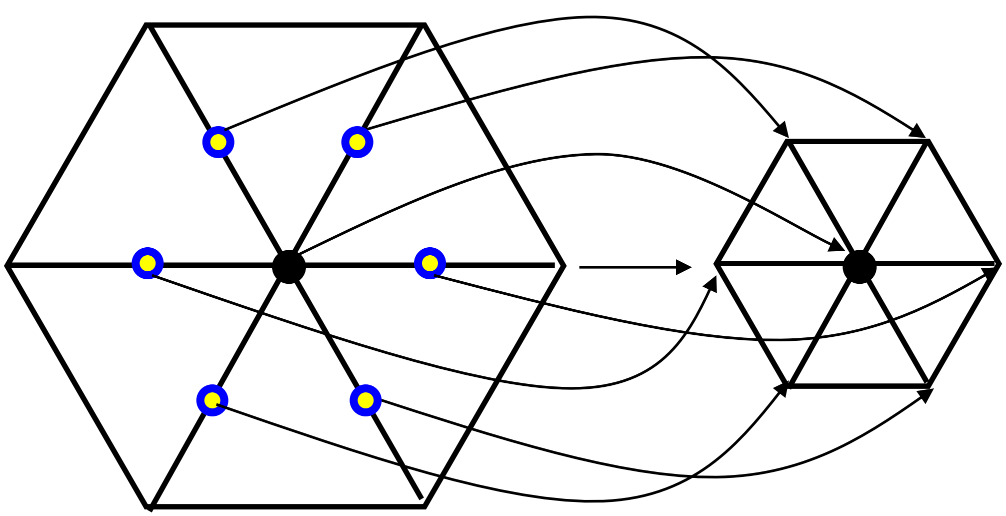

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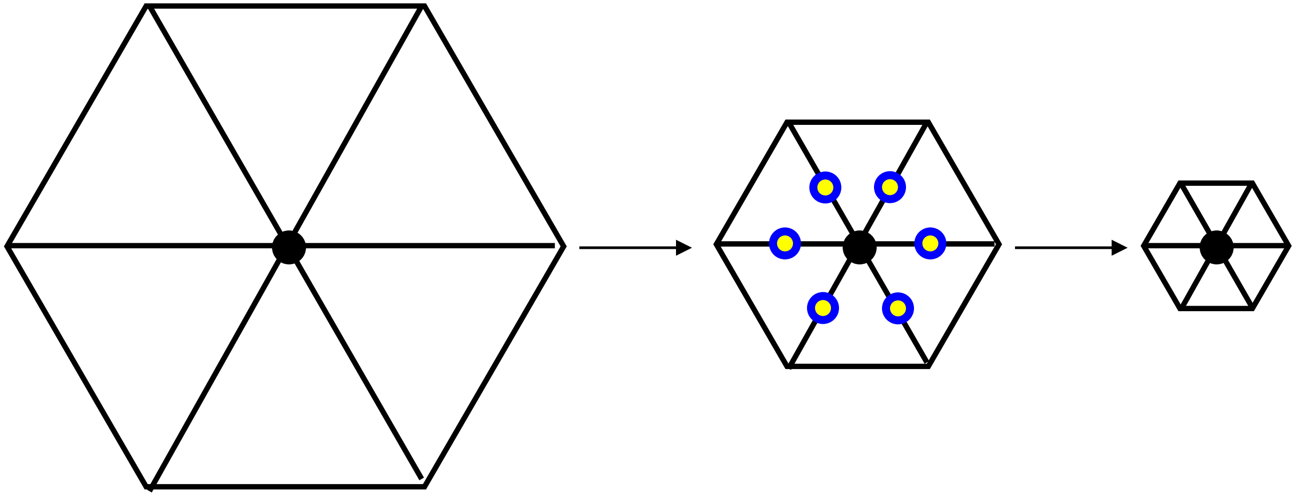

