600.657: Mesh Processing

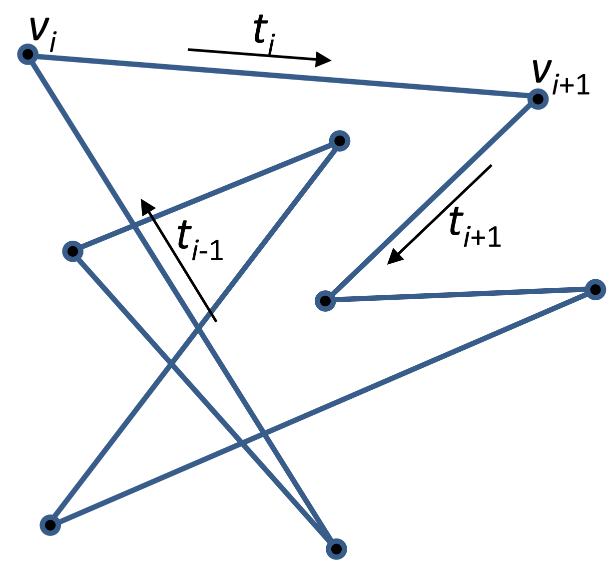
Chapter 3

Discrete Curves

Q: Where and how should we define curvature?

A: Since the tangent only changes at vertices we should define the curvature as a vertex value.

We should define the value of the curvature as the change in the tangent as we move through the vertex.

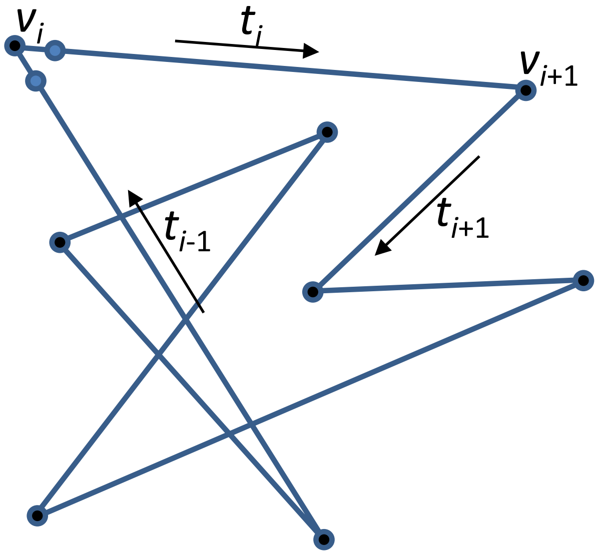


Discrete Curves

Note that we cannot define the curvature as the result of a limiting process:

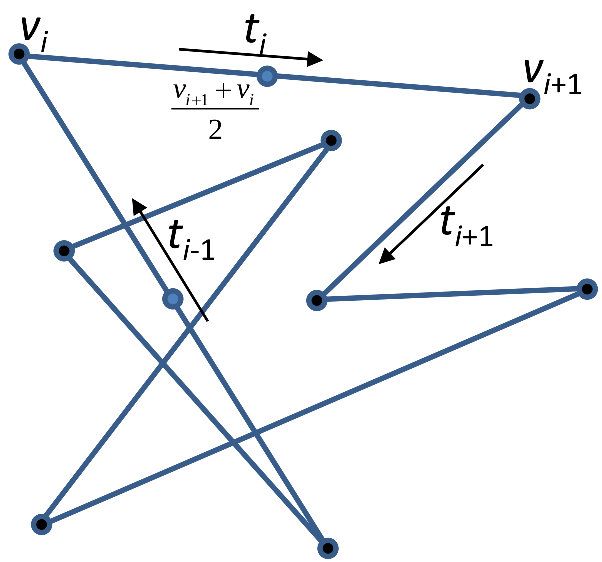
$$\kappa_i = \lim_{\Delta t \to 0} \frac{|t_i - t_{i-1}|}{\left\| \left[(1-\Delta t)v_i + \Delta t v_{i+1}\right] - \left[(1-\Delta t)v_i + \Delta t v_{i-1}\right] \right\|}$$

