600.657: Mesh Processing

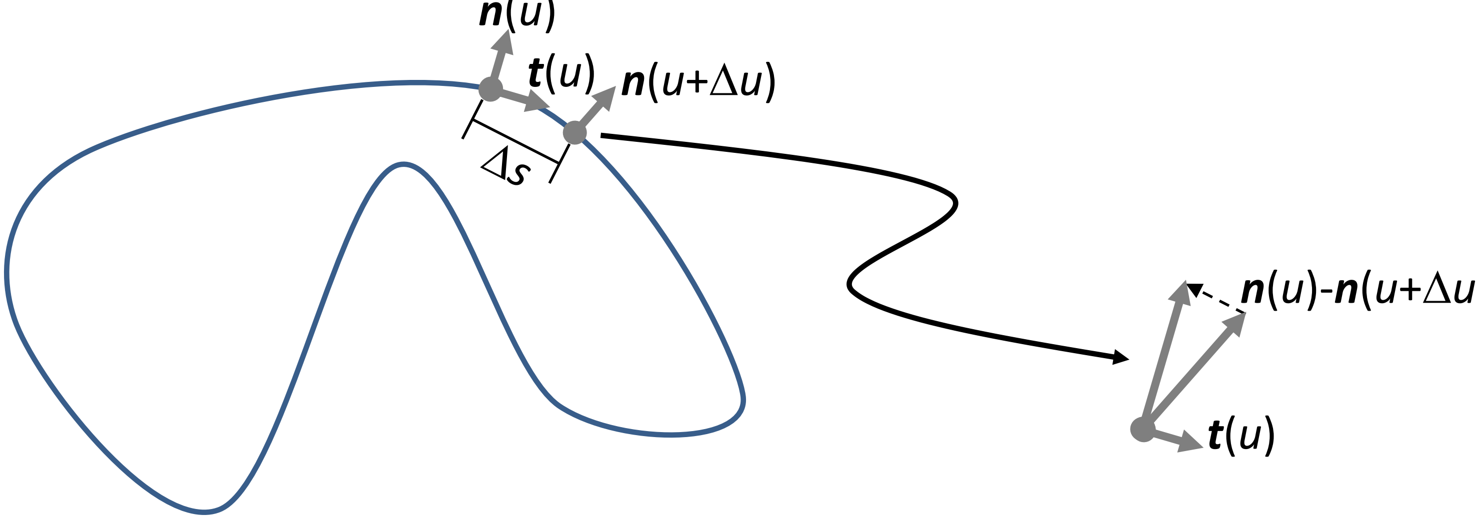
Chapter 3

Regular Curves

<u>Recall</u>:

The *curvature* at $\boldsymbol{x}(u)$ is the (negative)change in normal vector along the tangent direction relative to change in distance along the curve:

$$\kappa(u) = \left\langle \lim_{\Delta u \to 0} \frac{\boldsymbol{n}(u) - \boldsymbol{n}(u+\Delta u)}{\Delta s}, \boldsymbol{t}(u) \right\rangle$$



Regular Curves

$$\kappa(u) = \left\langle \lim_{\Delta u \to 0} \frac{\boldsymbol{n}(u) - \boldsymbol{n}(u+\Delta u)}{\Delta s}, \boldsymbol{t}(u) \right\rangle$$

<u>Note</u>:

If $\boldsymbol{x}$ is parameterized by arc-length, then $\Delta s = \Delta u$ so the curvature becomes:

$$\kappa(u) = \left\langle \lim_{\Delta u \to 0} \frac{\boldsymbol{n}(\Delta u) - \boldsymbol{n}(u+\Delta u)}{\Delta u}, \boldsymbol{t}(u) \right\rangle = -\langle \boldsymbol{n}'(u), \boldsymbol{t}(u) \rangle$$

Regular Curves

$$\kappa(u) = \left\langle \lim_{\Delta u \to 0} \frac{\boldsymbol{n}(u) - \boldsymbol{n}(u+\Delta u)}{\Delta s}, \boldsymbol{t}(u) \right\rangle$$

<u>Note</u>:

If $\boldsymbol{x}$ is parameterized by arc-length, then $\Delta s = \Delta u$ so the curvature becomes:

$$\kappa(u) = \left\langle \lim_{\Delta u \to 0} \frac{\boldsymbol{n}(\Delta u) - \boldsymbol{n}(u+\Delta u)}{\Delta u}, \boldsymbol{t}(u) \right\rangle = -\langle \boldsymbol{n}'(u), \boldsymbol{t}(u) \rangle$$

Otherwise, we have $\Delta s / \Delta u = |\boldsymbol{x}'(u)|$, so that:

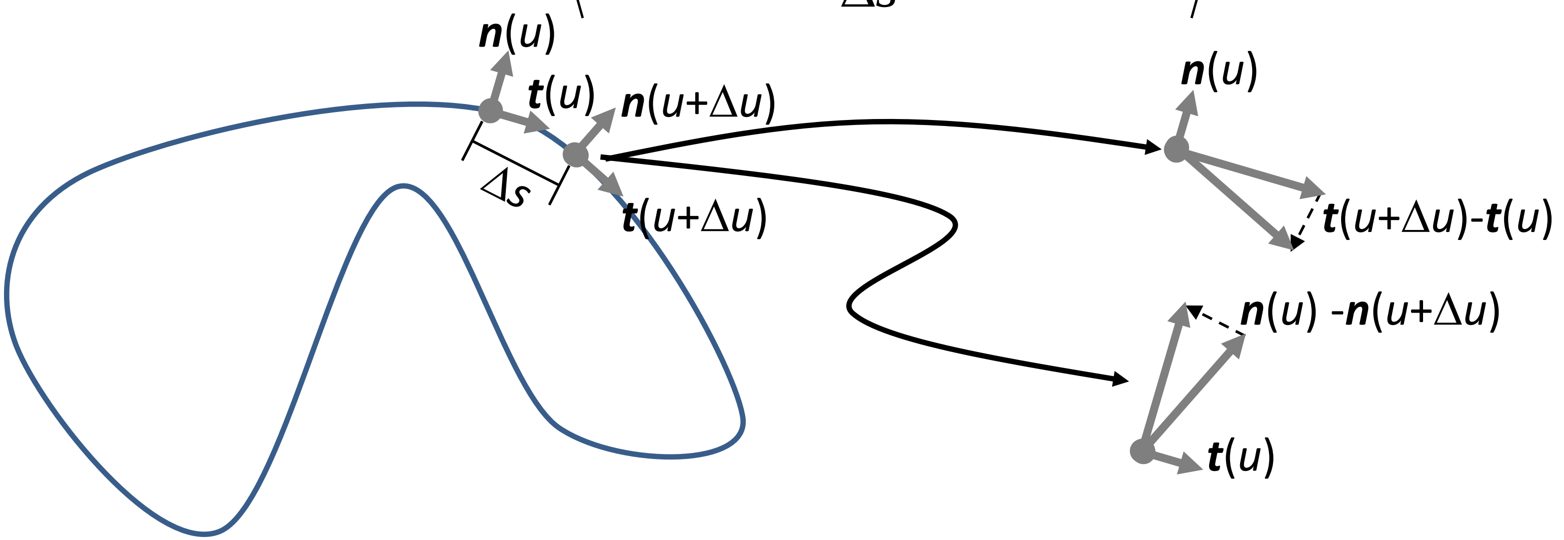
$$\kappa(u) = \left\langle \lim_{\Delta u \to 0} \frac{\boldsymbol{n}(u) - \boldsymbol{n}(u+\Delta u)}{\Delta u \cdot |\boldsymbol{x}'(u)|}, \boldsymbol{t}(u) \right\rangle = -\frac{\langle \boldsymbol{n}'(u), \boldsymbol{t}(u) \rangle}{|\boldsymbol{x}'(u)|} = -\frac{\langle \boldsymbol{n}'(u), \boldsymbol{x}'(u) \rangle}{\langle \boldsymbol{x}'(u), \boldsymbol{x}'(u) \rangle}$$

Regular Curves

<u>Alternate Interpretation</u>:

The *curvature* at $\boldsymbol{x}(u)$ is the (positive) change in the tangent vector along the normal direction relative to change in distance along the curve:

$$\kappa(u) = \left\langle \lim_{\Delta u \to 0} \frac{\boldsymbol{t}(u+\Delta u) - \boldsymbol{t}(u)}{\Delta s}, \boldsymbol{n}(u) \right\rangle$$



Regular Curves

Proof of Equivalence:

To show equivalence, we need to show that:

$$-\langle \boldsymbol{n}'(u), \boldsymbol{t}(u) \rangle = \langle \boldsymbol{n}(u), \boldsymbol{t}'(u) \rangle$$

Regular Curves

Proof of Equivalence:

To show equivalence, we need to show that:

$$-\left\langle \mathbf{n}'(u), \mathbf{t}(u) \right\rangle = \left\langle \mathbf{n}(u), \mathbf{t}'(u) \right\rangle$$

Taking the derivative of both sides:

$$0 = \left\langle \mathbf{n}(u), \mathbf{t}(u) \right\rangle$$

we get:

$$0 = \frac{d}{du}\left\langle \mathbf{n}(u), \mathbf{t}(u) \right\rangle = \left\langle \mathbf{n}'(u), \mathbf{t}(u) \right\rangle + \left\langle \mathbf{n}(u), \mathbf{t}'(u) \right\rangle$$

Regular Curves

Proof of Equivalence:

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giving:

$$-\langle \mathbf{n}'(u), \mathbf{t}(u) \rangle = \langle \mathbf{n}(u), \mathbf{t}'(u) \rangle$$

Regular Curves

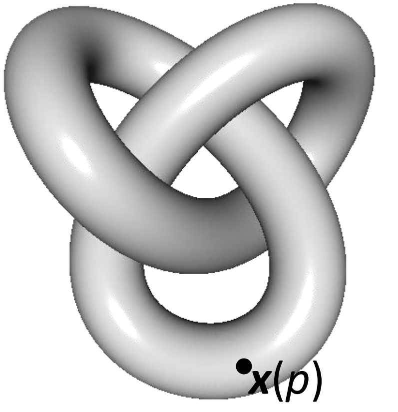
Thus, we can also express the curvature as:

$$\kappa(u) = -\frac{\langle \boldsymbol{n}'(u), \boldsymbol{t}(u) \rangle}{|\boldsymbol{x}'(u)|} = \frac{\langle \boldsymbol{n}(u), \boldsymbol{t}'(u) \rangle}{|\boldsymbol{x}'(u)|} = \ldots = \frac{\langle \boldsymbol{n}(u), \boldsymbol{x}''(u) \rangle}{\langle \boldsymbol{x}'(u), \boldsymbol{x}'(u) \rangle}$$

Regular Surfaces

Curvature:

We extend the notion to the curvature of a surface at the point $\boldsymbol{x}(p)$ by looking at the curvature of curves on the surface.

