# 600.657: Mesh Processing

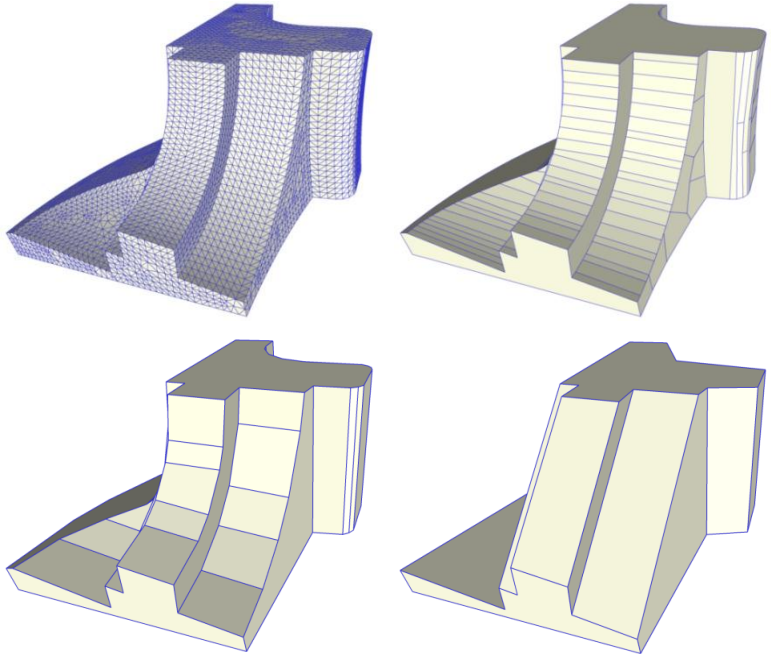
Chapter 7

# Variational Shape Approximation

<u>Goal</u>:

Compute lower-complexity polygonization(s) of a surface capturing the detail(s): $M_0 \rightarrow M_1 \rightarrow \ldots$

[Cohen-Steiner *et al*., 2004]



[Marinov and Kobbelt, 2005]

# Variational Shape Approximation

<u>Goal</u>:

Compute lower-complexity polygonization(s) of a surface capturing the detail(s): $M_0 \rightarrow M_1 \rightarrow \ldots$

- Capture detail $\rightarrow$ polygons align with anisotropy.
- Simple base models $\rightarrow$ topology is removed.

# Variational Shape Approximation

<u>Approach</u>:

Partition $M$ into $k$ regions $R=\{R_1,\ldots,R_k\}$ which are a best fit to $k$ planar proxies $P=\{P_1,\ldots,P_k\}$:

$$E(R,P)=\sum_{i=1}^{k}\int_{R_i} d^2(q,P_i)\,dq$$

where the planar proxy $P_i$ can be represented by a point $x_i$ on the plane and a unit-normal $n_i$.

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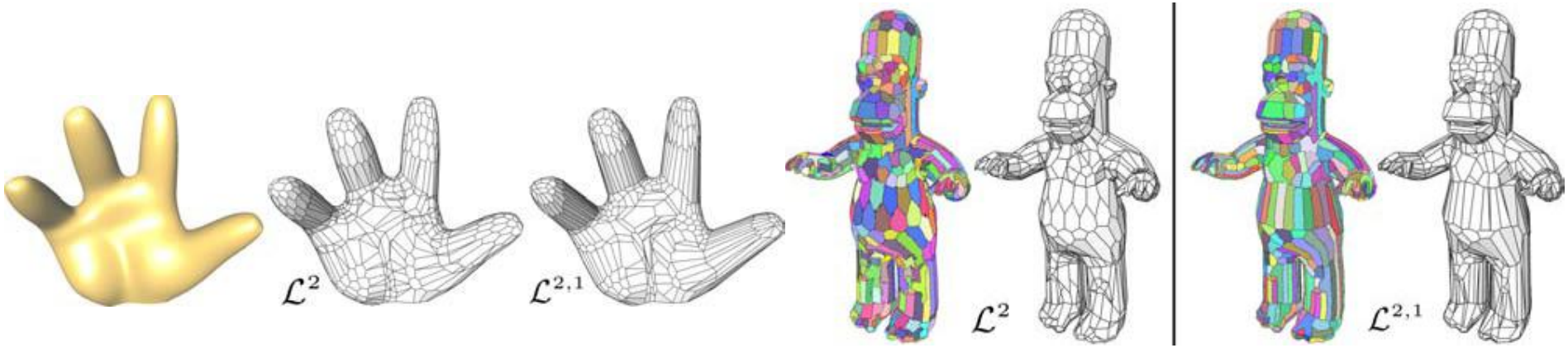
Need to simultaneously solve for the partition regions $R$ and the proxies $P$.

