Differential Geometry:

Conformal Maps

Linear Transformations

Definition:

We say that a linear transformation M:$\boldsymbol{R}^n \rightarrow \boldsymbol{R}^n$ *preserves angles* if M(v)$\neq$0 for all $v\neq$0 and:

$$\frac{\langle Mv, Mw \rangle}{|Mv||Mw|} = \frac{\langle v, w \rangle}{|v||w|}$$

for all v and w in $\boldsymbol{R}^n$.

Linear Transformations

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<u>Note</u>:

If we denote by e_i, the vector that has a one in the i-th place and zero everywhere else, then:

$$e_j^T M e_i = M_{ij}$$

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$$\Downarrow$$

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⇓

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So $M^T M$ must be a diagonal matrix.

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Similarly, since $\langle e_i - e_j, e_i + e_j \rangle = 0$, this implies that:

$$\langle M(e_i - e_j), M(e_i + e_j) \rangle = 0$$

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$$(M^T M)_{ii} - (M^T M)_{jj} = 0$$

Linear Transformations

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Linear Transformations

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Thus, if *M* preserves angles, then $M^T M$ must be of the form:

$$M^T M = \begin{pmatrix} \lambda & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda \end{pmatrix}$$

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If we set $N = M/\lambda^{1/2}$, then:

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So N must be a rotation/reflection.

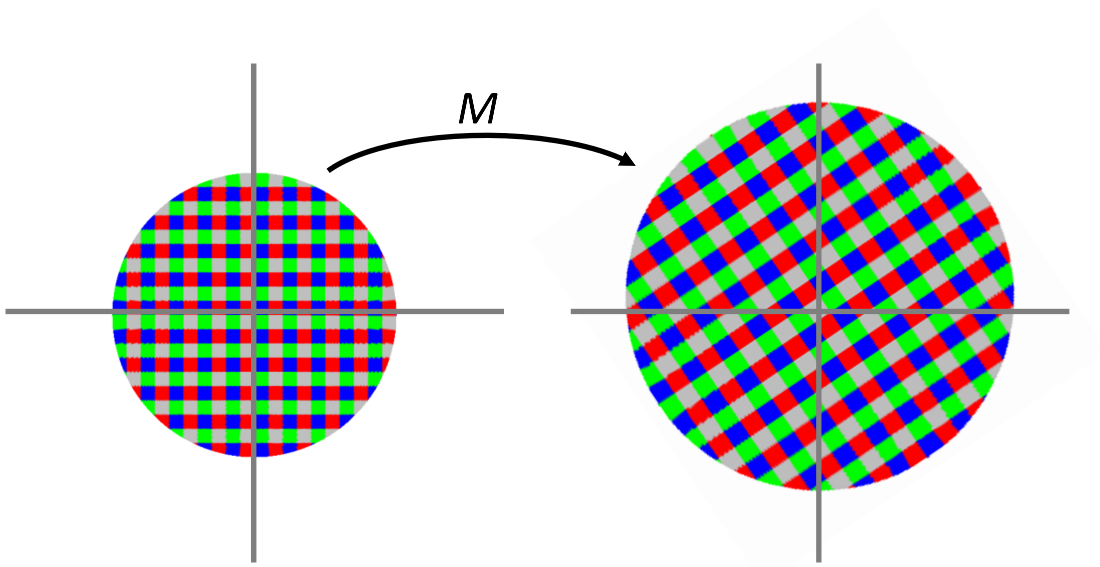
And M must be a rotation/reflection composed with a scaling transformation.

Linear Transformations

$$\frac{\langle Mv, Mw\rangle}{|Mv||Mw|} = \frac{\langle v, w\rangle}{|v||w|}$$

Note:

In particular, *M* preserves angles, if and only if it maps circles to circles.



Complex Numbers

A *complex number z* is any number that can be written as:

$$z = x + iy$$

where *x* and *y* are real numbers and *i* is the square root of -1: $i^2 = -1$

Complex Numbers

Given two complex numbers, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$:

- The sum of the numbers is:

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

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- The product of the numbers is:

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1 x_2 + iy_1 iy_2 + x_1 iy_2 + iy_1 x_2 \\ &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2) \end{aligned}$$

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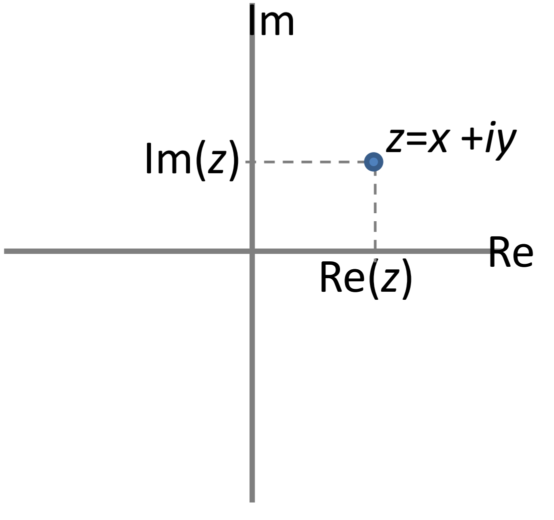
- The reciprocal of the number is:

$$\frac{1}{z} = \frac{1}{z}\frac{\bar{z}}{\bar{z}} = \frac{x}{\|z\|^2} - i\frac{y}{\|z\|^2}$$

Complex Numbers

Often, we think of the complex numbers as living in the (real) 2D plane.