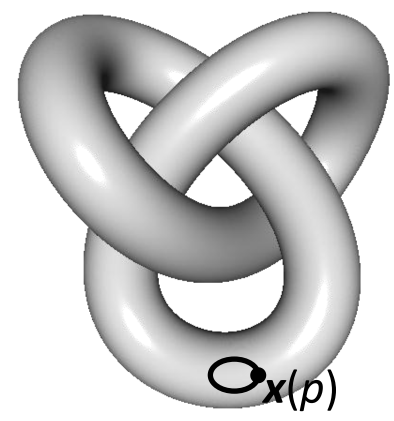


Regular Surfaces

<u>Curvature</u>:

We extend the notion to the curvature of a surface at the point $\boldsymbol{x}(p)$ by looking at the curvature of curves on the surface.

Using arbitrary curves, we don't get a sense of the curvature as we go "around" the surface, e.g. we can get the curvature to be arbitrarily small.

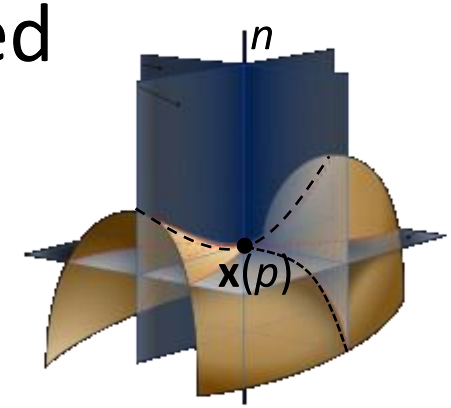


Regular Surfaces

Curvature:

We extend the notion to the curvature of a surface at the point $\boldsymbol{x}(p)$ by looking at the curvature of curves on the surface.

Instead, we look at the curvature of *normal curves* – curves through $\boldsymbol{x}(p)$ obtained by intersecting the surface with a plane containing the normal at $\boldsymbol{x}(p)$.



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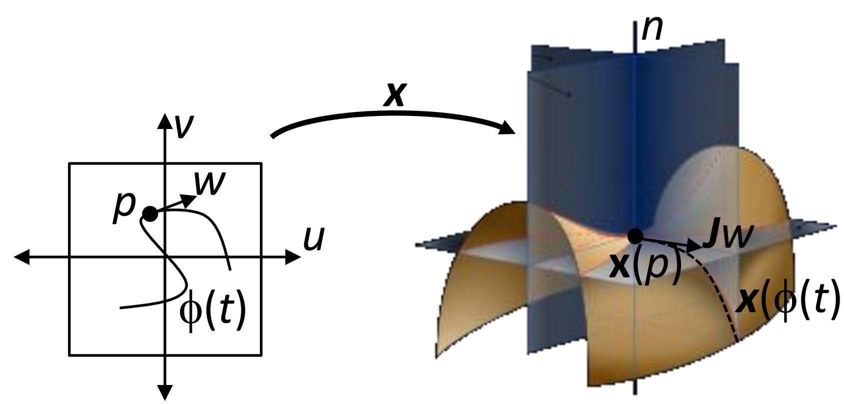
Regular Surfaces

<u>Curvature</u>:

Fix a normal plane by choosing $v=\boldsymbol{J}w$ in the tangent plane at $\boldsymbol{x}(p)$ and let $\phi(t)$ be the curve in the parameterization domain which maps to the intersection of the surface with the plane.



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Regular Surfaces

$$\kappa(u)=\frac{\langle \boldsymbol{n}(u),\boldsymbol{x}''(u)\rangle}{\langle \boldsymbol{x}'(u),\boldsymbol{x}'(u)\rangle}$$

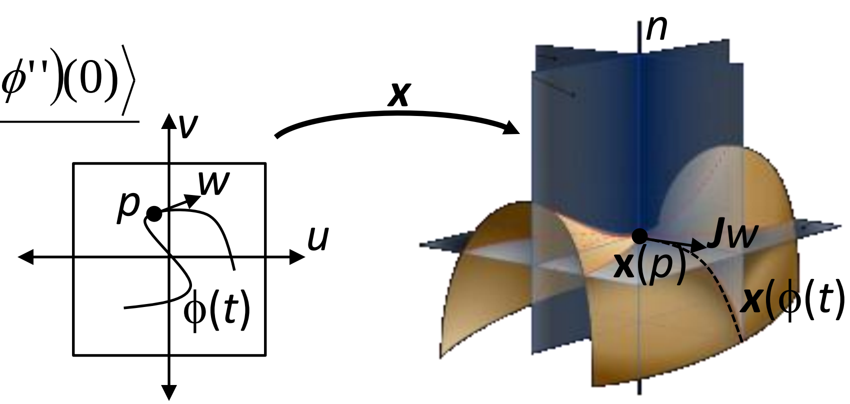
<u>Curvature</u>:

Computing the curvature of the curve $\boldsymbol{x}(\phi(t))$ at $\boldsymbol{x}(\phi(0))=\boldsymbol{x}(p)$ gives:

$$\kappa(0)=\frac{\langle \boldsymbol{n},(\boldsymbol{x}\circ\phi)''(0)\rangle}{\langle (\boldsymbol{x}\circ\phi)'(0),(\boldsymbol{x}\circ\phi)'(0)\rangle}$$

$$=\frac{\left\langle \boldsymbol{n},\left(\left(d^2\boldsymbol{x}\circ\phi\right)\cdot\phi'(0)\right)\cdot\phi'(0)+\left(\left(d\boldsymbol{x}\circ\phi\right)\cdot\phi''\right)(0)\right\rangle}{\langle \boldsymbol{J}w,\boldsymbol{J}w\rangle}$$

$$=\frac{w^t\begin{pmatrix}\langle \boldsymbol{n},\boldsymbol{x}_{uu}(p)\rangle & \langle \boldsymbol{n},\boldsymbol{x}_{vu}(p)\rangle\\ \langle \boldsymbol{n},\boldsymbol{x}_{uv}(p)\rangle & \langle \boldsymbol{n},\boldsymbol{x}_{vv}(p)\rangle\end{pmatrix}w}{w^t\boldsymbol{I}(p)w}$$



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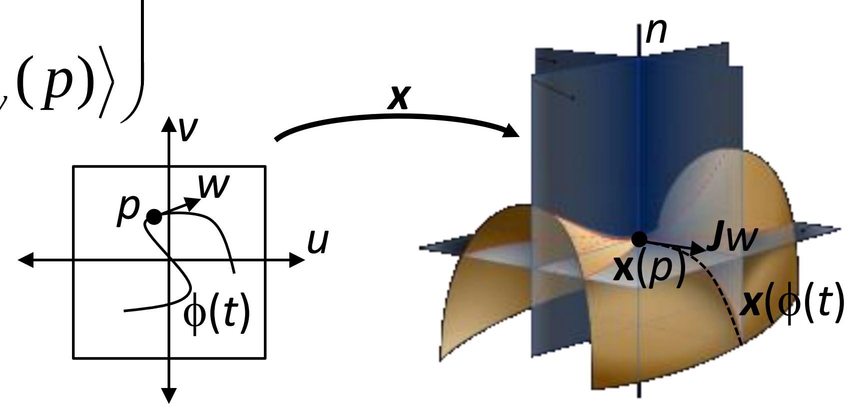
Regular Surfaces

$$\kappa(0) = \frac{w^t \begin{pmatrix} \langle \boldsymbol{n}, \boldsymbol{x}_{uu} \rangle & \langle \boldsymbol{n}, \boldsymbol{x}_{vu} \rangle \\ \langle \boldsymbol{n}, \boldsymbol{x}_{uv} \rangle & \langle \boldsymbol{n}, \boldsymbol{x}_{vv} \rangle \end{pmatrix} w}{w^t \boldsymbol{I} w}$$

Definition:

Given the parameterization ***x***, the *second fundamental form* 2x2 matrix:

$$\boldsymbol{II}(p) = \begin{pmatrix} \langle \boldsymbol{n}(p), \boldsymbol{x}_{uu}(p) \rangle & \langle \boldsymbol{n}(p), \boldsymbol{x}_{vu}(p) \rangle \\ \langle \boldsymbol{n}(p), \boldsymbol{x}_{uv}(p) \rangle & \langle \boldsymbol{n}(p), \boldsymbol{x}_{vv}(p) \rangle \end{pmatrix}$$