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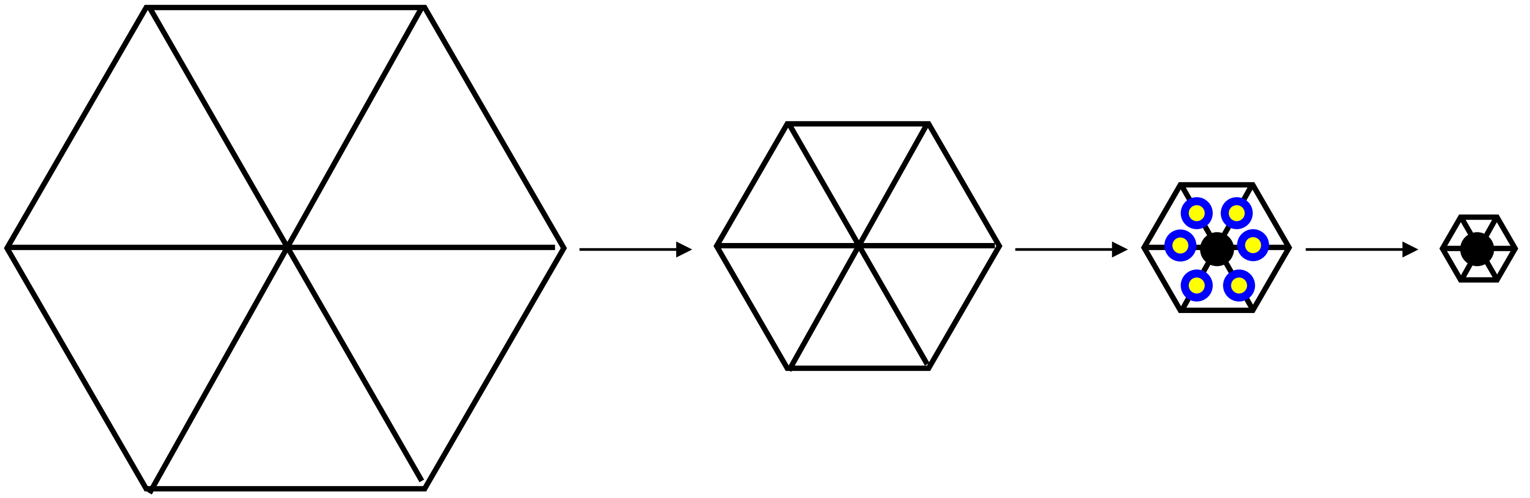
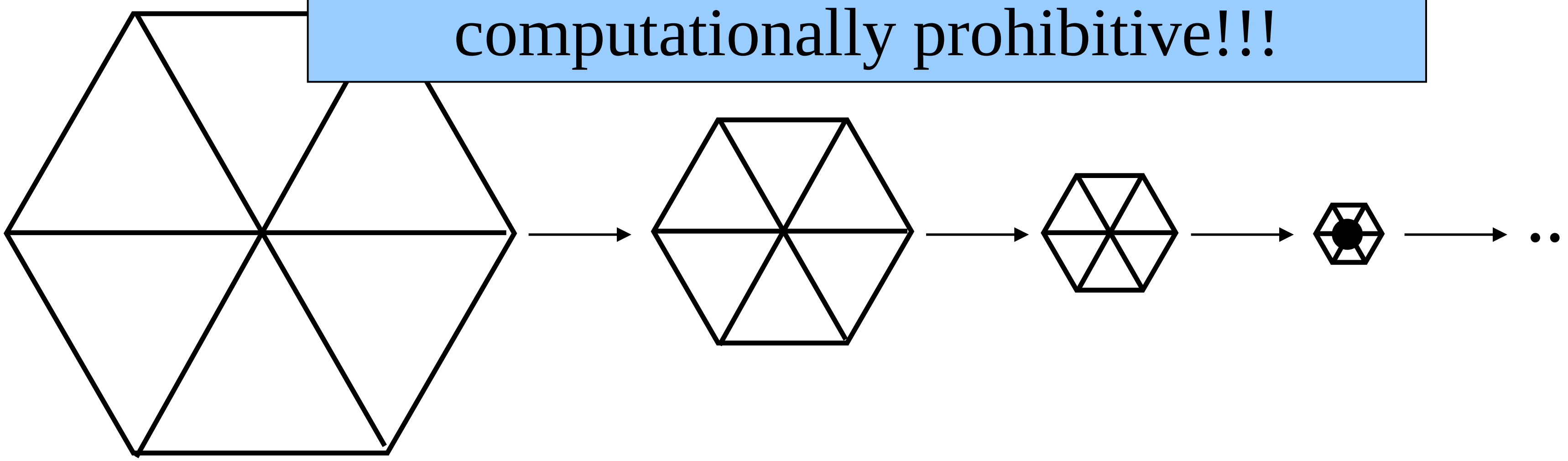


Subdivision Smoothness

To determine the smoothness of the subdivision:

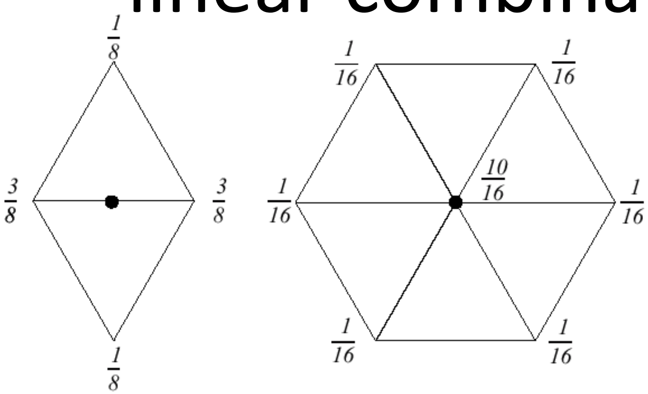
- Repeatedly apply the subdivision scheme
- Look a