$$= \lim_{\Delta t \to 0} \frac{|t_i - t_{i-1}|}{\Delta t |v_{i+1} - v_{i-1}|}$$



Discrete Curves

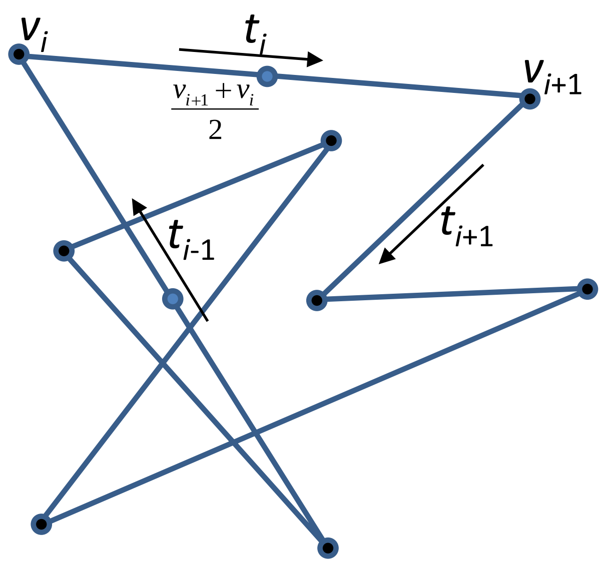
However, we can estimate it using the finite-differences using the edge centers $(v_i+v_{i+1})/2$.



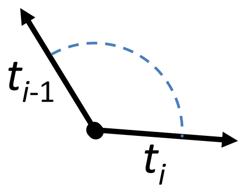
Discrete Curves

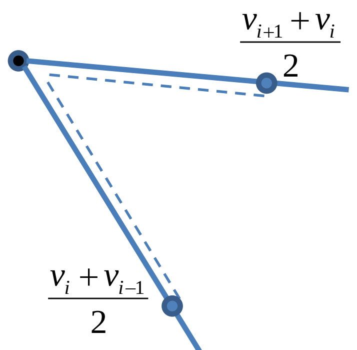
However, we can estimate it using the finite-differences using the edge centers $(v_i + v_{i+1})/2$.

Specifically, we define the curvature in terms of change in tangents angle divided by the arc-length between edge centers:



$$\kappa_i = \frac{\angle t_i t_{i+1}}{\left|(v_{i+1} - v_i)/2\right| + \left|(v_i - v_{i-1})/2\right|}$$

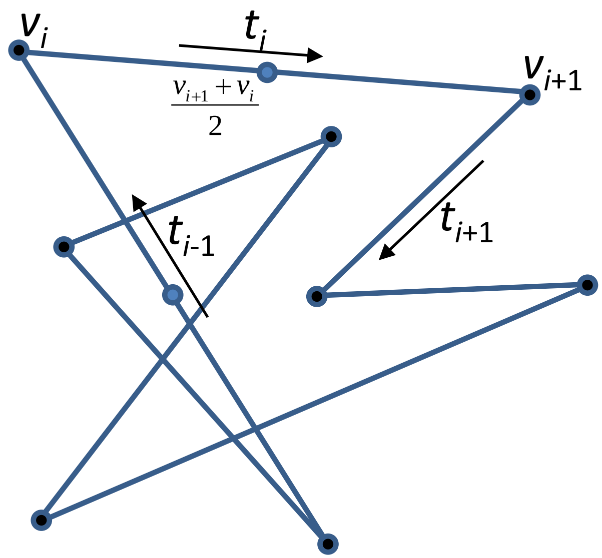




Discrete Curves

Since we are only storing the curvature at the vertices, we want the value to correspond to the "total curvature associated to the vertex":

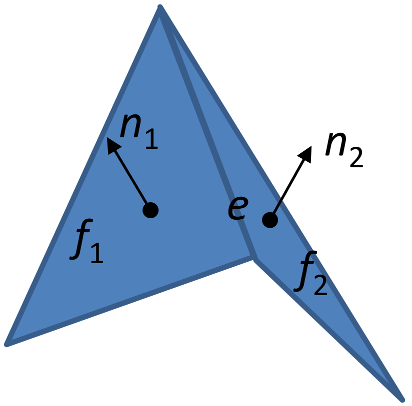
$$\kappa(v_i) = \frac{|v_{i+1} - v_i| + |v_i - v_{i-1}|}{2}\kappa_i = \angle t_i t_{i+1}$$



Discrete Surfaces

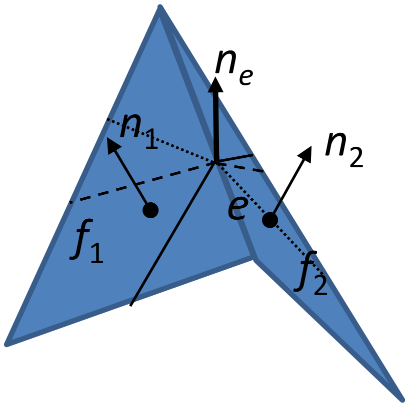
Q: Where and how should we compute the normal curvatures?