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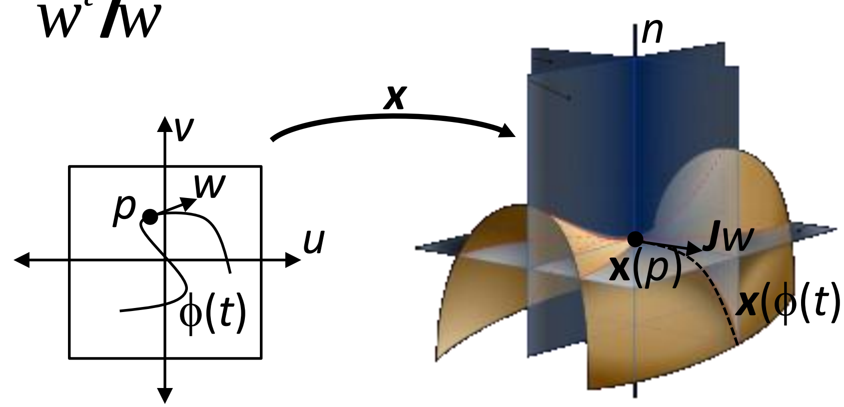
Regular Surfaces

$$\boldsymbol{II}(p)=\begin{pmatrix}\langle \boldsymbol{n}(p),\boldsymbol{x}_{uu}(p)\rangle & \langle \boldsymbol{n}(p),\boldsymbol{x}_{vu}(p)\rangle \\ \langle \boldsymbol{n}(p),\boldsymbol{x}_{uv}(p)\rangle & \langle \boldsymbol{n}(p),\boldsymbol{x}_{vv}(p)\rangle\end{pmatrix}$$

Definition:

Using this matrix, the curvature of the surface at the point q=$\boldsymbol{x}(p)$ in direction v=$\boldsymbol{J}w$ is:

$$\kappa_p(w)=\frac{w^t\boldsymbol{II}w}{w^t\boldsymbol{I}w}$$



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Regular Surfaces

$$\kappa_p(w) = \frac{w^t \boldsymbol{II} w}{w^t \boldsymbol{I} w}$$

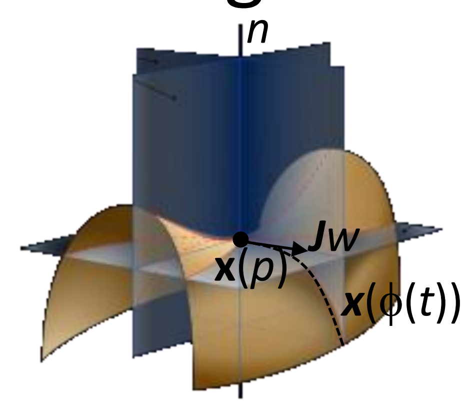
<u>Properties</u>:

1. The function $\kappa_p(w)$ is independent of scale.
2. It has minima and maxima at w_1 and w_2.
3. The images $v_1 = \boldsymbol{J}w_1$ and $v_2 = \boldsymbol{J}w_2$ are orthogonal:

$$\langle v_1, v_2 \rangle = \langle \boldsymbol{J}w_1, \boldsymbol{J}w_2 \rangle = w_1^t \boldsymbol{I} w_2 = 0$$

4. If $v = \boldsymbol{J}w$ is a tangent vector at $\boldsymbol{x}(p)$ making an angle α with v_1 then:

$$\kappa_p(w) = \kappa_p(w_1)\cos^2\alpha + \kappa_p(w_2)\sin^2\alpha$$



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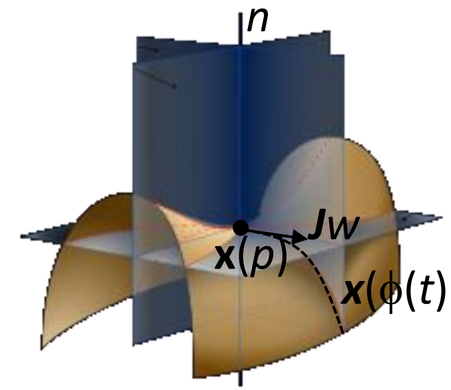
Regular Surfaces

$$\kappa_p(w) = \frac{w^t \mathbf{II} w}{w^t \mathbf{I} w}$$

<u>Definition</u>:

The (unit) directions $\boldsymbol{J}w_1$ and $\boldsymbol{J}w_2$ are called the *principal curvature directions* and the associated values $\kappa_1(p)=\kappa_p(w_1)$ and $\kappa_1(p)=\kappa_p(w_2)$ are called the *principal curvature* values.



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Regular Surfaces

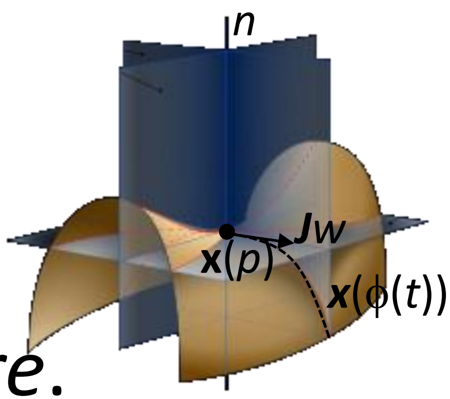
$$\kappa_p(w) = \frac{w^t \mathbf{II} w}{w^t \mathbf{I} w}$$

<u>Definition</u>:

The (unit) directions $\mathbf{J}w_1$ and $\mathbf{J}w_2$ are called the *principal curvature directions* and the associated values $\kappa_1(p)=\kappa_p(w_1)$ and $\kappa_1(p)=\kappa_p(w_2)$ are called the *principal curvature* values.

The sum of the principal curvatures is the *mean curvature*.

The product is the *Gaussian curvature*.



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Regular Surfaces

$$\kappa_p(w) = \frac{w^t \mathbf{II} w}{w^t \mathbf{I} w}$$

<u>Note</u>:

We can find the principal curvature values (and directions) by setting the derivative of $\kappa_p(w)$ to 0:

$$\nabla \kappa_p(w) = 0 \quad \Rightarrow \quad \frac{\left(w^t \mathbf{I} w\right)}{\left(w^t \mathbf{II} w\right)} \mathbf{II} w = \mathbf{I} w$$

Regular Surfaces

$$\kappa_p(w) = \frac{w^t \mathbf{II} w}{w^t \mathbf{I} w}$$

<u>Note</u>:

We can find the principal curvature values (and directions) by setting the derivative of $\kappa_p(w)$ to 0:

$$\nabla \kappa_p(w) = 0 \quad \Rightarrow \quad \frac{(w^t \mathbf{I} w)}{(w^t \mathbf{II} w)} \mathbf{II} w = \mathbf{I} w$$

Thus, the principal curvature values (and directions) can be obtained by solving:

$$\mathbf{I}^{-1} \mathbf{II} w = \lambda w$$

Regular Surfaces

$$\mathbf{I}^{-1}\mathbf{II}w_1 = \kappa_1 w_1 \qquad \mathbf{I}^{-1}\mathbf{II}w_2 = \kappa_2 w_2$$

<u>Note</u>:

In particular, this implies that mean and Gaussian curvatures are the trace and determinant of this matrix:

$$\text{mean curvature} = H = \text{Tr}\left(\mathbf{I}^{-1}\mathbf{II}\right)$$

$$\text{Gaussian curvature} = K = \text{Det}\left(\mathbf{I}^{-1}\mathbf{II}\right)$$

Regular Surfaces

$$\kappa_p(w) = \kappa_1(p)\cos^2\alpha + \kappa_2(p)\sin^2\alpha$$

<u>Definition</u>:

Given the principal curvatures/directions, $\kappa_1/\boldsymbol{J}w_1$ and $\kappa_2/\boldsymbol{J}w_2$, the *curvature tensor* is a 3x3 symmetric matrix associated to each point on the surface, defined by:

$$\boldsymbol{C}(\boldsymbol{x}(p)) = \kappa_1 \boldsymbol{J}w_1 \boldsymbol{J}w_1^t + \kappa_2 \boldsymbol{J}w_2 \boldsymbol{J}w_2^t$$

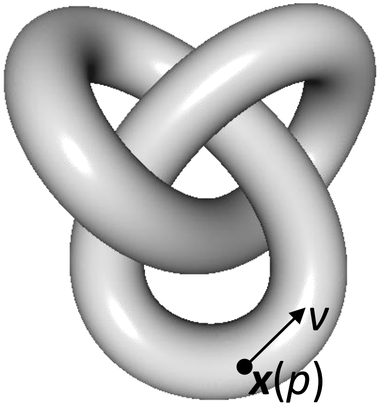
Regular Surfaces

$$\boldsymbol{C}(\boldsymbol{x}(p)) = \kappa_1 \boldsymbol{J}w_1 \boldsymbol{J}w_1^t + \kappa_2 \boldsymbol{J}w_2 \boldsymbol{J}w_2^t$$

Note:

Given a (non-tangent) vector *v* at the point ***x***(*p*), we can express *v* as:

$$v = \alpha \boldsymbol{J}w_1 + \beta \boldsymbol{J}w_2 + \gamma \boldsymbol{n}(p)$$



Regular Surfaces

$$\boldsymbol{C}(\boldsymbol{x}(p)) = \kappa_1 \boldsymbol{J}w_1 \boldsymbol{J}w_1^t + \kappa_2 \boldsymbol{J}w_2 \boldsymbol{J}w_2^t$$

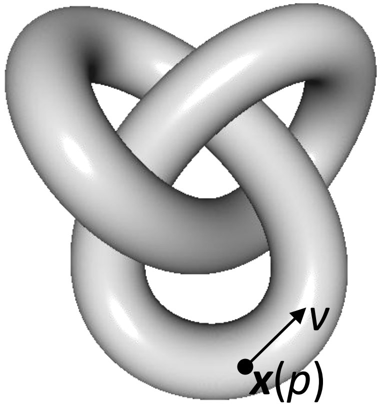
Note:

Given a (non-tangent) vector *v* at the point ***x***(*p*), we can express *v* as:

$$v = \alpha \boldsymbol{J}w_1 + \beta \boldsymbol{J}w_2 + \gamma \boldsymbol{n}(p)$$

Applying the curvature tensor to *v*, gives:

$$v^t \boldsymbol{C}(\boldsymbol{x}(p)) v = \alpha^2 \kappa_1 + \beta^2 \kappa_2$$



Regular Surfaces

$$\boldsymbol{C}(\boldsymbol{x}(p)) = \kappa_1 \boldsymbol{J} w_1 \boldsymbol{J} w_1^t + \kappa_2 \boldsymbol{J} w_2 \boldsymbol{J} w_2^t$$

<u>Note</u>:

Given a (non-tangent) vector *v* at the point ***x***(*p*), we can express *v* as:

$$v = \alpha \boldsymbol{J} w_1 + \beta \boldsymbol{J} w_2 + \gamma \boldsymbol{n}(p)$$

Applying the curvature tensor to *v*, gives:

$$v^t \boldsymbol{C}(\boldsymbol{x}(p)) v = \alpha^2 \kappa_1 + \beta^2 \kappa_2$$

So the curvature tensor gives the curvature in the tangent component (scaled by square length).