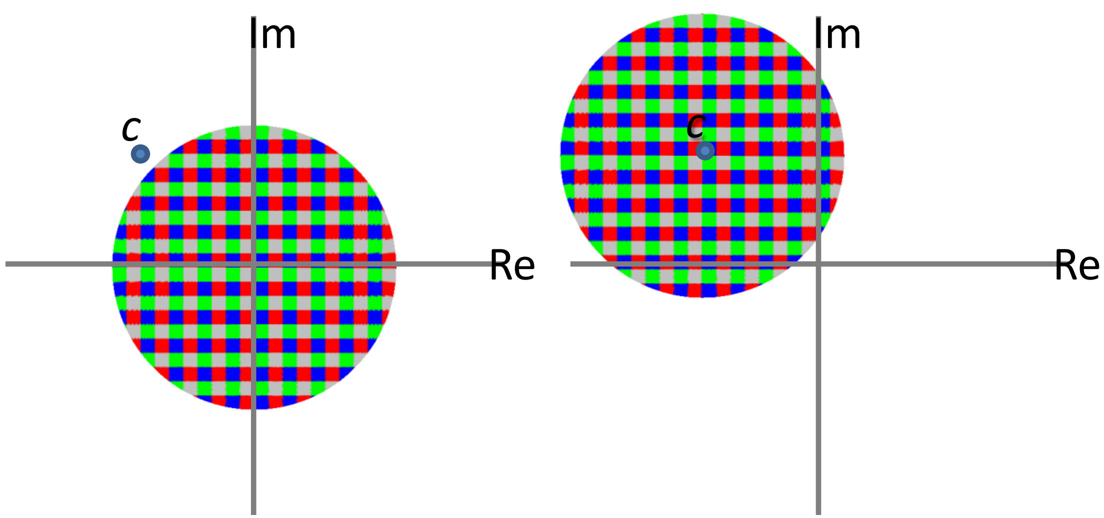
Complex Numbers

Often, we think of the complex numbers as living in the (real) 2D plane.

Then if *c* is a complex number, the function:

$$f(z) = z + c$$

is a translation in the complex plane.



Complex Exponentials

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And any complex number, $z=x+iy$, can be expressed in terms of its radius and angle:

$$z = re^{i\theta}$$

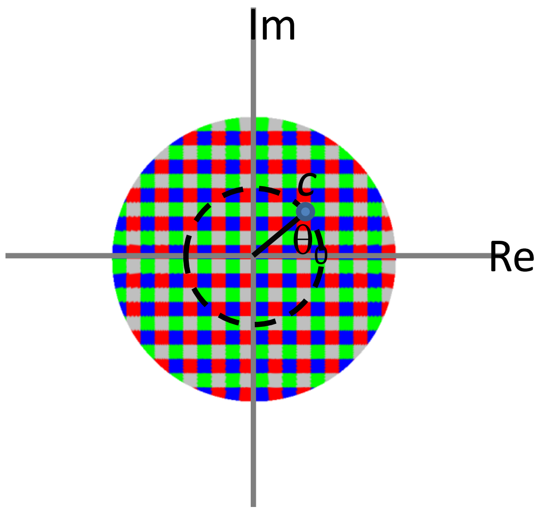
where $r=|z|$, and θ=ArcTan2(y,x).

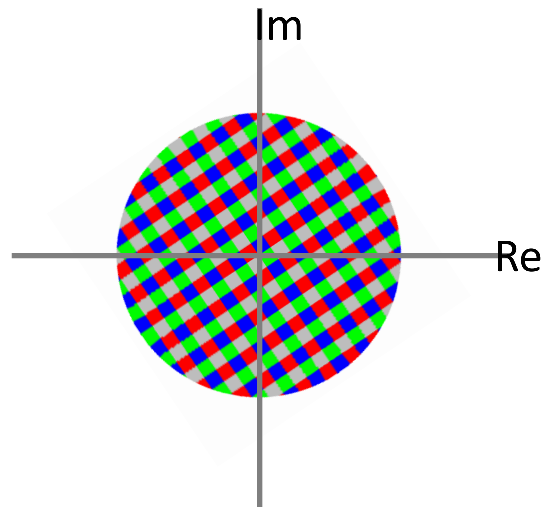
Complex Numbers

If $c=e^{i\theta_0}$ is a complex exponential, then the function:

$$f(z) = cz \qquad re^{i\theta} \mapsto re^{i(\theta+\theta_0)}$$

is a rotation by the angle θ_0.



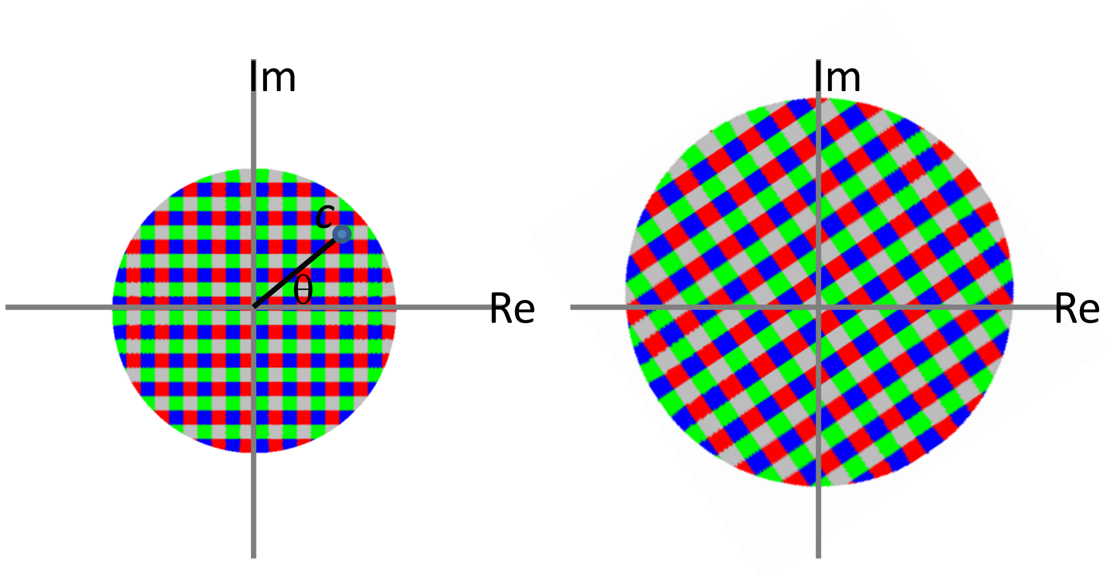


Complex Numbers

More generally, if $c=r_0e^{i\theta_0}$ is any complex number, then the function:

$$f(z) = cz \qquad re^{i\theta} \mapsto (r \cdot r_0)e^{i(\theta+\theta_0)}$$

is a rotation by the angle θ_0 followed/preceded by a scaling by r_0.