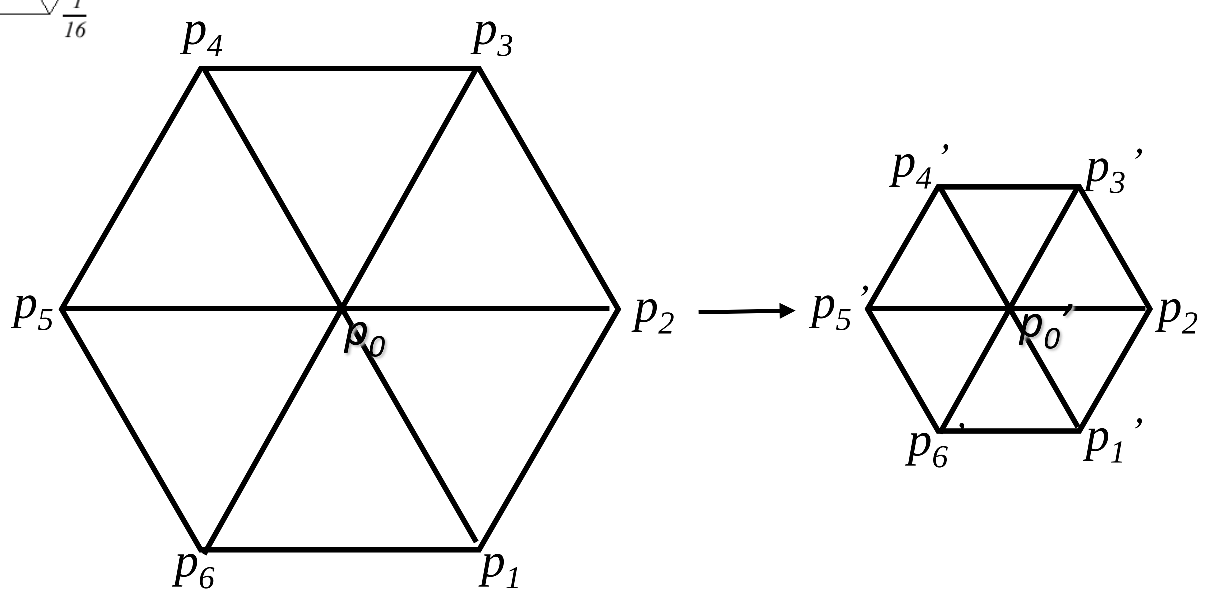
Computing infinitely many iterations is computationally prohibitive!!!



Subdivision Matrix

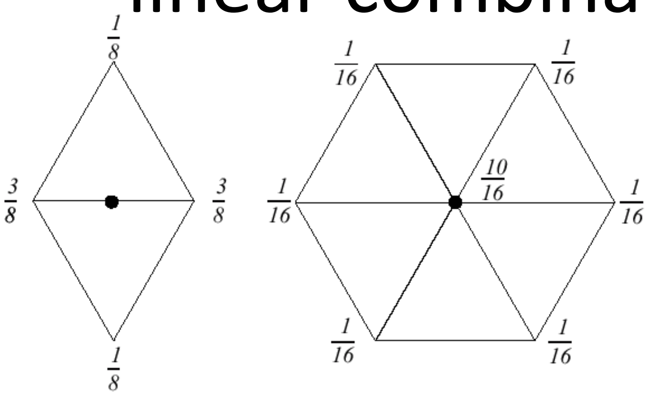
- Compute the new positions/vertices as a linear combination of previous ones.