A: Since the normals only changes at edges we should compute the normal curvature at the edges.



Discrete Surfaces

Picking a (good) normal at the edge and looking at the set of planes passing through the normal, we get a family of curves.

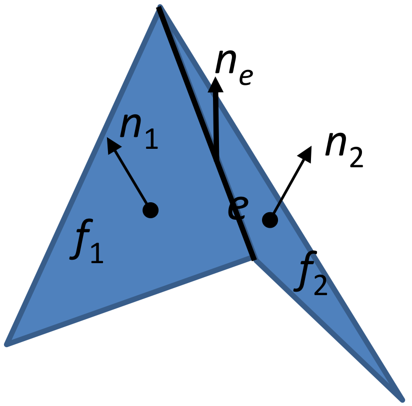


Discrete Surfaces

Picking a (good) normal at the edge and looking at the set of planes passing through the normal, we get a family of curves.

Defining the curvature of the curve in terms of the angle between curve segments:

- The min/max curvature is 0, with principal curvature direction along *e*.

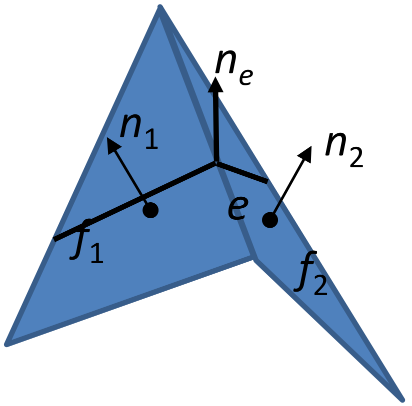


Discrete Surfaces

Picking a (good) normal at the edge and looking at the set of planes passing through the normal, we get a family of curves.

Defining the curvature of the curve in terms of the angle between curve segments:

- The min/max curvature is 0, with principal curvature direction along e.
- The max/min curvature is equal to the dihedral angle ($\beta(e)=\angle n_1 n_2$), with principal curvature direction along $n_e x e$.



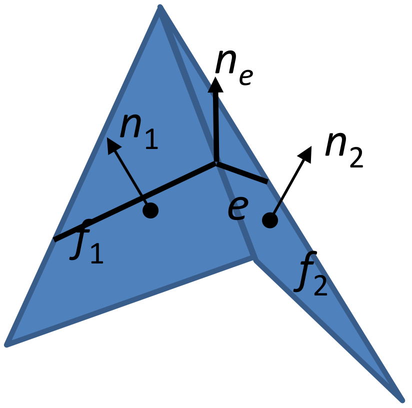
Discrete Surfaces

Thus, the mean curvature at a point on the edge is equal to the dihedral angle between the faces:

$$H(p \in e) = \beta(e)$$

Moreover, since the principal curvatures at $p \in e$ are 0 and $\beta(e)$, with directions $e/|e|$ and $n_e x e/|n_e x e|$, we can use this to define a curvature tensor at $p \in e$:

$$\boldsymbol{C}(p) = \beta(e) \frac{1}{\|n_e \times e\|^2} (n_e \times e)(n_e \times e)^t$$

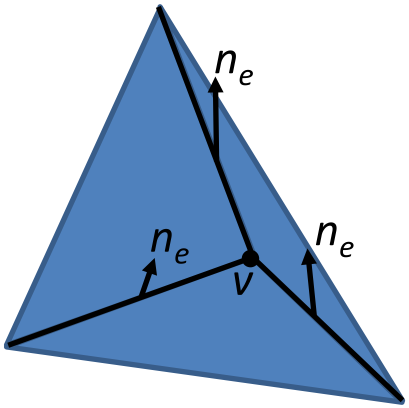


Discrete Surfaces

$$\boldsymbol{C}(p) = \beta(e)\frac{1}{\|n_e \times e\|^2}(n_e \times e)(n_e \times e)^t$$