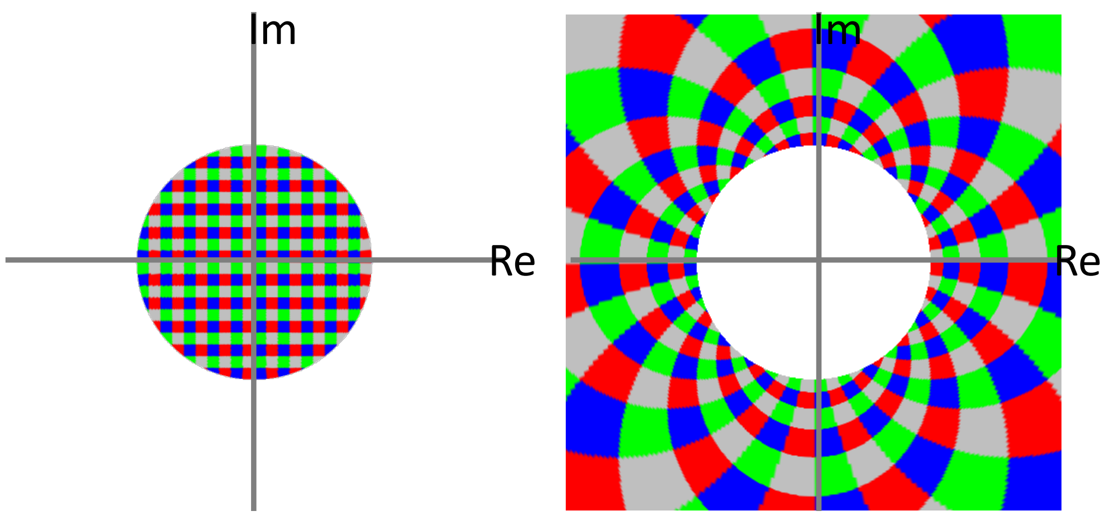


Complex Numbers

Finally, if we consider the reciprocal function, then the function:

$$f(z) = \frac{1}{z} \qquad re^{i\theta} \mapsto \frac{1}{r} e^{-i\theta}$$

is an (orientation-preserving) inversion.



Conformal Maps

Definition:

Given a domain $\Omega \subset \boldsymbol{R}^2$, the map $F{:}\Omega \rightarrow \boldsymbol{R}^2$ is *conformal* if it preserves oriented angles.

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That is, if $F(x,y)=(f_1(x,y),f_2(x,y))$ then the derivative of F preserves oriented angles:

$$\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \lambda R$$

where R is a rotation.

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where R is a rotation.

Thus, a map is conformal if it sends infinitesimally small circles to circles.

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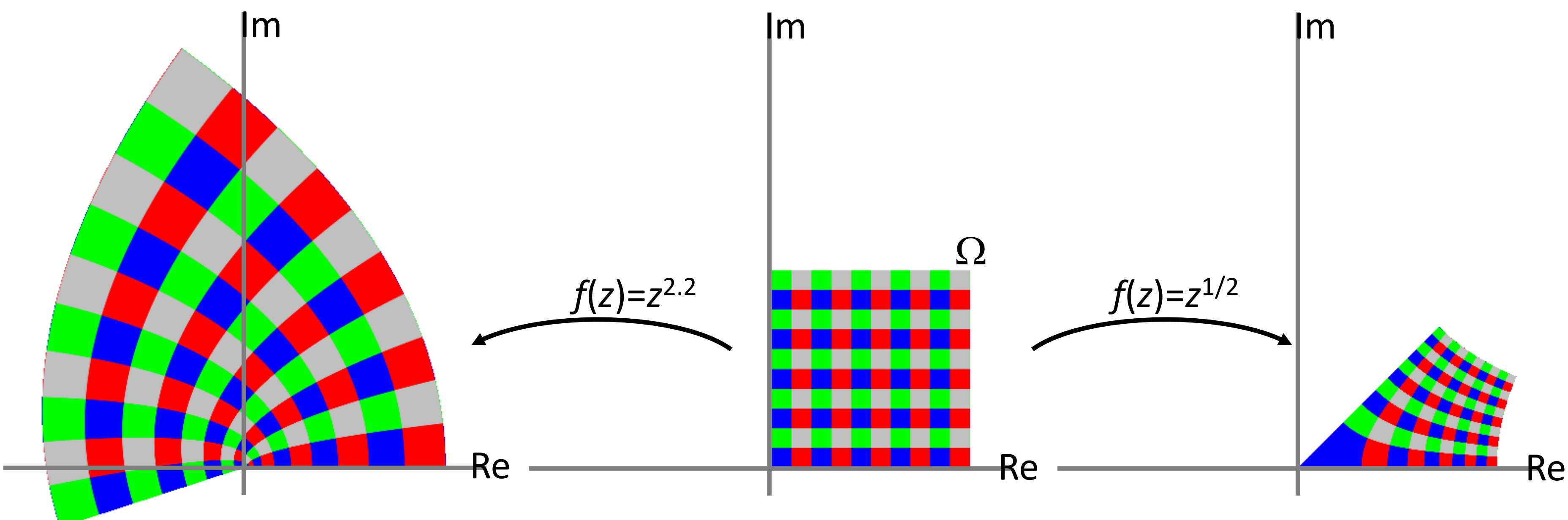