# Variational Shape Approximation

<u>Approach</u>:

The distance function *d* can be:

- The $L^2$ distance: $d^2(q, P_i) = \min_{p \in P_i} \|q - p\|^2 = \langle q - x_i, n_i \rangle^2$
- The $L^{2,1}$ distance: $d^2(q, P_i) = \|n(q) - n_i\|^2$




[Cohen-Steiner *et al*., 2004]

# Variational Shape Approximation

<u>Implementation</u>:

- [Cohen-Steiner *et al*., 2004]: Use a Lloyd's algorithm type approach – iterate between growing regions and fitting proxies.
- [Marinov and Kobbelt, 2005]: Use a decimation type approach, merging triangles into regions, and using the regions to define the proxies.

# [Cohen-Steiner *et al*., 2004]

<u>Initialization</u>:

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- Choose $k$ triangles at random, and set $P_i$ to be the plane going through triangle $t_i$.

<u>Iteratively</u>:

- Grow the regions $R_i$.
- Fit the proxies $P_i$.
- Seed new triangles by choosing the triangle $t_i \in R_i$ that is closest to $P_i$.

# [Cohen-Steiner *et al*., 2004]

<u>Growing $R_i$</u>:

- For each assigned triangle $t$:
  - For each unassigned neighbor $t'$:
    - Compute the fit of $t'$ to the proxy of $t$.
- Assign the best fitting $t'$.

# [Cohen-Steiner *et al*., 2004]

<u>Fitting $P_i$</u>:

Given a region $R_i$, we would like to solve for the position $x_i$ and normal $n_i$ minimizing:

$$E_{R_i}(x_i, n_i) = \int_{R_i} d^2(q, P_i)\,dq$$

# [Cohen-Steiner *et al*., 2004]

<u>Fitting $P_i$</u>:

If $d^2$ is the $L^2$ distance:

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$$\frac{\partial E_{R_i}}{\partial x_i} = -2 \int_{R_i} \langle q - x_i, n_i \rangle n_i \, dq = -2 \left\langle \int_{R_i} q \, dq - |R_i| x_i, n_i \right\rangle n_i$$

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So the distance is minimized when $x_i$ is the average of the points in $R_i$:

$$\frac{\partial E_{R_i}}{\partial x_i} = 0 \Leftarrow \int_{R_i} q \, dq - |R_i| x_i = 0 \Leftrightarrow x_i = \frac{1}{|R_i|} \int_{R_i} q \, dq$$

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Thus, the unit-vector $n_i$ minimizing the distance is the smallest eigenvector of:

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Note that $C_{R_i}$ is the covariance matrix of $R_i$.

# [Cohen-Steiner *et al*., 2004]

<u>Fitting $P_i$</u>:

If $d^2$ is the $L^{2,1}$ distance:

$$E_{R_i}(x_i, n_i) = \int_{R_i} \left\| n(q) - n_i \right\|^2 dq$$

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Since $R_i$ is made up of triangles, the normal of a point is the normal of its triangle.

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Since $R_i$ is made up of triangles, the normal of a point is the normal of its triangle.

So, the normal $n_i$ minimizing the distance is the average of the triangle's normals:

$$n_i = \frac{\sum_{t \in R_i} N(t)|t|}{\left\| \sum_{t \in R_i} N(t)|t| \right\|}$$

# [Cohen-Steiner *et al*., 2004]

<u>In Sum</u>:

Iterating this process, we get a set of connected regions $R_i$ that partition the original surface.