Averaging the curvature tensor over the edges coming out of a vertex *v*, we get a curvature tensor associated with vertices:

$$\boldsymbol{C}(v) = \sum_{v \in e}\frac{|e|}{2}\beta(e)\frac{1}{\|n_e \times e\|^2}(n_e \times e)(n_e \times e)^t$$

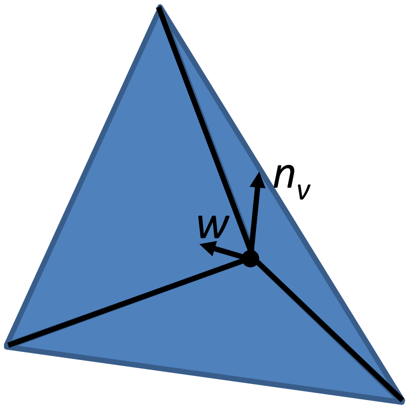


Discrete Surfaces

$$\boldsymbol{C}(v) = \sum_{v \in e} \frac{|e|}{2} \beta(e) \frac{1}{\|n \times e\|^2} (n \times e)(n \times e)^t$$

For any "tangent direction" *w* at *v*, we can now compute the normal curvature along *w* as:

$$\kappa_v(w) = \frac{w^t \boldsymbol{C}(v) w}{w^t w}$$



Discrete Surfaces

$$\boldsymbol{C}(v) = \sum_{v \in e} \frac{|e|}{2} \beta(e) \frac{1}{\|n \times e\|^2} (n \times e)(n \times e)^t$$

For any "tangent direction" *w* at *v*, we can now compute the normal curvature along *w* as:

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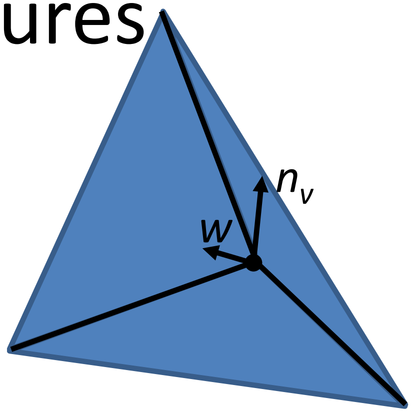
allowing us to compute principal curvatures values and directions at a vertex.



Discrete Surfaces

$$\boldsymbol{C}(v) = \sum_{v \in e} \frac{|e|}{2} \beta(e) \frac{1}{\|n \times e\|^2} (n \times e)(n \times e)^t$$

Using the principal curvature information, we can define mean and Gaussian curvatures at the vertices.

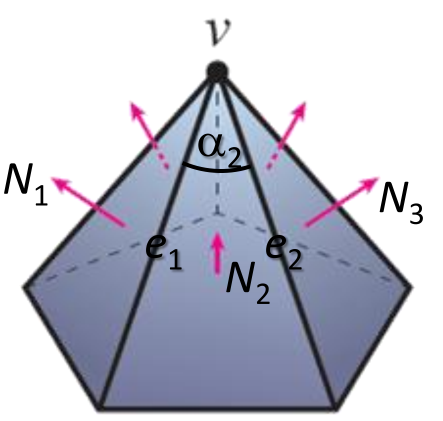
Discrete Surfaces

Note:

This discretization of curvature information and does not have to conform with others.

For example, the computed Gaussian curvature is not the same as the one defined using angle deficits.

$$\kappa(v) = 2\pi - \sum_{i=1}^{k} \alpha_i$$



Discrete Surfaces

Note:

This discretization of curvature information and does not have to conform with others.

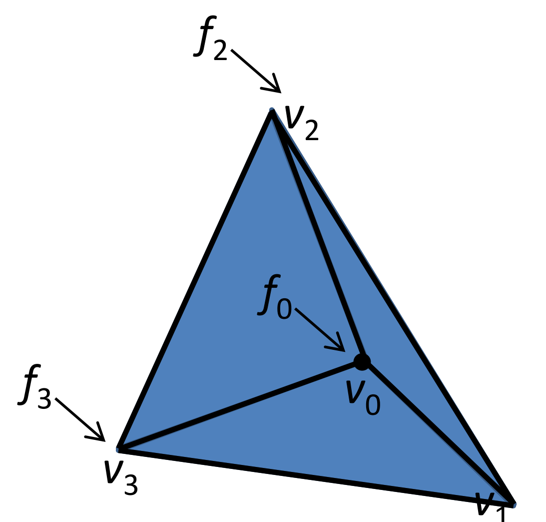
Similarly, we can use the cotangent Laplacian and the fact that:

$$\Delta_S f = -2H\boldsymbol{n}$$

to define the mean curvature at a vertex, but this also won't agree with the mean-curvature defined by the curvature tensor.