Conformal Maps

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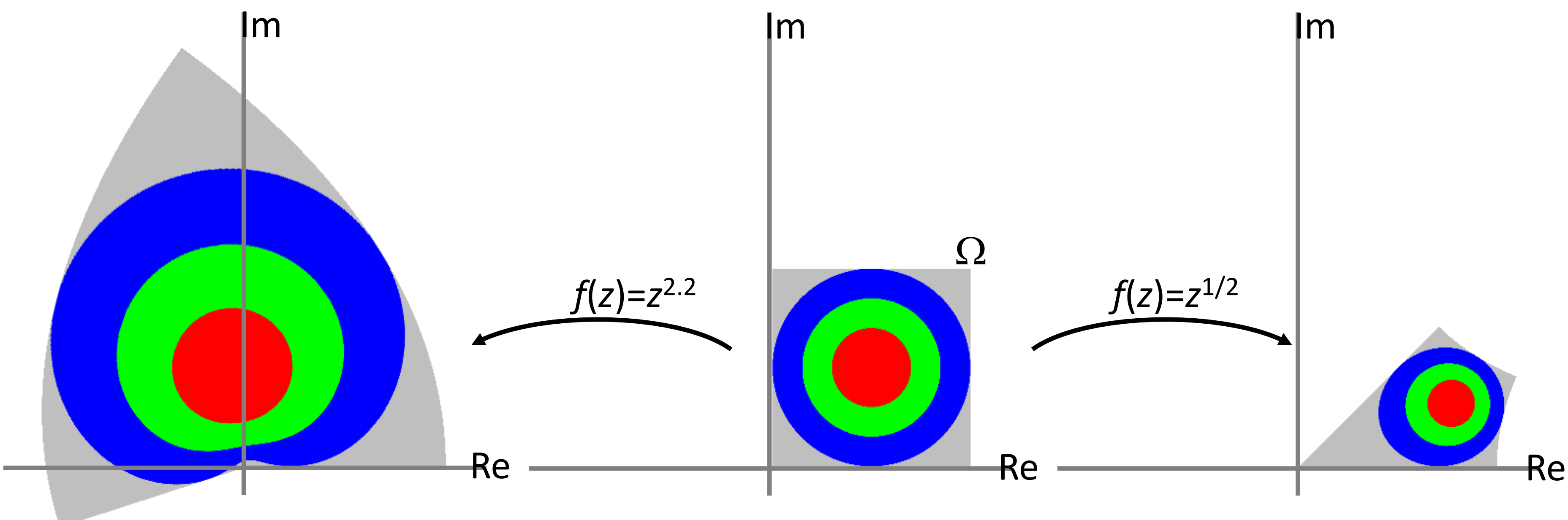
Given a domain $\Omega \subset \boldsymbol{R}^2$, the map $F:\Omega \rightarrow \boldsymbol{R}^2$ is *conformal* if it preserves oriented angles.

It <u>does not</u> map circles to circles.



Conformal Maps

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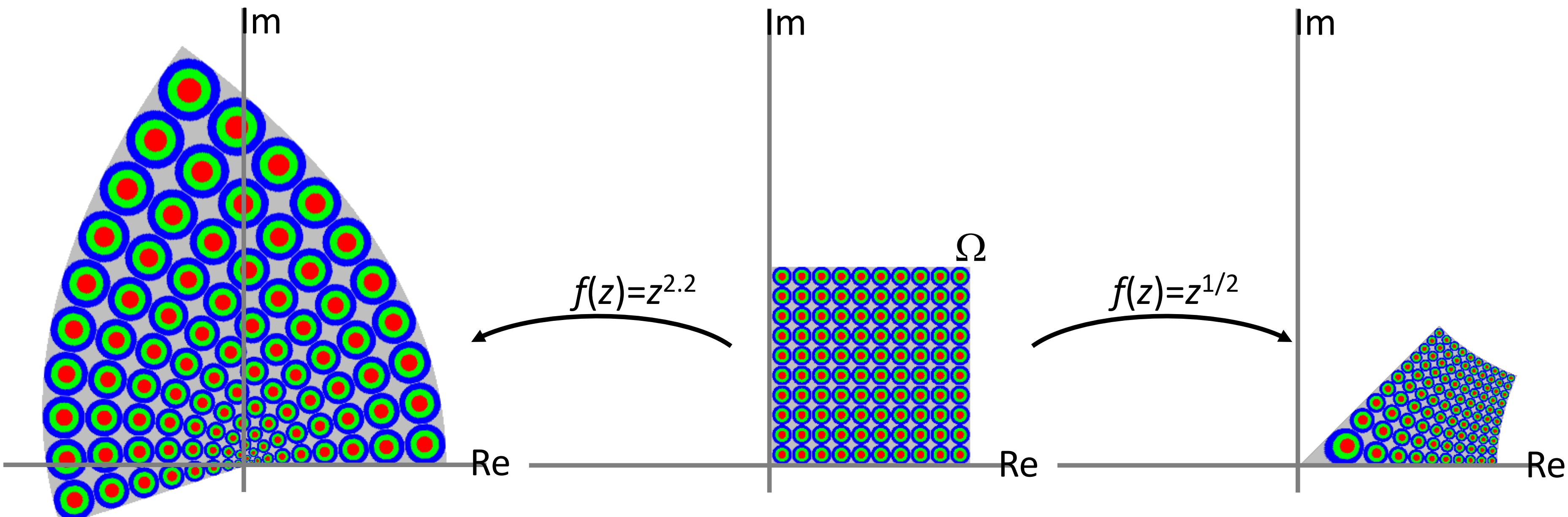
Given a domain $\Omega \subset \boldsymbol{R}^2$, the map $F: \Omega \rightarrow \boldsymbol{R}^2$ is *conformal* if it preserves oriented angles.

But it does map "tiny" circles to circles.