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- Identify corners
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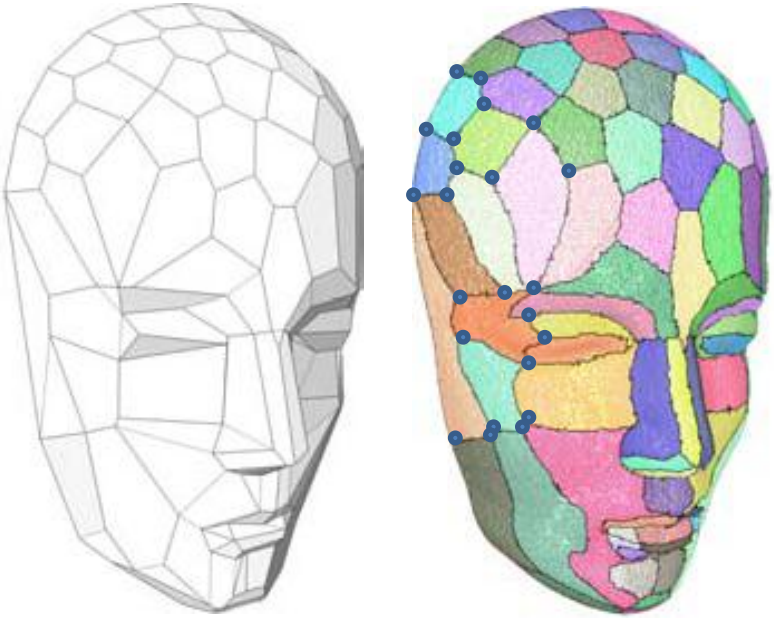

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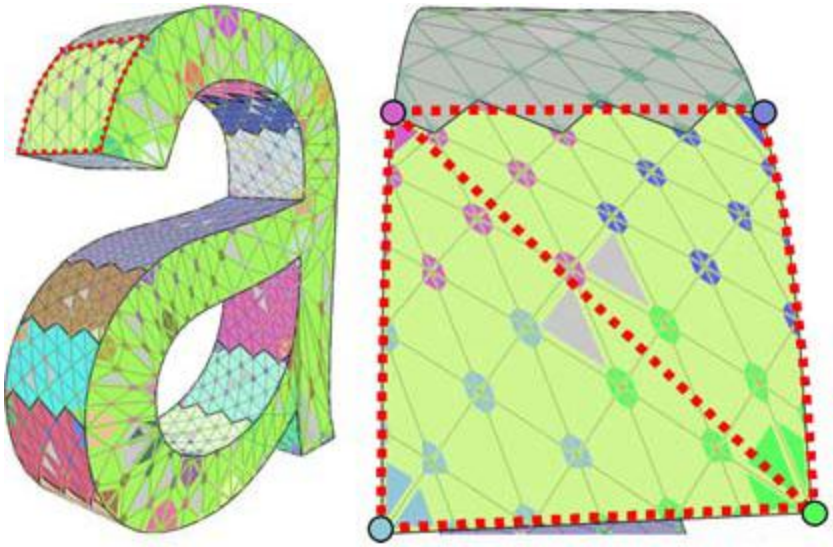
- Connect corners
  - May need to divide edges
- Triangulate polygons
  - Using "discrete CDT"



[Cohen-Steiner *et al*., 2004]