Regular Surfaces

Example (Sphere):

Parameterizing the sphere (almost everywhere) by the map:

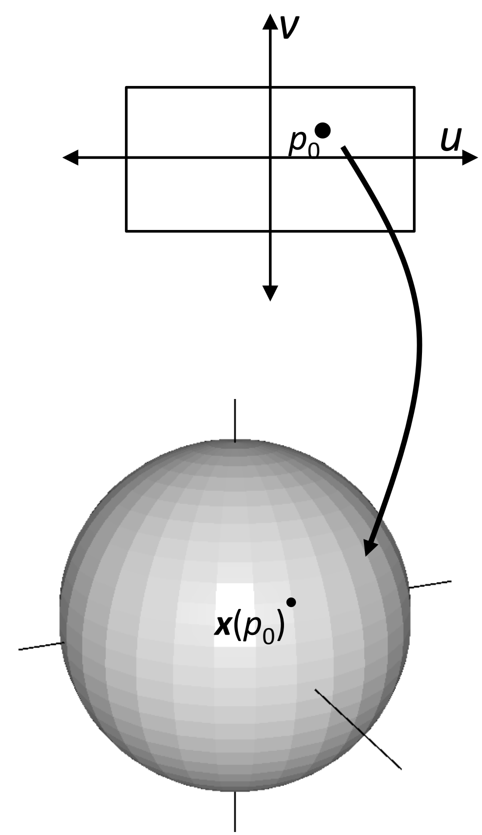
$$\boldsymbol{x}(u,v) = \begin{pmatrix}\cos u\cos v & \sin v & \sin u\cos v\end{pmatrix}^t$$

The 1st partial derivatives are:

$$\boldsymbol{x}_u(u,v) = \begin{pmatrix}-\sin u\cos v & 0 & \cos u\cos v\end{pmatrix}$$
$$\boldsymbol{x}_v(u,v) = \begin{pmatrix}-\cos u\sin v & \cos v & -\sin u\sin v\end{pmatrix}$$

Which gives:

$$\boldsymbol{I}(u,v) = \begin{pmatrix}\cos^2 v & 0 \\ 0 & 1\end{pmatrix}$$



Regular Surfaces

$$\boldsymbol{x}(u,v) = \begin{pmatrix} \cos u \cos v & \sin v & \sin u \cos v \end{pmatrix}^t \qquad \boldsymbol{I}(u,v) = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

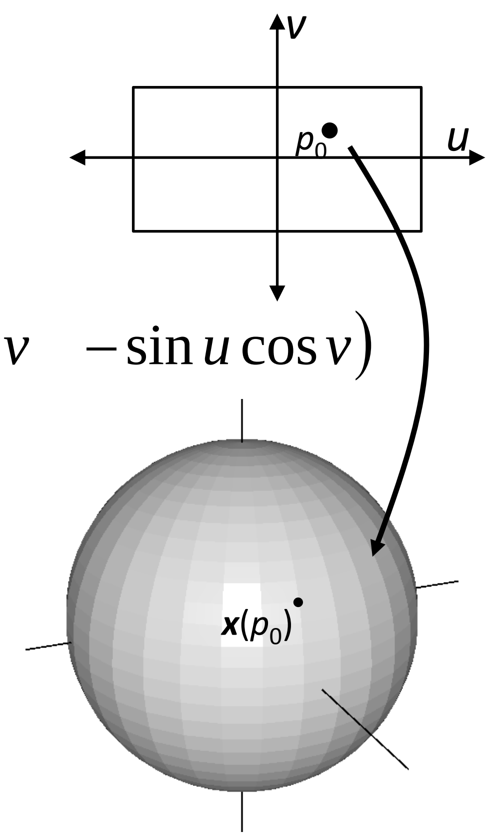
$$\boldsymbol{x}_u(u,v) = \begin{pmatrix} -\sin u \cos v & 0 & \cos u \cos v \end{pmatrix}$$

$$\boldsymbol{x}_v(u,v) = \begin{pmatrix} -\cos u \sin v & \cos v & -\sin u \sin v \end{pmatrix}$$

<u>Example (Sphere)</u>:

The normal is defined as:

$$\boldsymbol{n}(u,v) = \frac{\boldsymbol{x}_u(u,v) \times \boldsymbol{x}_u(u,v)}{\left|\boldsymbol{x}_u(u,v) \times \boldsymbol{x}_u(u,v)\right|} = \begin{pmatrix} -\cos u \cos v & -\sin v & -\sin u \cos v \end{pmatrix}$$



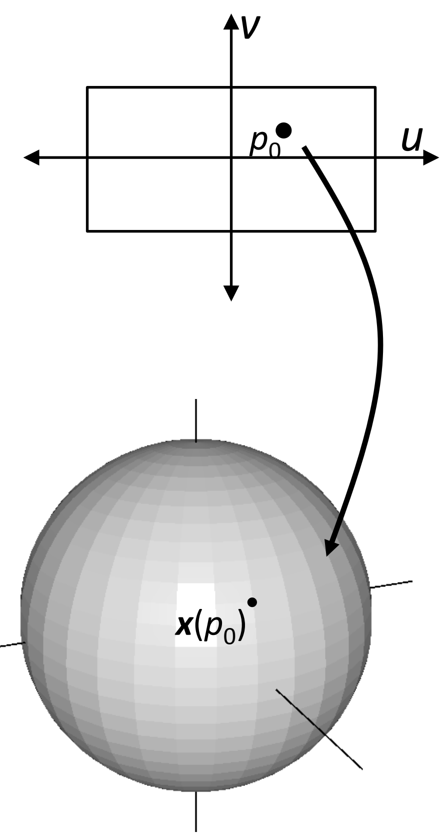
Regular Surfaces

$$\boldsymbol{x}(u,v) = \begin{pmatrix}\cos u\cos v & \sin v & \sin u\cos v\end{pmatrix}^t \qquad \boldsymbol{I}(u,v) = \begin{pmatrix}\cos^2 v & 0 \\ 0 & 1\end{pmatrix}$$

$$\boldsymbol{x}_u(u,v) = \begin{pmatrix}-\sin u\cos v & 0 & \cos u\cos v\end{pmatrix}$$

$$\boldsymbol{x}_v(u,v) = \begin{pmatrix}-\cos u\sin v & \cos v & -\sin u\sin v\end{pmatrix}$$

$$\boldsymbol{n}(u,v) = \begin{pmatrix}-\cos u\cos v & -\sin v & -\sin u\cos v\end{pmatrix}$$



<u>Example (Sphere)</u>:

The 2nd partial derivatives are:

$$\boldsymbol{x}_{uu}(u,v) = \begin{pmatrix}-\cos u\cos v & 0 & -\sin u\cos v\end{pmatrix}$$

$$\boldsymbol{x}_{uv}(u,v) = \boldsymbol{x}_{vu}(u,v) = \begin{pmatrix}\sin u\sin v & 0 & -\cos u\sin v\end{pmatrix}$$

$$\boldsymbol{x}_{vv}(u,v) = \begin{pmatrix}-\cos u\cos v & -\sin v & -\sin u\cos v\end{pmatrix}$$

Regular Surfaces

$$\boldsymbol{x}(u,v) = \begin{pmatrix}\cos u\cos v & \sin v & \sin u\cos v\end{pmatrix}^t \qquad \boldsymbol{I}(u,v) = \begin{pmatrix}\cos^2 v & 0 \\ 0 & 1\end{pmatrix}$$

$$\boldsymbol{x}_u(u,v) = \begin{pmatrix}-\sin u\cos v & 0 & \cos u\cos v\end{pmatrix}$$

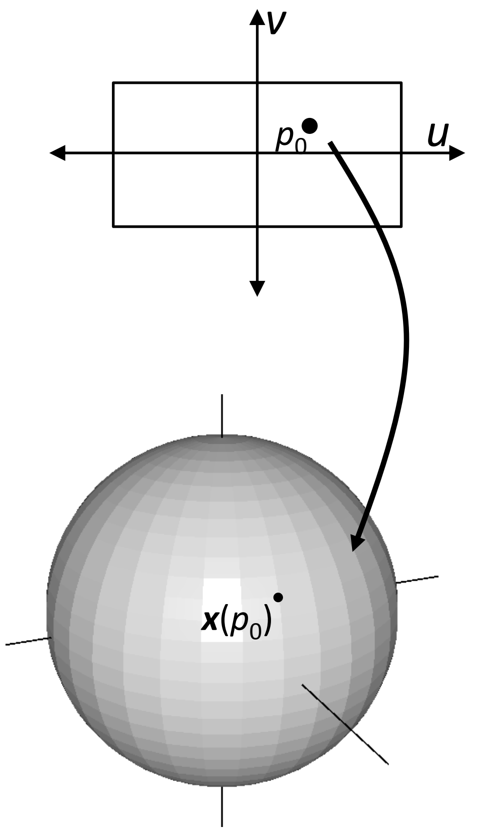
$$\boldsymbol{x}_v(u,v) = \begin{pmatrix}-\cos u\sin v & \cos v & -\sin u\sin v\end{pmatrix}$$

$$\boldsymbol{n}(u,v) = \begin{pmatrix}-\cos u\cos v & -\sin v & -\sin u\cos v\end{pmatrix}$$

$$\boldsymbol{x}_{uu}(u,v) = \begin{pmatrix}-\cos u\cos v & 0 & -\sin u\cos v\end{pmatrix}$$

$$\boldsymbol{x}_{uv}(u,v) = \boldsymbol{x}_{vu}(u,v) = \begin{pmatrix}\sin u\sin v & 0 & -\cos u\sin v\end{pmatrix}$$

$$\boldsymbol{x}_{vv}(u,v) = \begin{pmatrix}-\cos u\cos v & -\sin v & -\sin u\cos v\end{pmatrix}$$