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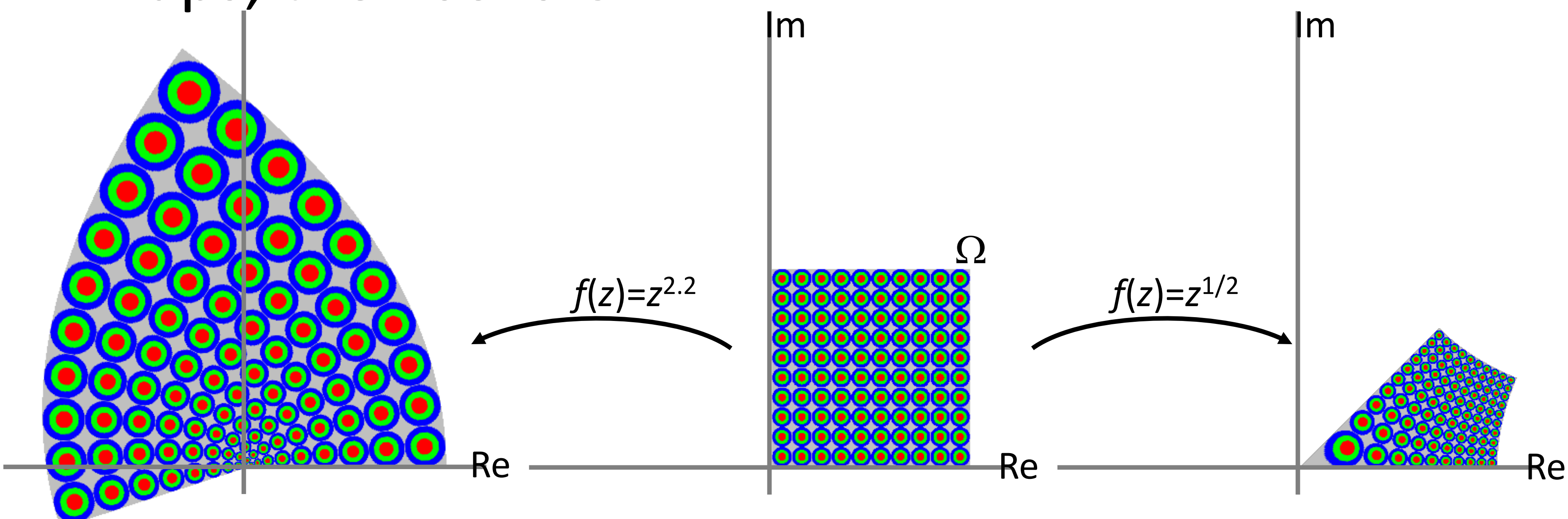
Note, if $F{:}\Omega \rightarrow \boldsymbol{R}^2$ and $G{:}F(\Omega) \rightarrow \boldsymbol{R}^2$ are conformal maps, then so is $G \circ F$.

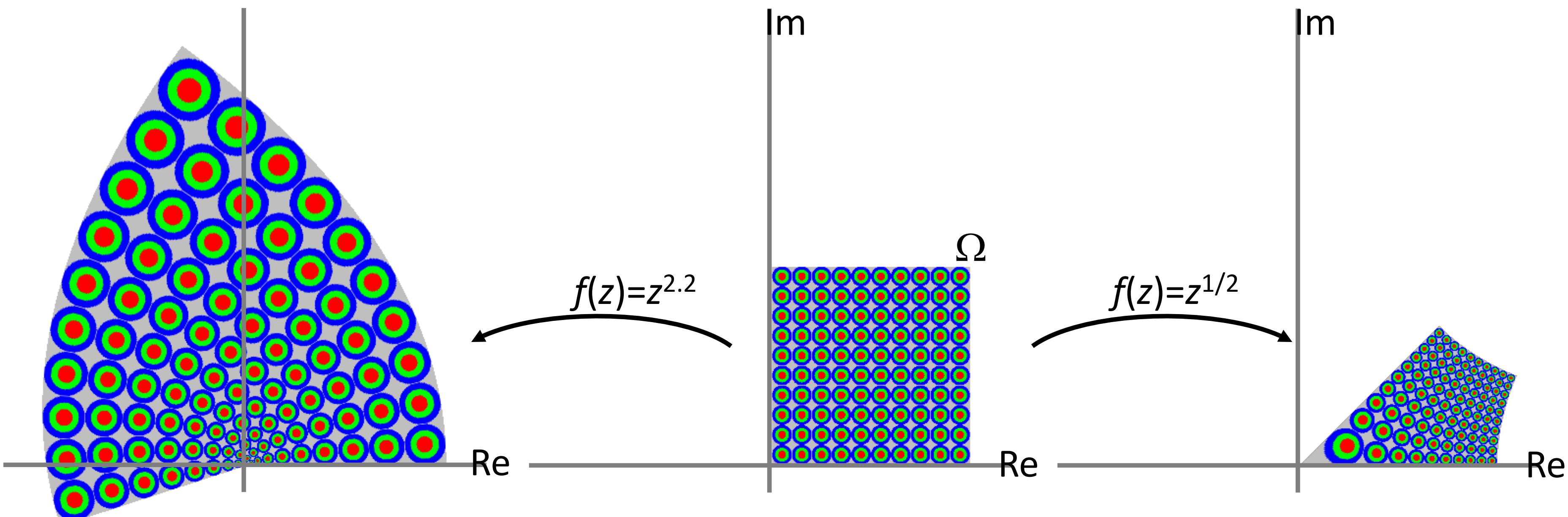


Conformal Maps

Definition:

Given a domain $\Omega \subset \boldsymbol{R}^2$, the map $F: \Omega \to \boldsymbol{R}^2$ is *conformal* if it preserves oriented angles.