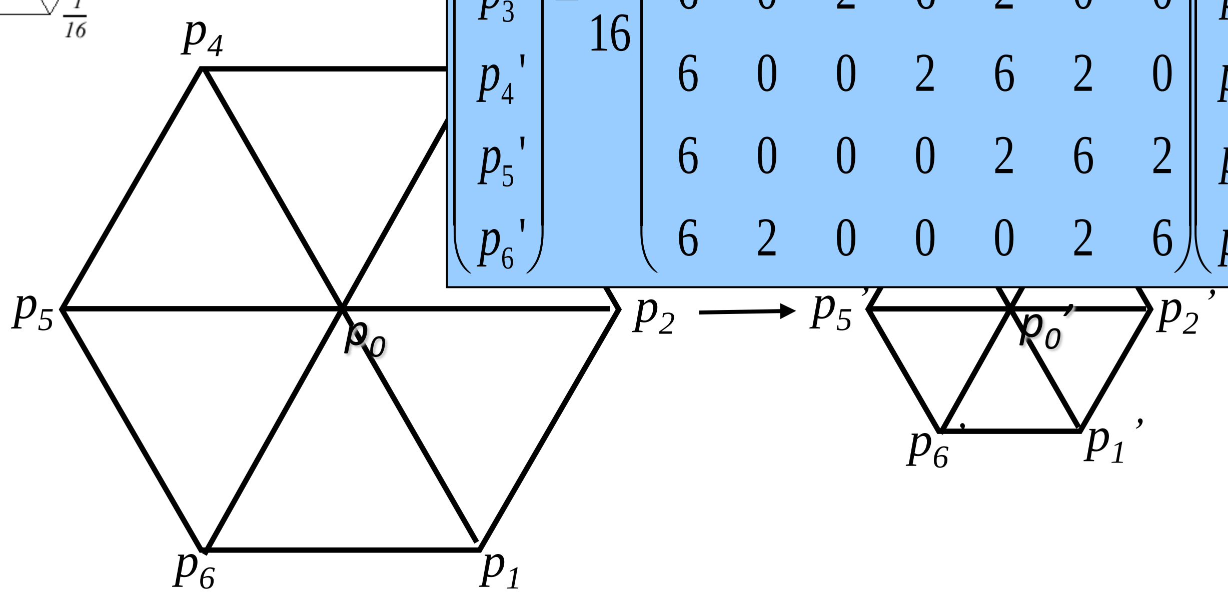
Subdivision Matrix

- Compute the new positions/vertices as a linear combination of p



Subdivision Matrix

$$\begin{pmatrix} p_0' \\ p_1' \\ p_2' \\ p_3' \\ p_4' \\ p_5' \\ p_6' \end{pmatrix} = \frac{1}{16} \begin{pmatrix} 10 & 1 & 1 & 1 & 1 & 1 & 1 \\ 6 & 6 & 2 & 0 & 0 & 0 & 2 \\ 6 & 2 & 6 & 2 & 0 & 0 & 0 \\ 6 & 0 & 2 & 6 & 2 & 0 & 0 \\ 6 & 0 & 0 & 2 & 6 & 2 & 0 \\ 6 & 0 & 0 & 0 & 2 & 6 & 2 \\ 6 & 2 & 0 & 0 & 0 & 2 & 6 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{pmatrix}$$



Subdivision Matrix

- Compute the new positions/vertices as a linear combination of previous ones.
- To find the limit position of p_0, repeatedly apply the subdivision matrix.

$$\begin{pmatrix} p_0^{(n)} \\ p_1^{(n)} \\ p_2^{(n)} \\ p_3^{(n)} \\ p_4^{(n)} \\ p_5^{(n)} \\ p_6^{(n)} \end{pmatrix} = \left[\frac{1}{16} \begin{pmatrix} 10 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 6 & 0 & 0 & 0 & 6 \\ 2 & 6 & 2 & 6 & 0 & 0 & 0 \\ 2 & 0 & 6 & 2 & 6 & 0 & 0 \\ 2 & 0 & 0 & 6 & 2 & 6 & 0 \\ 2 & 0 & 0 & 0 & 6 & 2 & 6 \\ 2 & 6 & 0 & 0 & 0 & 6 & 2 \end{pmatrix} \right]^n \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{pmatrix}$$

Subdivision Matrix

- Compute the new positions/vertices as a linear combination of previous ones.
- To find the limit position of p_0, repeatedly apply the subdivision matrix.
- Use eigen-value decomposition to compute the n^{th} power of the matrix efficiently.

$$\begin{pmatrix} p_0^{(n)} \\ p_1^{(n)} \\ p_2^{(n)} \\ p_3^{(n)} \\ p_4^{(n)} \\ p_5^{(n)} \\ p_6^{(n)} \end{pmatrix} = \left[\frac{1}{16} \begin{pmatrix} 10 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 6 & 0 & 0 & 0 & 6 \\ 2 & 6 & 2 & 6 & 0 & 0 & 0 \\ 2 & 0 & 6 & 2 & 6 & 0 & 0 \\ 2 & 0 & 0 & 6 & 2 & 6 & 0 \\ 2 & 0 & 0 & 0 & 6 & 2 & 6 \\ 2 & 6 & 0 & 0 & 0 & 6 & 2 \end{pmatrix} \right]^n \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{pmatrix}$$

Subdivision Matrix

- Compute the new positions/vertices as a linear combination of previous ones.
- To find the limit position of p_0, repeatedly apply the subdivision matrix.
- Use eigen-value

$$\begin{pmatrix} p_0^{(n)} \end{pmatrix} = \left[\begin{pmatrix} 10 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \right]^n \begin{pmatrix} p_0 \end{pmatrix}$$

If, after a change of basis we have $M=A^{-1}DA$, where D is a diagonal matrix, then:

$$M^n=A^{-1}D^nA,$$

Since D is diagonal, raising D to the n-th power just amounts to raising each of the diagonal entries of D to the n-th power.