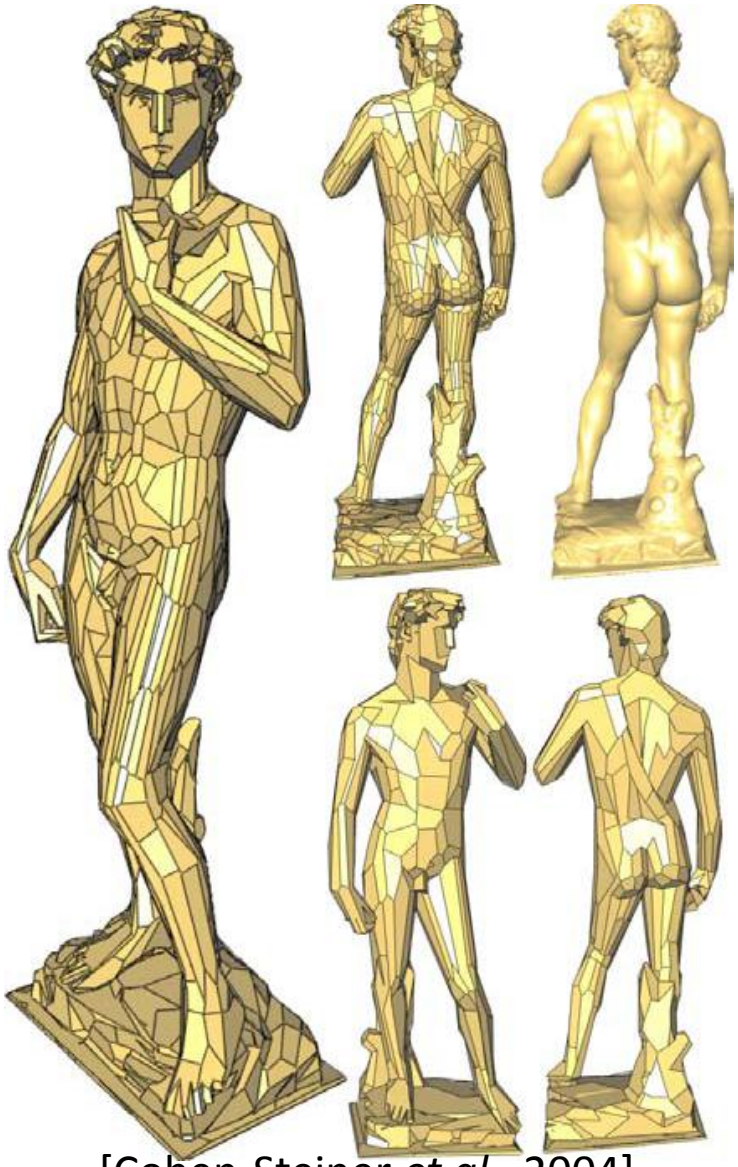
# [Cohen-Steiner *et al*., 2004]

<u>Advantages</u>:

- Fast
- Can be made hierarchical*

<u>Disadvantage</u>:

- Convergence?
- Not guaranteed to be manifold



[Cohen-Steiner *et al*., 2004]

*Can grow regions made up of polygons, not just triangles

# [Marinov and Kobbelt, 2005]

<u>Approach</u>:

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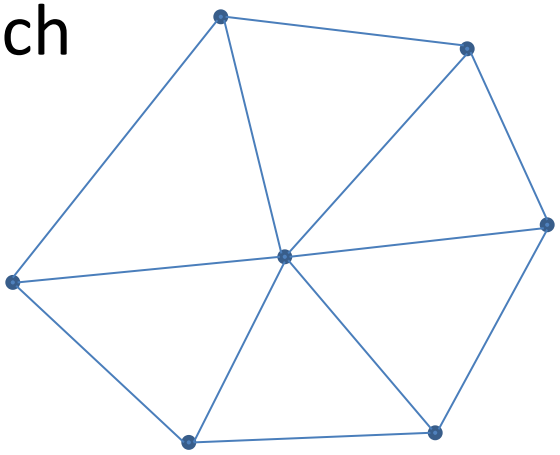
- Each triangle starts in its own patch



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Instead of using a Lloyd's algorithm approach to grow regions, use face-merge to collapse them.

Two regions can be collapsed into each other iff. their projection onto their proxy is injective.