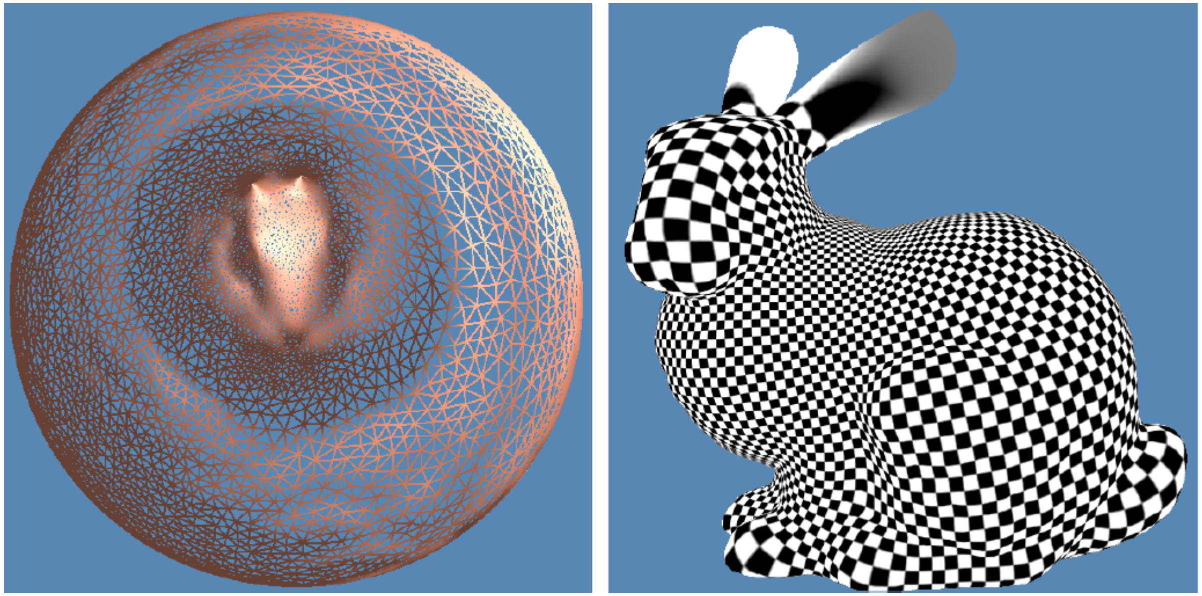
And, if $F: \Omega \to \boldsymbol{R}^2$ is conformal, then so is F^{-1}.



Conformal Maps

<u>Definition</u>:

In a similar manner, we say that a map between two surfaces is conformal if it preserves angles.



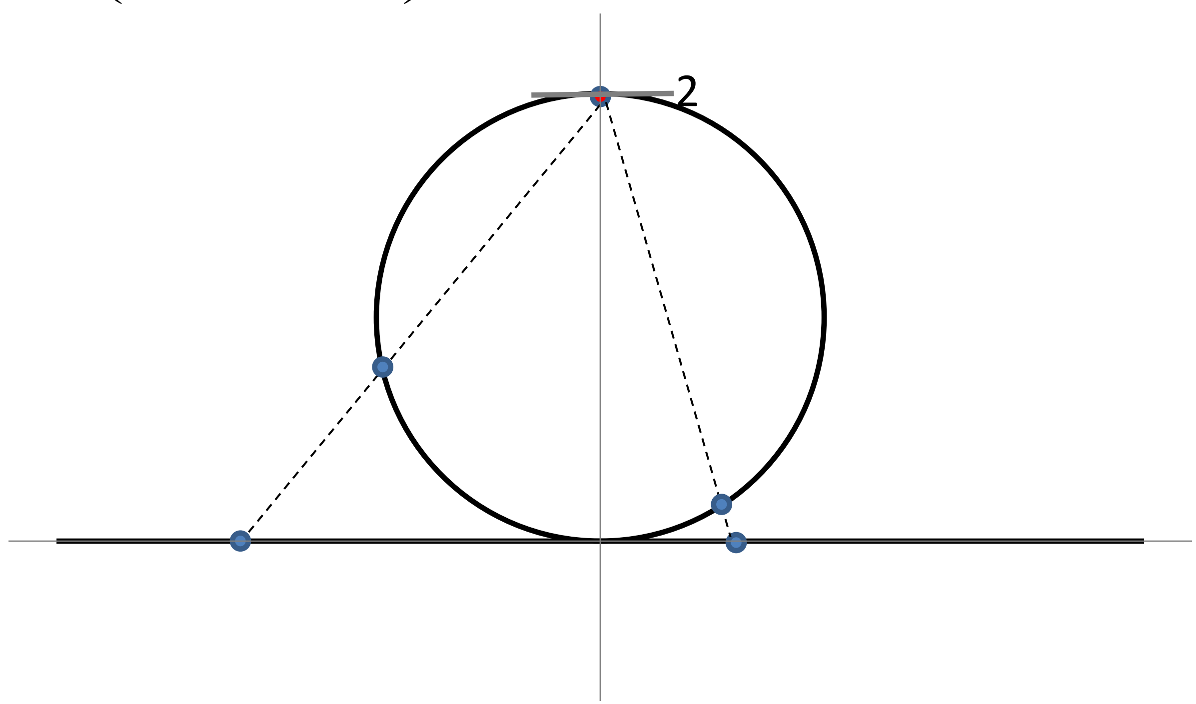
[*Global Conformal Surface Parameterization*, Gu and Yau]

The Complex Plane and the Sphere

Stereographic Projection:

A bijective map from the unit sphere onto $\boldsymbol{R}^2+\infty$:

$$\pi(x,y,z)=\left(\frac{x}{y-2},\frac{z}{y-2}\right)$$



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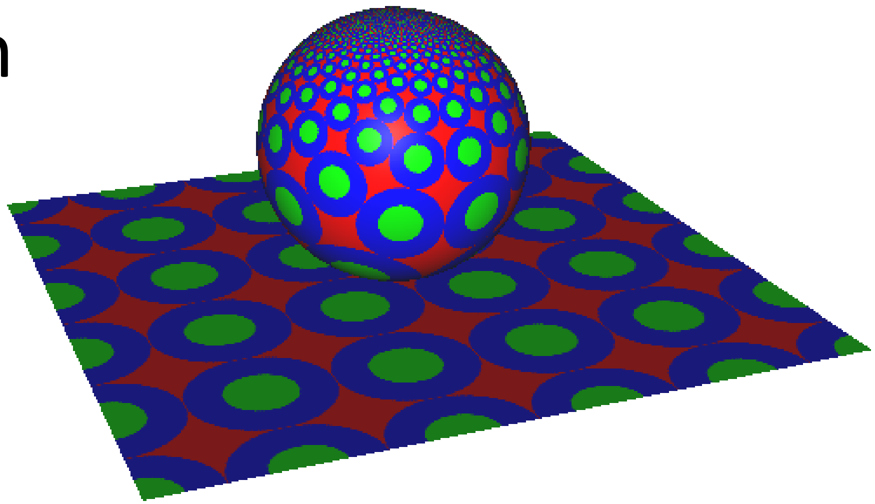
Circles/lines on the plane get mapped to circles on the sphere, so the map is conformal.