Discretizing the Laplacian

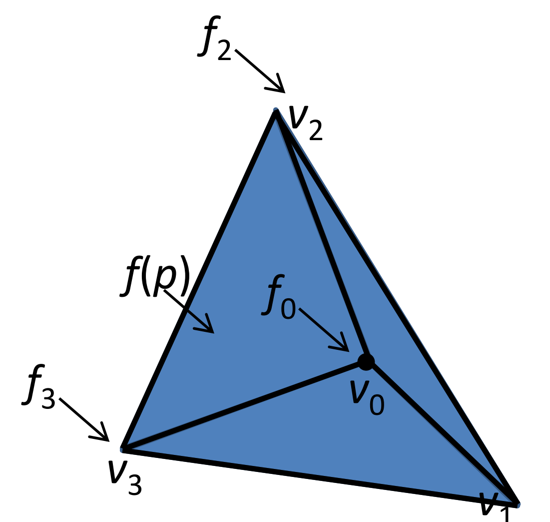
In considering functions on the a mesh, we will associate values with each vertex.



Discretizing the Laplacian

In considering functions on the a mesh, we will associate values with each vertex.

We then extend the functions to the interior of the triangles using barycentric interpolation.

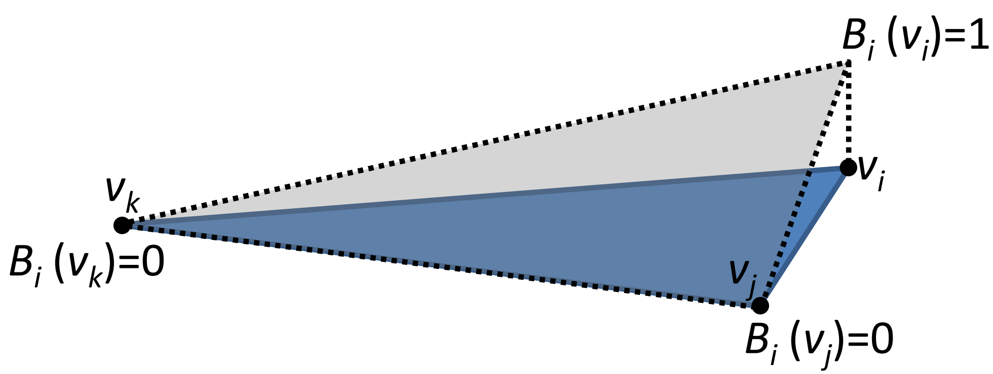


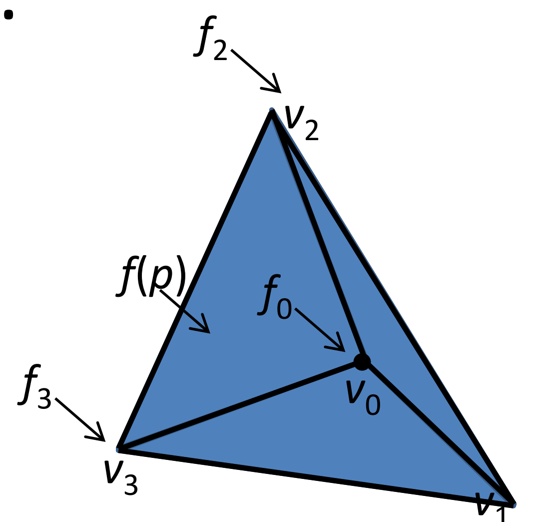
Discretizing the Laplacian

This allows us to think of the function $f(p)$ as the sum:

$$f(p) = \sum_i f_i B_i(p)$$

where B_i is the "tent" function centered at vertex v_i and supported in the 1-ring.