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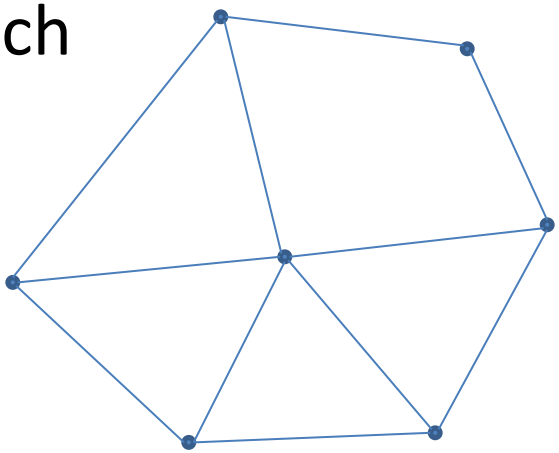

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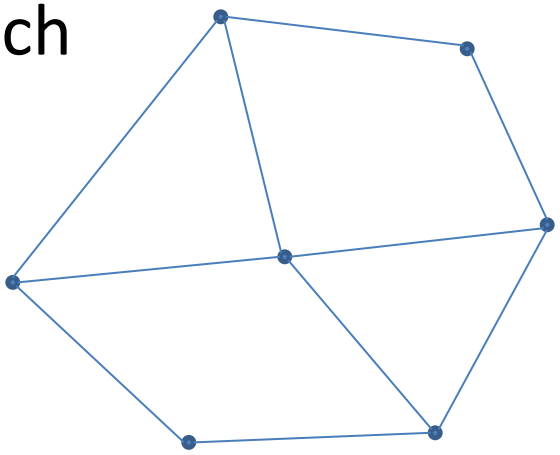

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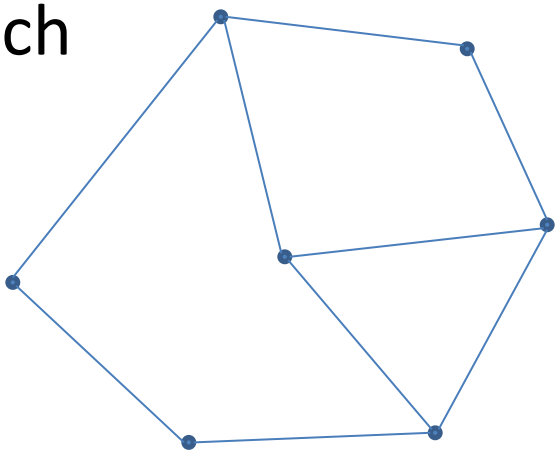

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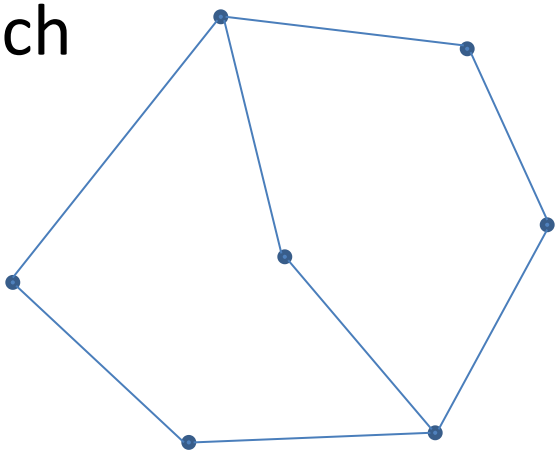

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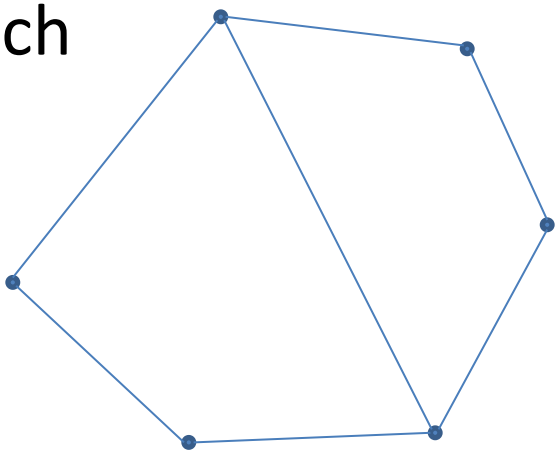
- Edges are successively removed
- Valence-2 vertices are removed