Example (Sphere):

This gives:

$$\boldsymbol{II}(u,v) = \begin{pmatrix}\cos^2 v & 0 \\ 0 & 1\end{pmatrix}$$

Regular Surfaces

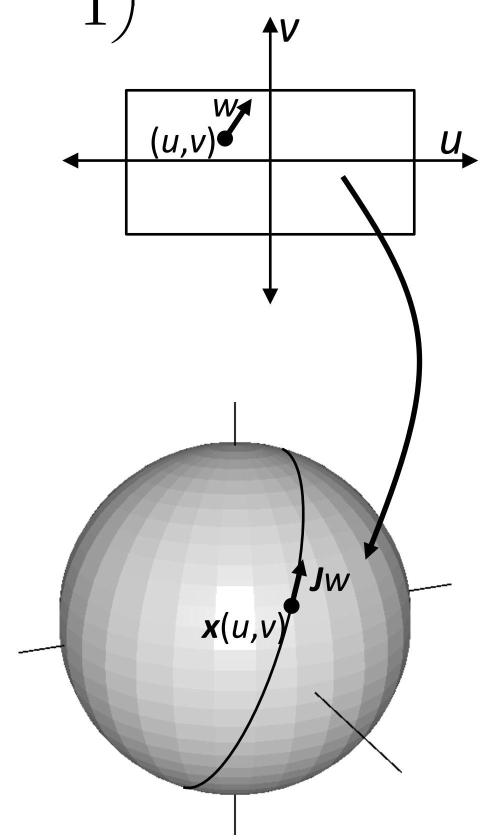
$$\boldsymbol{x}(u,v) = \begin{pmatrix} \cos u \cos v & \sin v & \sin u \cos v \end{pmatrix}^t$$

$$\boldsymbol{I}(u,v) = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix} \qquad \boldsymbol{II}(u,v) = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

<u>Example (Sphere)</u>:

Thus, for any point $\boldsymbol{x}(u,v)$ on the sphere, and any tangent direction $\boldsymbol{J}w$, the curvature at $\boldsymbol{x}(u,v)$ in the direction $\boldsymbol{J}w$ is:

$$\kappa_{\boldsymbol{x}(u,v)}(w) = \frac{w^t \boldsymbol{II}(u,v) w}{w^t \boldsymbol{I}(u,v) w} = 1$$



Regular Surfaces

$$\boldsymbol{x}(u,v) = \begin{pmatrix}\cos u \cos v & \sin v & \sin u \cos v\end{pmatrix}^t$$

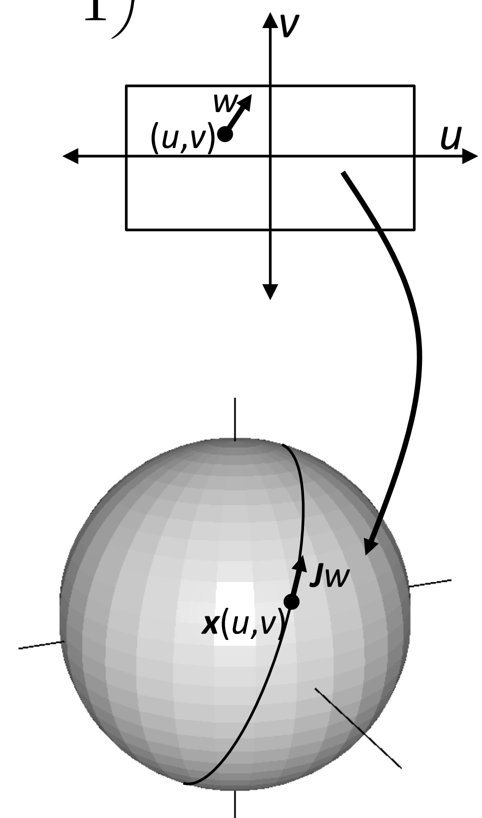
$$\boldsymbol{I}(u,v) = \begin{pmatrix}\cos^2 v & 0 \\ 0 & 1\end{pmatrix} \qquad \boldsymbol{II}(u,v) = \begin{pmatrix}\cos^2 v & 0 \\ 0 & 1\end{pmatrix}$$

<u>Example (Sphere)</u>:

Taking the trace and determinant:

$$H = \mathrm{Tr}\left(\boldsymbol{I}^{-1}\boldsymbol{II}\right) = 2$$

$$K = \mathrm{Det}\left(\boldsymbol{I}^{-1}\boldsymbol{II}\right) = 1$$



Regular Surfaces

Example (Torus):

Parameterizing the torus by the map:

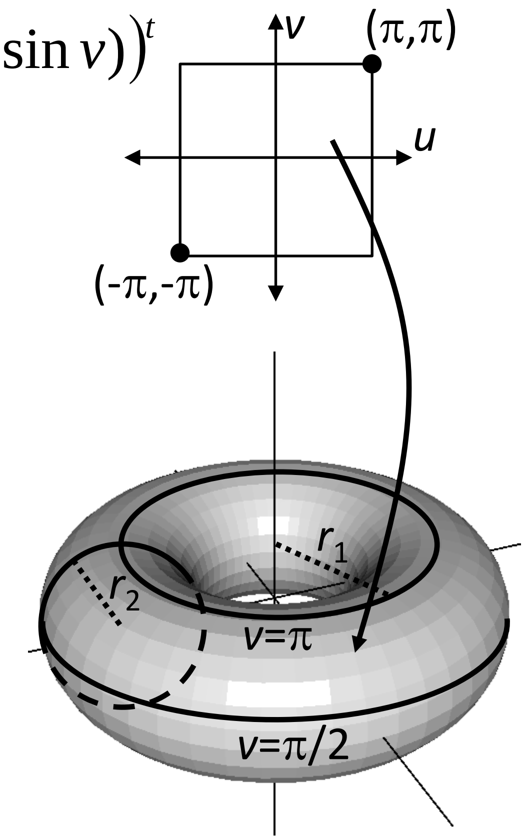
$$\boldsymbol{x}(u,v) = \left(\cos u(r_1 + r_2 \sin v) \quad -r_2 \cos v \quad \sin u(r_1 + r_2 \sin v)\right)^t$$

The 1st partial derivatives are:

$$\boldsymbol{x}_u(u,v) = (r_1 + r_2 \sin v)\left(-\sin u \quad 0 \quad \cos u\right)$$
$$\boldsymbol{x}_v(u,v) = r_2\left(\cos u \cos v \quad \sin v \quad \sin u \cos v\right)$$

Which gives:

$$\boldsymbol{I}(u,v) = \begin{pmatrix} (r_1 + r_2 \sin v)^2 & 0 \\ 0 & r_2^2 \end{pmatrix}$$



Regular Surfaces

$$\boldsymbol{x}(u,v) = \big((r_1 + r_2 \sin v)\cos u \quad -r_2 \cos v \quad (r_1 + r_2 \sin v)\sin u\big)^t \quad \boldsymbol{I}(u,v) = \begin{pmatrix} (r_1 + r_2 \sin v)^2 & 0 \\ 0 & r_2^2 \end{pmatrix}$$

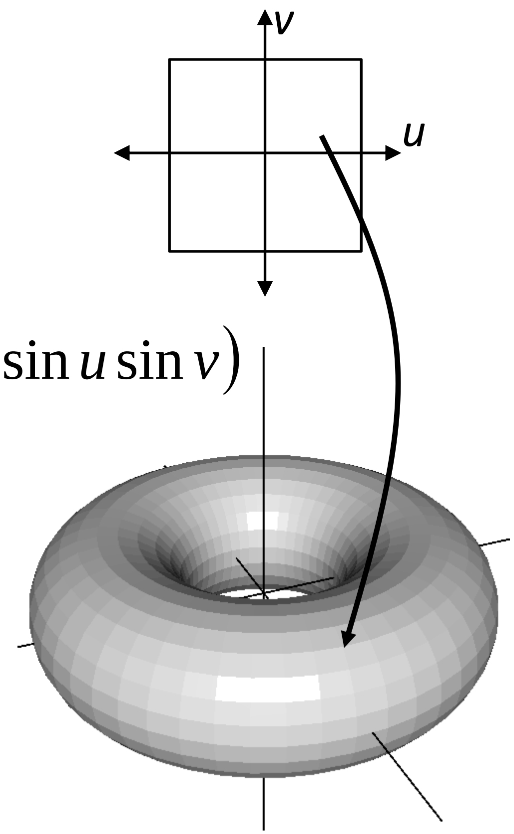
$$\boldsymbol{x}_u(u,v) = (r_1 + r_2 \sin v)\big(-\sin u \quad 0 \quad \cos u\big)$$

$$\boldsymbol{x}_v(u,v) = r_2\big(\cos u \cos v \quad \sin v \quad \sin u \cos v\big)$$

<u>Example (Torus)</u>:

The normal is defined as:

$$\boldsymbol{n}(u,v) = \frac{\boldsymbol{x}_u(u,v) \times \boldsymbol{x}_u(u,v)}{\left|\boldsymbol{x}_u(u,v) \times \boldsymbol{x}_u(u,v)\right|} = \big(-\cos u \sin v \quad \cos v \quad -\sin u \sin v\big)$$