Möbius Transformations

The group of invertible conformal maps from the plane into itself is called the *Möbius group*.

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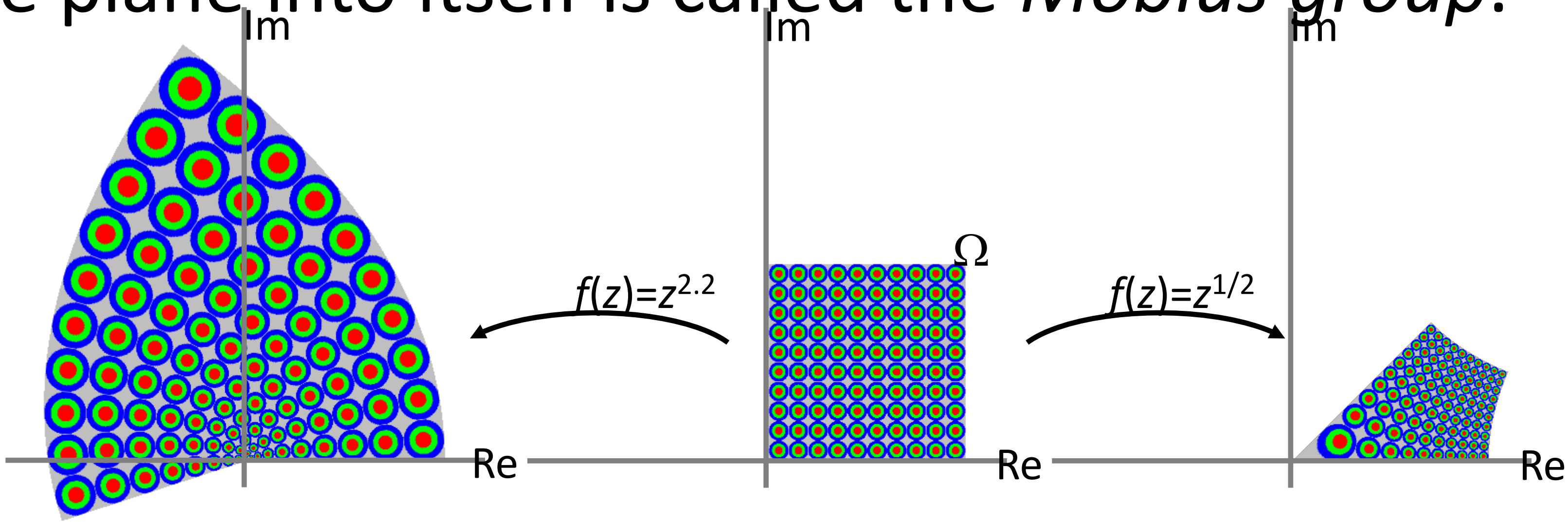

Q: Are these maps part of the Möbius group?

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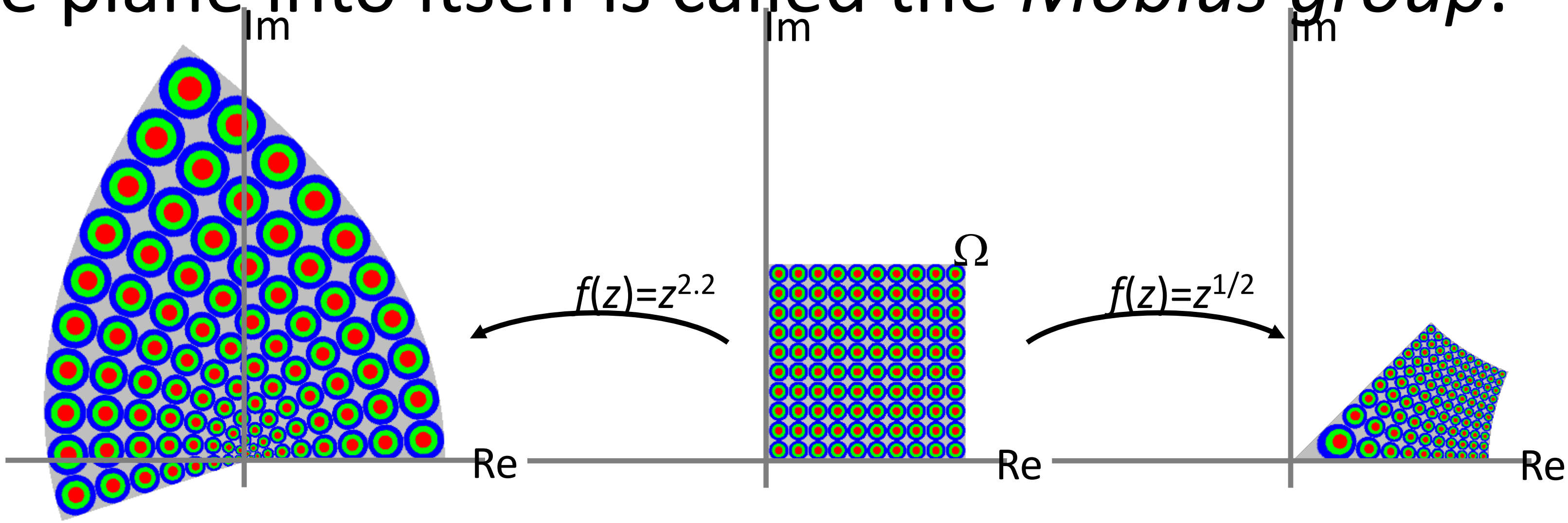

Q: Are these maps part of the Möbius group?

A: No, they aren't invertible on the whole plane.