# [Marinov and Kobbelt, 2005]

<u>Invariants</u>:

At each step of the iteration:

- The region $R_i$ tracks its:
  - Boundary: $b(R_i)$
  - Area: $a(R_i)$
  - Centroid: $x(R_i)$
  - Normal: $n(R_i)$
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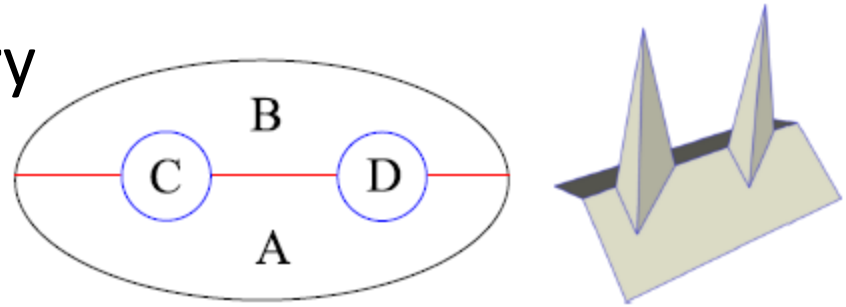
<u>Note</u>: The area of $R_i$ is not tracked!

# [Marinov and Kobbelt, 2005]

<u>Injectivity</u>:

Since only the boundary of a region $R_i$ is tracked, injectivity is determined by testing that the projection of the boundary is injective:

- That the boundary curves do not self-intersect
- That the exterior boundary contains interior boundaries.
- That the interior boundary curves do not nest.

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<u>Initialization</u>:

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<u>Merging</u>:

When merging adjacent regions $R \leftarrow (R_i \cup R_j)$:

- $b(R_i \cup R_j) = b(R_i) \cup a(R_j) - b(R_i) \cap a(R_j)$

  $a(R_i \cup R_j) = a(R_i) + a(R_j)$
- $x(R_i \cup R_j) = \dfrac{x(R_i) \cdot a(R_i) + x(R_j) \cdot a(R_j)}{a(R_i) + a(R_j)}$
- $n(R_i \cup R_j) = \dfrac{n(R_i) \cdot a(R_i) + n(R_j) \cdot a(R_j)}{\left| n(R_i) \cdot a(R_i) + n(R_j) \cdot a(R_j) \right|}$

# [Marinov and Kobbelt, 2005]

<u>Initialization</u>:

- Set each triangle to be its own region.

<u>Iteratively</u>:

- Evaluate the cost of every feasible merge
- Perform the lowest-cost merge
- Repeat

# [Marinov and Kobbelt, 2005]

<u>Cost of a Merge</u>:

In order to evaluate the merge, we need to associate a surface with each region $R_i$.

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<u>Cost of a Merge</u>:

In order to evaluate the merge, we need to associate a surface with each region $R_i$.

We do this by:

- Projecting the boundary of $R_i$ onto the proxy
- Computing the CDT of the boundary
- Raising the triangulation to 3D.

# [Marinov and Kobbelt, 2005]

<u>Cost of a Merge</u>:

If $d^2$ is the $L^2$ distance, merging $R_i$ and $R_j$ costs:

$$E(R_i, R_j) = \sum_{t \in R_i \cup R_j} \int_{p \in t} \left\langle p - x(R_i \cup R_j), n(R_i \cup R_j) \right\rangle^2 dp$$

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If $d^2$ is the $L^{2,1}$ distance, merging $R_i$ and $R_j$ costs:

$$E(R_i, R_j) = a(R_i) \left\| n(R_i) - n(R_i \cup R_j) \right\|^2 + a(R_j) \left\| n(R_j) - n(R_i \cup R_j) \right\|^2$$