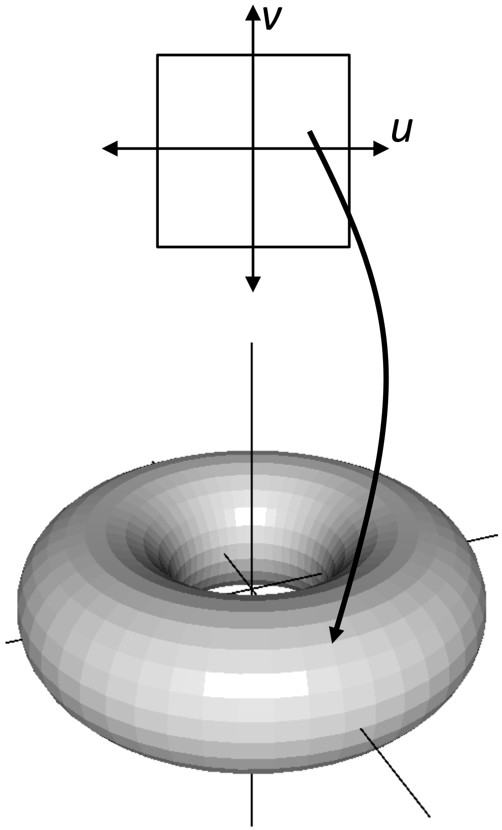
Regular Surfaces

$$\boldsymbol{x}(u,v) = \big((r_1 + r_2 \sin v)\cos u \quad -r_2 \cos v \quad (r_1 + r_2 \sin v)\sin u\big)^t \qquad \boldsymbol{I}(u,v) = \begin{pmatrix} (r_1 + r_2 \sin v)^2 & 0 \\ 0 & r_2^2 \end{pmatrix}$$

$$\boldsymbol{x}_u(u,v) = (r_1 + r_2 \sin v)\big(-\sin u \quad 0 \quad \cos u\big)$$

$$\boldsymbol{x}_v(u,v) = r_2\big(\cos u \cos v \quad \sin v \quad \sin u \cos v\big)$$

$$\boldsymbol{n}(u,v) = \big(-\cos u \sin v \quad \cos v \quad -\sin u \sin v\big)$$



<u>Example (Torus)</u>:

The 2nd partial derivatives are:

$$\boldsymbol{x}_{uu}(u,v) = (r_1 + r_2 \sin v)\big(-\cos u \quad 0 \quad -\sin u\big)$$

$$\boldsymbol{x}_{uv}(u,v) = \boldsymbol{x}_{vu}(u,v) = r_2 \cos v\big(-\sin u \quad 0 \quad \cos u\big)$$

$$\boldsymbol{x}_{vv}(u,v) = r_2\big(-\cos u \sin v \quad \cos v \quad -\sin u \sin v\big)$$

Regular Surfaces

$$\boldsymbol{x}(u,v) = \left((r_1 + r_2 \sin v)\cos u \quad -r_2 \cos v \quad (r_1 + r_2 \sin v)\sin u\right)^t \quad \boldsymbol{I}(u,v) = \begin{pmatrix} (r_1 + r_2 \sin v)^2 & 0 \\ 0 & r_2^2 \end{pmatrix}$$

$$\boldsymbol{x}_u(u,v) = (r_1 + r_2 \sin v)\left(-\sin u \quad 0 \quad \cos u\right)$$

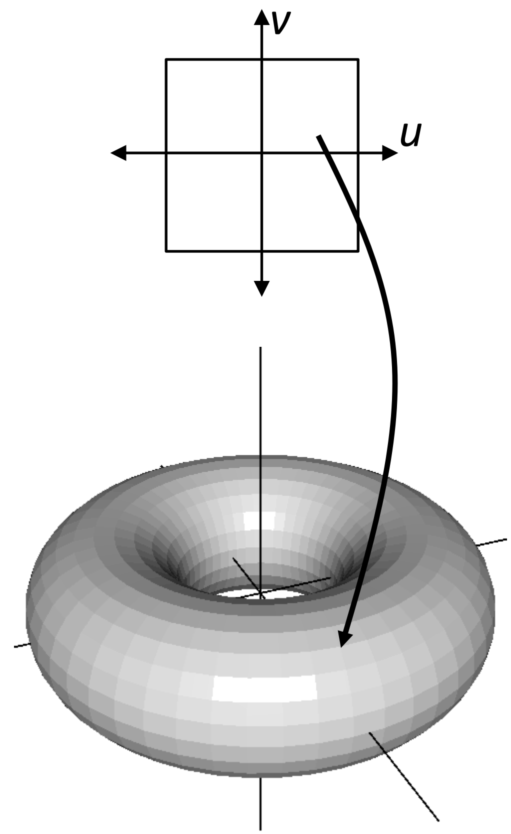
$$\boldsymbol{x}_v(u,v) = r_2\left(\cos u \cos v \quad \sin v \quad \sin u \cos v\right)$$

$$\boldsymbol{n}(u,v) = \left(-\cos u \sin v \quad \cos v \quad -\sin u \sin v\right)$$

$$\boldsymbol{x}_{uu}(u,v) = (r_1 + r_2 \sin v)\left(-\cos u \quad 0 \quad -\sin u\right)$$

$$\boldsymbol{x}_{uv}(u,v) = \boldsymbol{x}_{vu}(u,v) = r_2 \cos v\left(-\sin u \quad 0 \quad \cos u\right)$$

$$\boldsymbol{x}_{vv}(u,v) = r_2\left(-\cos u \sin v \quad \cos v \quad -\sin u \sin v\right)$$



<u>Example (Torus)</u>:

This gives:

$$\boldsymbol{II}(u,v) = \begin{pmatrix} (r_1 + r_2 \sin v)\sin v & 0 \\ 0 & r_2 \end{pmatrix}$$

Regular Surfaces

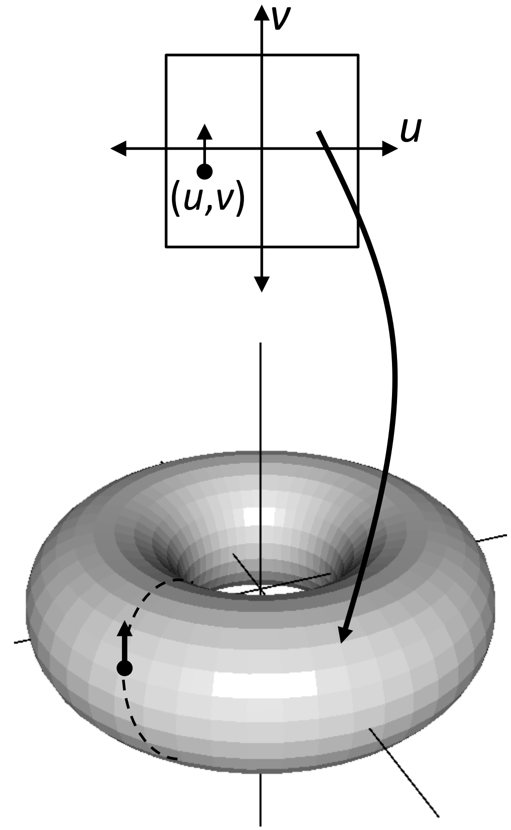
$$\boldsymbol{x}(u,v) = \big((r_1 + r_2 \sin v)\cos u \quad r_2 \cos v \quad (r_1 + r_2 \sin v)\sin u\big)^t$$

$$\boldsymbol{I}(u,v) = \begin{pmatrix} (r_1 + r_2 \sin v)^2 & 0 \\ 0 & r_2^2 \end{pmatrix} \qquad \boldsymbol{II}(u,v) = \begin{pmatrix} (r_1 + r_2 \sin v)\sin v & 0 \\ 0 & r_2 \end{pmatrix}$$

<u>Example (Torus)</u>:

And, for a point $\boldsymbol{x}(u,v)$ on the torus and for $w = (0,1)$, the curvature at $\boldsymbol{x}(u,v)$ in the direction $\boldsymbol{J}w$ is:

$$\kappa_{\boldsymbol{x}(u,v)}(\boldsymbol{J}w) = \frac{w^t \boldsymbol{II}(u,v) w}{w^t \boldsymbol{I}(u,v) w} = \frac{1}{r_2}$$



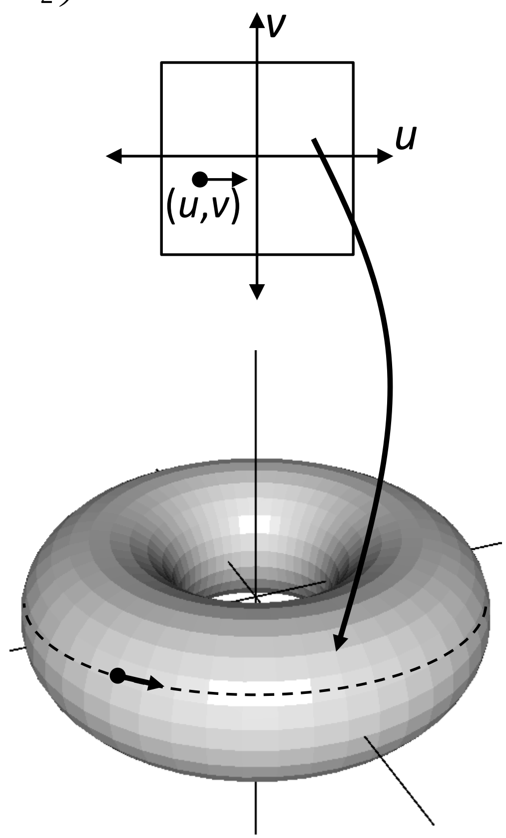
Regular Surfaces

$$\boldsymbol{x}(u,v) = \left((r_1 + r_2 \sin v)\cos u \quad r_2 \cos v \quad (r_1 + r_2 \sin v)\sin u\right)^t$$

$$\boldsymbol{I}(u,v) = \begin{pmatrix} (r_1 + r_2 \sin v)^2 & 0 \\ 0 & r_2^2 \end{pmatrix} \qquad \boldsymbol{II}(u,v) = \begin{pmatrix} (r_1 + r_2 \sin v)\sin v & 0 \\ 0 & r_2 \end{pmatrix}$$

<ins>Example (Torus)</ins>:

Thus, for a point $\boldsymbol{x}(u,v)$ on the torus and for $w = (1,0)$, the curvature at $\boldsymbol{x}(u,v)$ in the direction $\boldsymbol{J}w$ is:

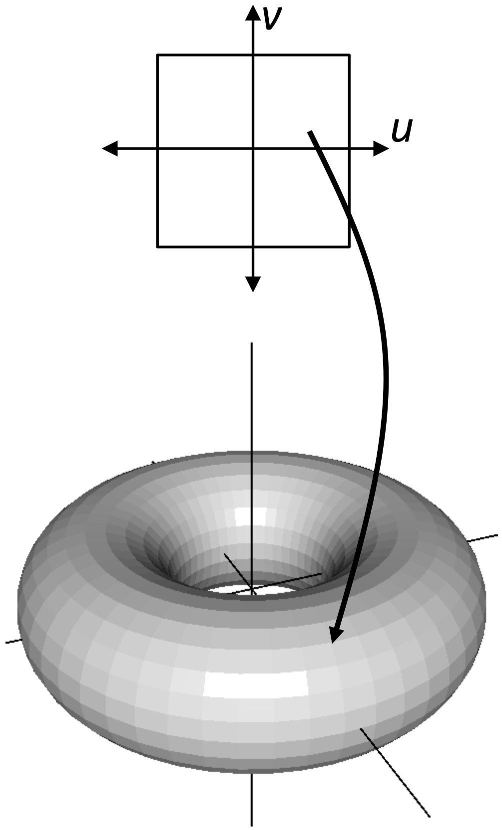


$$\kappa_{\boldsymbol{x}(u,v)}(\boldsymbol{J}w) = -\frac{w^t \boldsymbol{II}(u,v) w}{w^t \boldsymbol{I}(u,v) w} = \frac{\sin v}{r_1 + r_2 \sin v}$$

Regular Surfaces

$$\boldsymbol{X}(u,v) = \left((r_1 + r_2 \sin v)\cos u \quad r_2 \cos v \quad (r_1 + r_2 \sin v)\sin u\right)^t$$

$$\boldsymbol{I}(u,v) = \begin{pmatrix} (r_1 + r_2 \sin v)^2 & 0 \\ 0 & r_2^2 \end{pmatrix} \qquad \boldsymbol{II}(u,v) = \begin{pmatrix} (r_1 + r_2 \sin v)\sin v & 0 \\ 0 & r_2 \end{pmatrix}$$



Example (Torus):

Taking the trace and determinant:

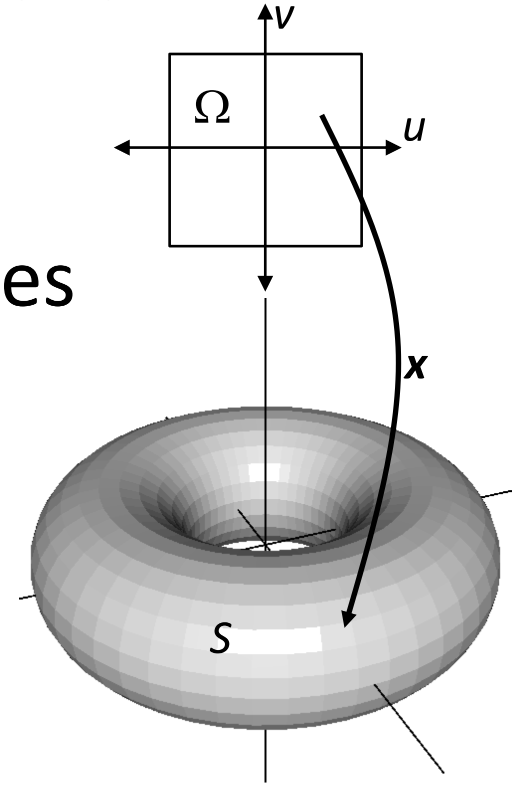
$$H = \mathrm{Tr}\left(\boldsymbol{I}^{-1}\boldsymbol{II}\right) = \frac{\sin v}{(r_1 + r_2 \sin v)} + \frac{1}{r_2}$$

$$K = \mathrm{Det}\left(\boldsymbol{I}^{-1}\boldsymbol{II}\right) = \frac{\sin v}{(r_1 + r_2 \sin v) r_2}$$

Regular Surfaces

<u>Note</u>:

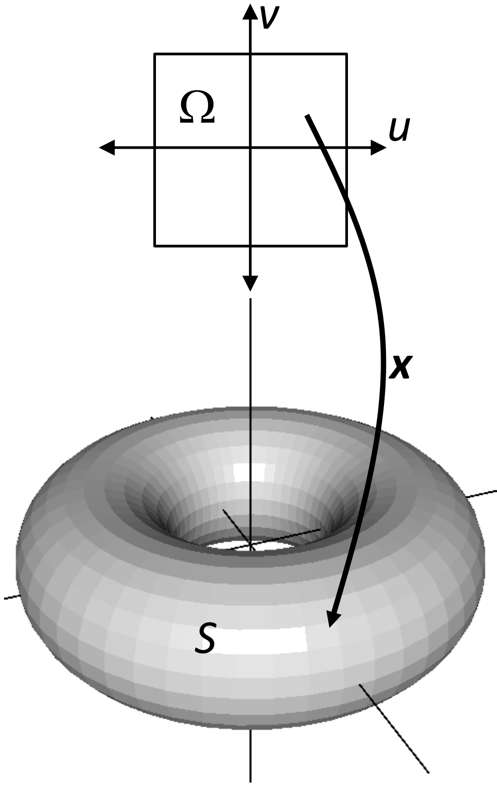
Though when we talk about a parameterization $\boldsymbol{x}:\Omega\rightarrow S\subset\boldsymbol{R}^3$, we compute the first/second fundamental forms using the map $\boldsymbol{x}$, these forms allows us to compute distances, angles, areas, and curvatures over the parameterization domain.



Regular Surfaces

Gradients:

Similarly, when we talk about functions defined on the surface, we will actually work with functions defined over the parameter domain, $f:\Omega\rightarrow\boldsymbol{R}$.

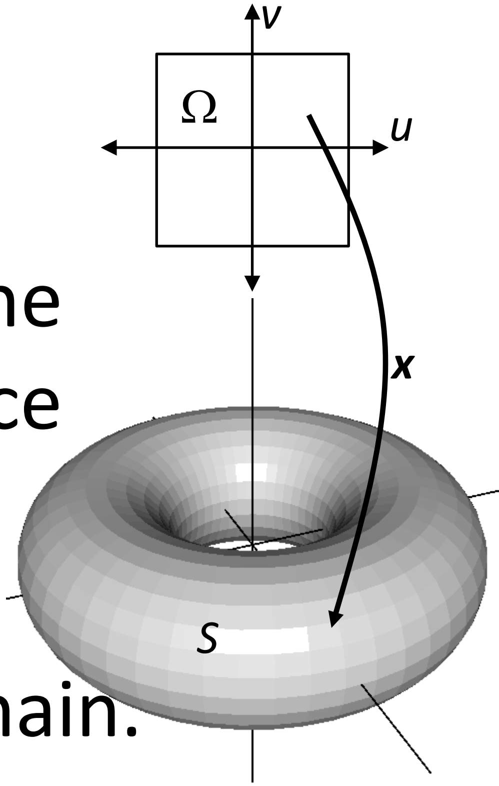


Regular Surfaces

<u>Gradients</u>:

Similarly, when we talk about functions defined on the surface, we will actually work with functions defined over the parameter domain, $f:\Omega\rightarrow\boldsymbol{R}$.

So while we think of the gradient of the gradient as a vector field on the surface pointing in the direction of greatest change, we will actually define it as a vector field over the parameter domain.

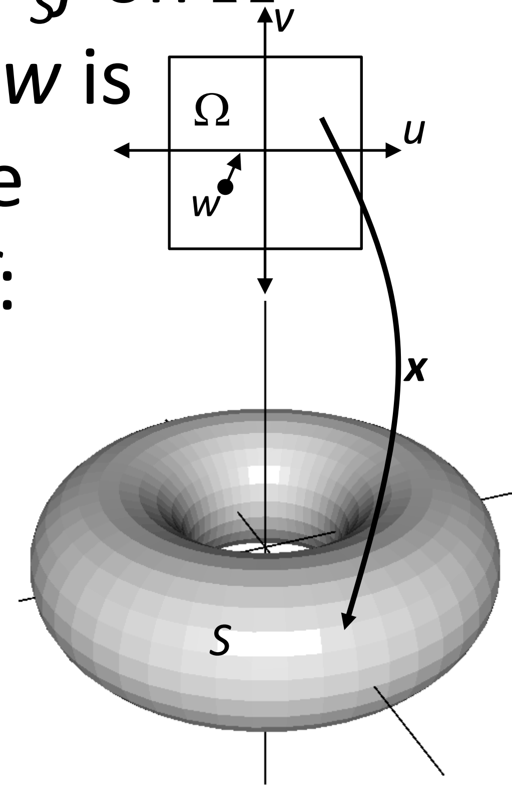


Regular Surfaces

Definition:

Given a smooth function $f:\Omega\rightarrow\mathbf{R}$, the *gradient* of f with respect to S is the vector field $\nabla_S f$ on Ω such that the change of f in direction w is the inner-product (with respect to the first fundamental form) of w with $\nabla_S f$:

$$df(w) = \langle \nabla_S f, w \rangle_I$$



Regular Surfaces

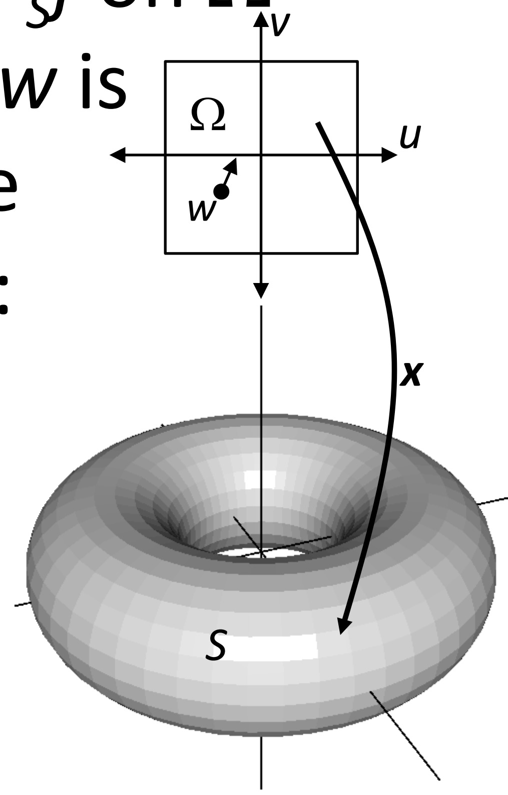
<u>Definition</u>:

Given a smooth function $f:\Omega\to\boldsymbol{R}$, the *gradient* of f with respect to S is the vector field $\nabla_S f$ on Ω such that the change of f in direction w is the inner-product (with respect to the first fundamental form) of w with $\nabla_S f$:

$$df(w)=\left\langle \nabla_S f, w\right\rangle_{\boldsymbol{I}}$$

Note that since $df(w)=\langle\nabla f,w\rangle$, we get:

$$\nabla_S f=\boldsymbol{I}^{-1}\nabla f$$



Regular Surfaces

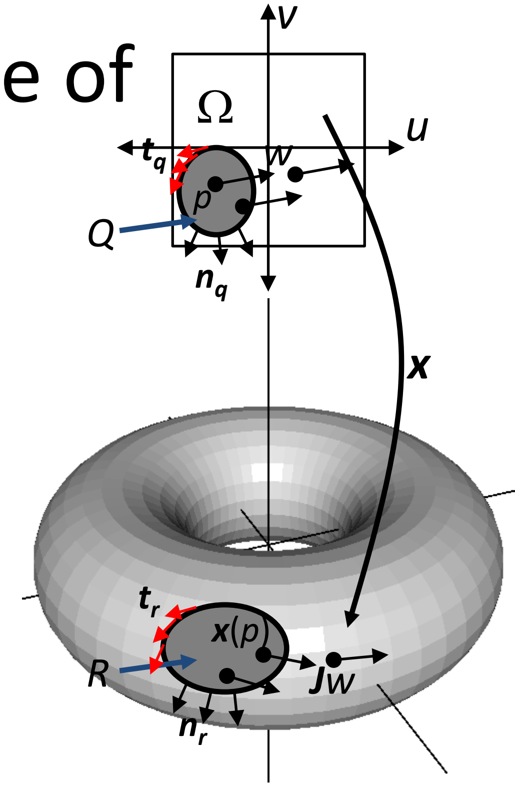
<u>Definition</u>:

Given a smooth vector field $w:\Omega\rightarrow\boldsymbol{R}^2$, the *divergence* of w with respect to S is the function $\text{div}_S w$ on Ω measuring the "magnitude of the vector fields sources and sinks at each point":

$$\text{div}_S w(p) = \lim_{R\rightarrow \boldsymbol{x}(p)} \frac{1}{\text{Area}(R)} \int_{\partial R} \langle n_r, w\rangle$$

$$\approx \lim_{Q\rightarrow p} \frac{1}{\text{Area}(Q)\sqrt{\det \boldsymbol{I}}} \int_{\partial Q} \frac{\langle \boldsymbol{I}^{-1} n_q, w\rangle_{\boldsymbol{I}}}{\left|\boldsymbol{I}^{-1} n_q\right|_{\boldsymbol{I}}} \left|t_q\right|_{\boldsymbol{I}}$$

$$= \frac{1}{\sqrt{\det \boldsymbol{I}(p)}} \text{div}\left(\sqrt{\det \boldsymbol{I}(p)}\, w\right)$$

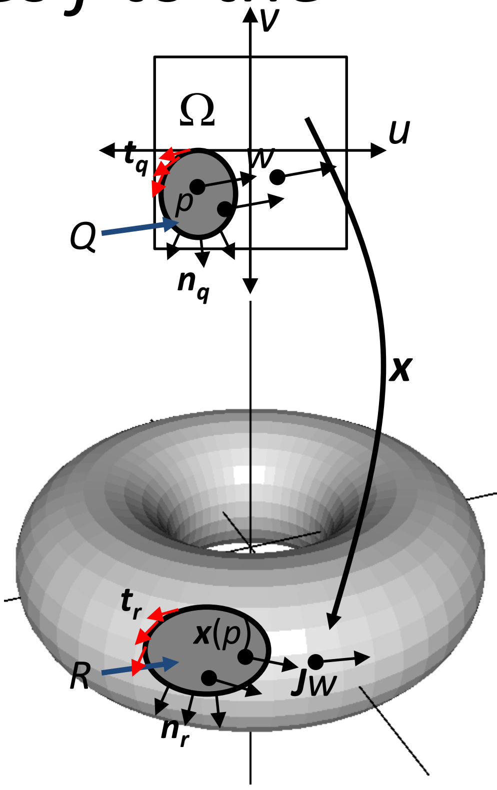


Regular Surfaces

<u>Definition</u>:

Given a smooth function $f:\Omega \rightarrow \boldsymbol{R}^2$, the *Laplace-Beltrami operator* is the map that takes *f* to the divergence of the gradient field of *f*:

$$\begin{aligned}
\Delta_S f &= \mathrm{div}_S(\nabla_S f) \\
&= \frac{1}{\sqrt{\det \boldsymbol{I}(p)}}\mathrm{div}\left(\sqrt{\det \boldsymbol{I}(p)}\nabla_S f\right) \\
&= \frac{1}{\sqrt{\det \boldsymbol{I}(p)}}\mathrm{div}\left(\sqrt{\det \boldsymbol{I}(p)}\boldsymbol{I}^{-1}\nabla f\right)
\end{aligned}$$



Regular Surfaces

Definition:

Given a smooth function $f\colon\Omega\to\boldsymbol{R}^2$, the *Laplace-Beltrami operator* is the map that takes f to the divergence of the gradient field of f:

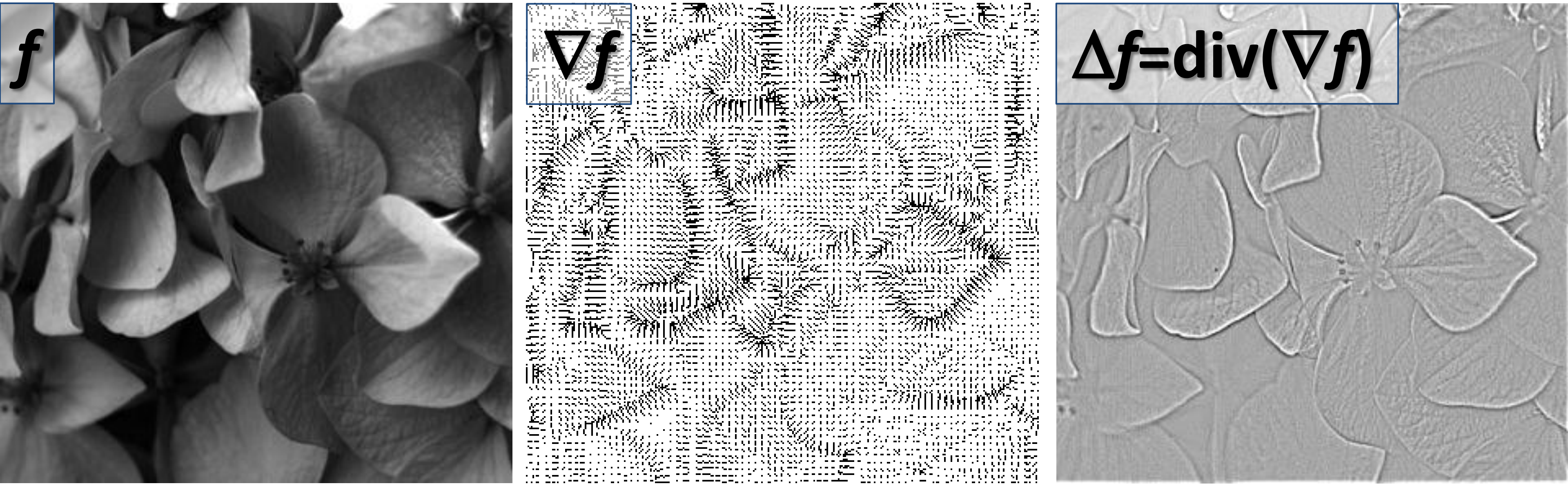
$$
\begin{aligned}
\Delta_S f &= \mathrm{div}_S(\nabla_S f) \\
&= \frac{1}{\sqrt{\det \boldsymbol{I}(p)}}\,\mathrm{div}\left(\sqrt{\det \boldsymbol{I}(p)}\,\nabla_S f\right) \\
&= \frac{1}{\sqrt{\det \boldsymbol{I}(p)}}\,\mathrm{div}\left(\sqrt{\det \boldsymbol{I}(p)}\,\boldsymbol{I}^{-1}\nabla f\right)
\end{aligned}
$$

Since the definition only depends on the first fundamental form, the operator is intrinsic.

Regular Surfaces

<u>Less Formally</u>:

The Laplacian measures how the gradients of *f* converge/diverge near a point *p*.



Regular Surfaces

Less Formally:

The Laplacian measures how the gradients of *f* converge/diverge near a point *p*.

If the gradients converge, the value at *p* is larger than (the average of) its neighbors' values and the Laplacian is positive.

If the gradients diverge, the value at *p* is smaller than (the average of) its neighbors' values and the Laplacian is negative.

Regular Surfaces

Less Formally:

The Laplacian measures how the gradients of f converge/diverge near a point p.

So, the Laplacian is a measure of the difference between the value at a point and the average value of its neighbors.

Regular Surfaces

Property:

Applying the Laplace-Beltrami operator to the (coordinates of) function $f=\boldsymbol{x}:\Omega\to\boldsymbol{R}^3$ gives:

$$\Delta_S f = -2H\boldsymbol{n}$$