Discretizing the Laplacian

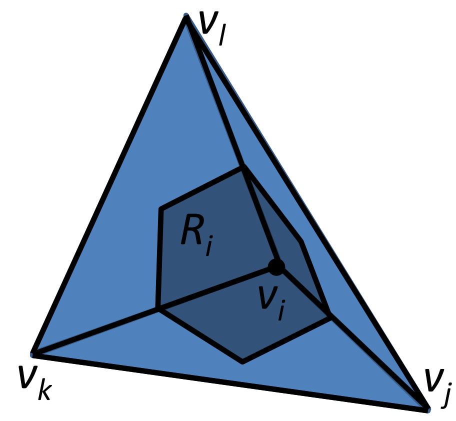
Since the Laplace-Beltrami operator is linear, to compute the Laplacian of *f* it suffices to be able to compute the Laplacians of $B_i(p)$:

$$\Delta f(p) = \sum_i f_i \Delta B_i(p)$$

Discretizing the Laplacian

Since also want to represent the Laplacian of *f* just by prescribing vertex values, we will set the Laplacian of *f* at vertex *v* to be the average of the Laplacian in a neighborhood around *v*:

$$\left(\Delta B_j\right)_i = \frac{1}{\left|R_i\right|}\int_{R_i} \Delta B_j(r)\,dr = \int_{R_i} \mathrm{div}\left(\nabla B_j(r)\right)dr$$



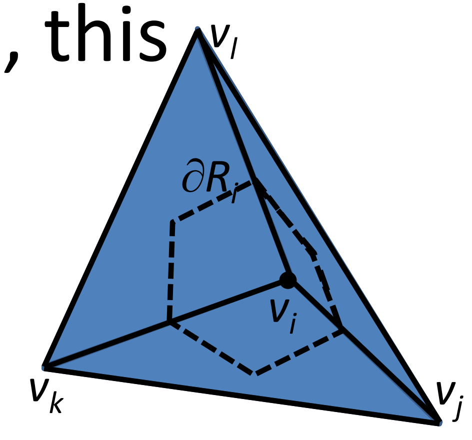
Discretizing the Laplacian

Since also want to represent the Laplacian of *f* just by prescribing vertex values, we will set the Laplacian of *f* at vertex *v* to be the average of the Laplacian in a neighborhood around *v*:

$$\left(\Delta B_j\right)_i = \frac{1}{|R_i|}\int_{R_i} \Delta B_j(r)\,dr = \int_{R_i} \mathrm{div}\left(\nabla B_j(r)\right)dr$$

But by the definition of the divergence, this is just the integral over the boundary:

$$\left(\Delta B_j\right)_i = \frac{1}{|R_i|}\int_{\partial R_i} \left\langle \nabla B_j(r), n_r \right\rangle dr$$

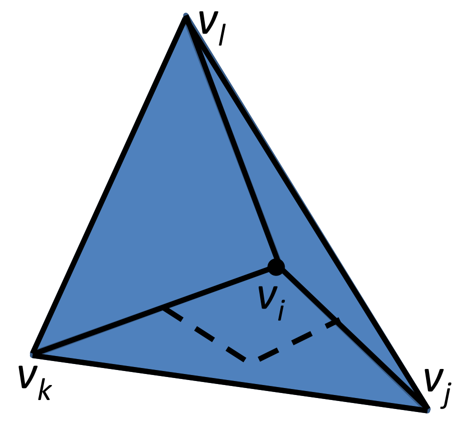


Discretizing the Laplacian

$$\left(\Delta B_j\right)_i = \frac{1}{|R_i|}\int_{\partial R_i}\left\langle \nabla B_j(r), n_r \right\rangle dr$$

Breaking up the R_i per triangle, we get:

$$\left(\Delta B_j\right)_i = \frac{1}{|R_i|}\sum_{v_i \in T}\int_{\partial R_i \cap T}\left\langle \nabla B_j(r), n_r \right\rangle dr$$