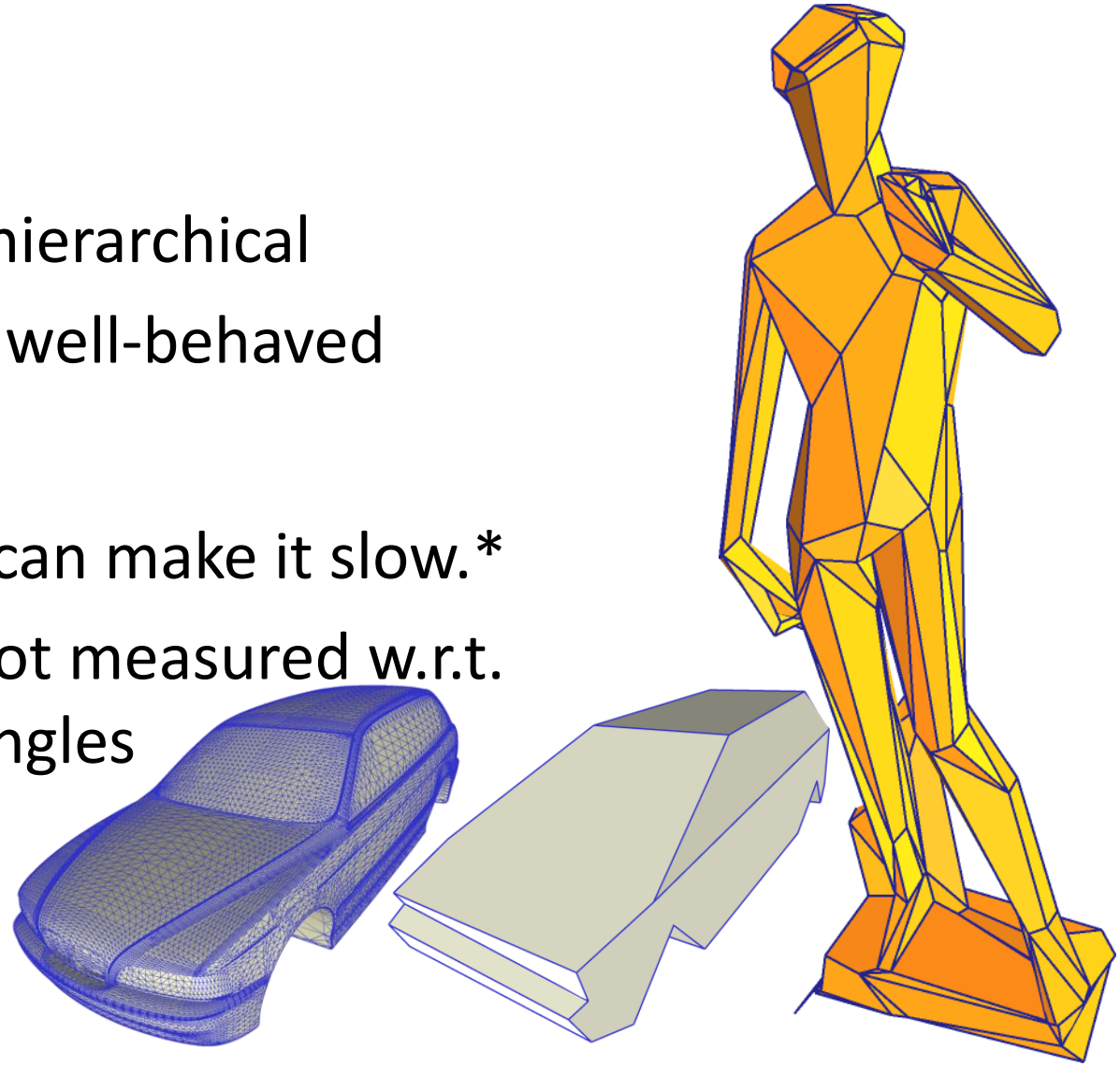
# [Marinov and Kobbelt, 2005]

<u>Advantages</u>:

- Partition is hierarchical
- Patches are well-behaved

<u>Disadvantage</u>:

- Use of CDT can make it slow.*
- Errors are not measured w.r.t. original triangles



*May be able to get by without using a <u>CDT</u>

[Marinov and Kobbelt, 2004]