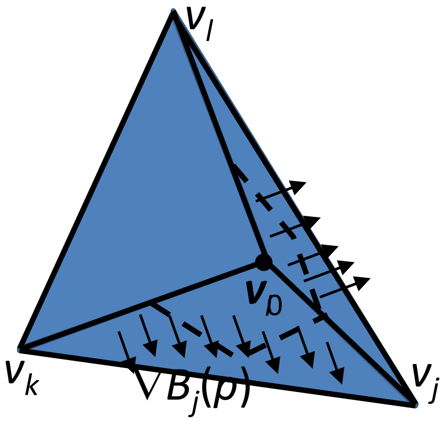
Discretizing the Laplacian

$$\left(\Delta B_j\right)_i = \frac{1}{\left|R_i\right|} \sum_{v_i \in T} \int_{\partial R_i \cap T} \left\langle \nabla B_j(r), n_r \right\rangle dr$$

Computing the gradient of B_j we get:

$$\nabla B_j\left(p\right) = \frac{(v_i - v_k)^{\perp}}{2\text{Area}(T_{ijk})}$$

for all p in triangle T_{ijk}.



Discretizing the Laplacian

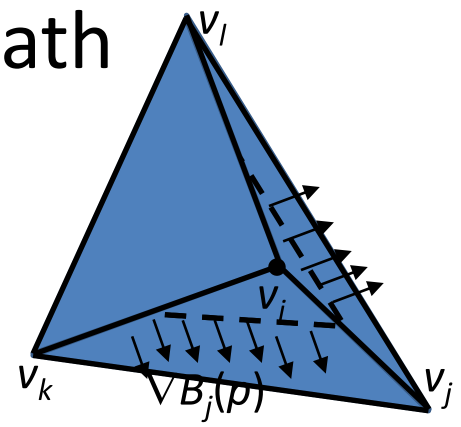
$$\left(\Delta B_j\right)_i = \frac{1}{|R_i|} \sum_{v_i \in T} \int_{\partial R_i \cap T} \left\langle \nabla B_j(r), n_r \right\rangle dr$$

Computing the gradient of B_j we get:

$$\nabla B_j(p) = \frac{(v_i - v_k)^{\perp}}{2\mathrm{Area}(T_{ijk})}$$

for all p in triangle T_{ijk}.

Note that since the gradient is constant over the interior of the triangle, the integral is path independent.



Discretizing the Laplacian

$$\left(\Delta B_j\right)_i = \frac{1}{|R_i|}\sum_{v_i\in T}\int_{\partial R_i\cap T}\left\langle \nabla B_j(r), n_r\right\rangle dr$$

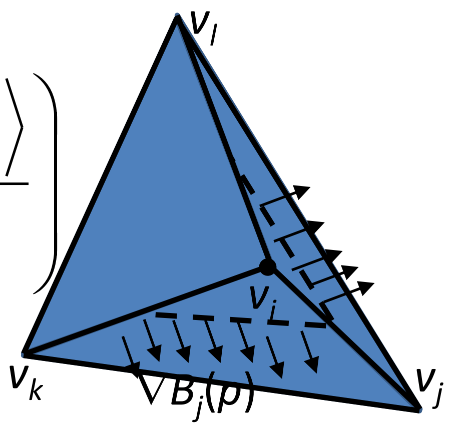
Computing the gradient of B_j we get:

$$\nabla B_j(p) = \frac{(v_i - v_k)^{\perp}}{2\text{Area}(T_{ijk})}$$

for all p in triangle T_{ijk}.

Computing the integral over the new boundary gives:

$$\left(\Delta B_j\right)_i = \frac{1}{|R_i|}\left(\frac{\left\langle (v_j - v_k)^{\perp}, (v_k - v_i)^{\perp}\right\rangle}{4\text{Area}(T_{ikj})} + \frac{\left\langle (v_l - v_j)^{\perp}, (v_i - v_l)^{\perp}\right\rangle}{4\text{Area}(T_{ijl})}\right)$$



Discretizing the Laplacian

$$\left(\Delta B_j\right)_i = \frac{1}{|R_i|}\left(\frac{\left\langle \left(v_j - v_k\right)^{\perp}, \left(v_k - v_i\right)^{\perp}\right\rangle}{4\mathrm{Area}(T_{ikj})} + \frac{\left\langle \left(v_l - v_j\right)^{\perp}, \left(v_i - v_l\right)^{\perp}\right\rangle}{4\mathrm{Area}(T_{ijl})}\right)$$

With a little bit of trigonometric manipulation, this gives:

$$\left(\Delta B_j\right)_i = \frac{1}{2|R_i|}\left(\cot \gamma_{jki} + \cot \gamma_{ilj}\right)$$