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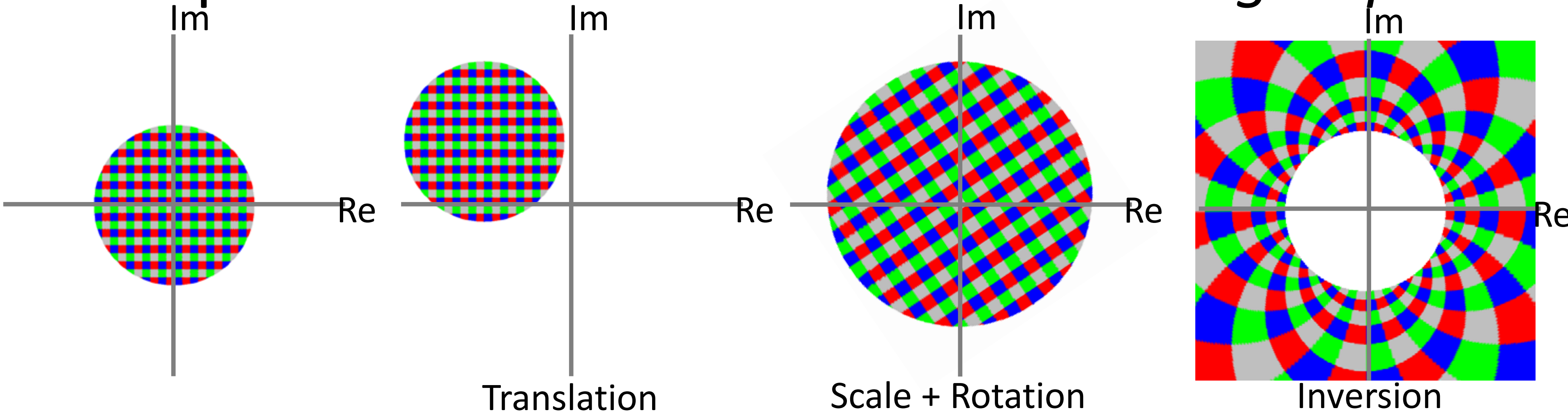

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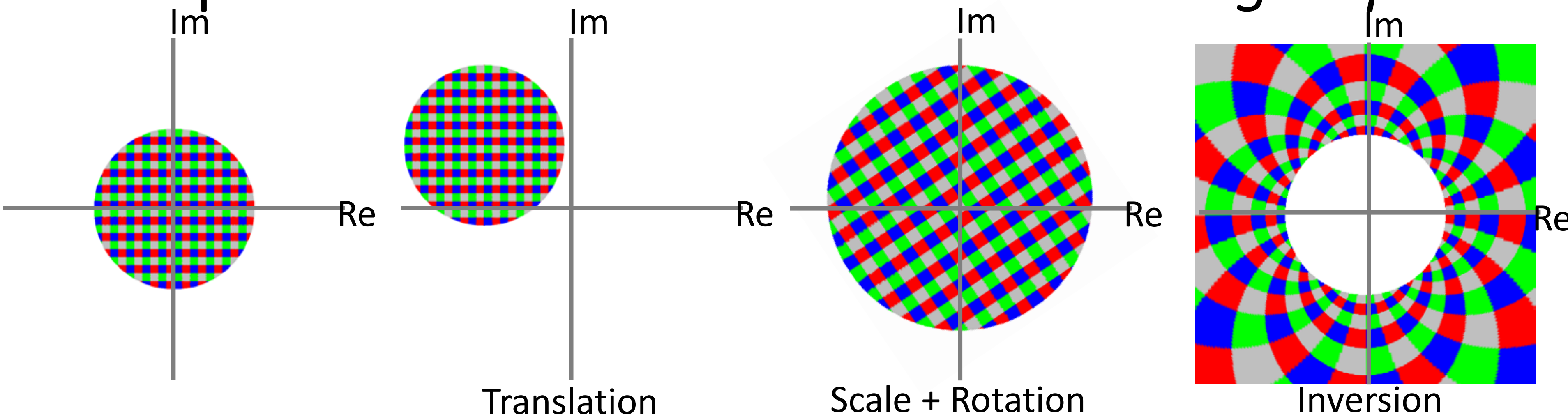

Q: Are these maps part of the Möbius group?

A: Yes. In fact, the composition of these maps comprises the entire group.

Möbius Transformations

If we think of the plane as the set of complex numbers, any Möbius transformation can be expressed as a *fractional linear transformation*:

$$f(z) = \frac{az+b}{cz+d}$$

with $ad\text{-}bc \neq 0$.

Möbius Transformations

$$f(z) = \frac{az+b}{cz+d}$$

If c=0, then:

$$f(z) = \frac{az+b}{d} = \left(\frac{a}{d}z\right) + \frac{b}{d}$$

Scale+Rotation   Translation

Möbius Transformations

$$f(z)=\frac{az+b}{cz+d}$$

If *c*=0, then:

$$f(z)=\frac{az+b}{d}=\left(\frac{a}{d}z\right)+\frac{b}{d}$$

If *a*=0, then:

Scale+Rotation

$$f(z)=\frac{b}{cz+d}=b\left(\frac{1}{((cz)+d)}\right)$$

Inversion

Scale+Rotation

Translation

Möbius Transformations

$$f(z) = \frac{az+b}{cz+d}$$

Otherwise, if $a,c \neq 0$:

$$f(z) = \frac{az+b}{cz+d} = \frac{\frac{c}{a}(az+b)}{\frac{c}{a}(cz+d)}$$

Möbius Transformations

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$$=\frac{a}{c}\frac{\left(cz+\frac{cb}{a}\right)}{\left(cz+d\right)}$$

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Möbius Transformations

$$f(z)=\frac{az+b}{cz+d}$$

Otherwise, if $a,c\neq 0$:

$$f(z)=\frac{az+b}{cz+d}=\frac{a}{c}+\left(\left(\frac{cb}{a}-d\right)\frac{1}{((cz)+d)}\right)$$

Translation: $\frac{a}{c}$

Scale+Rotation: $\left(\frac{cb}{a}-d\right)$

Inversion: $\frac{1}{((cz)+d)}$

Scale+Rotation: (cz)

Translation: $+d$

Möbius Transformations

$$f(z) = \frac{az+b}{cz+d}$$

Note:

1. Scaling the four coefficients does not change the transformation:

$$\frac{az+b}{cz+d} = \frac{f \cdot az + f \cdot b}{f \cdot cz + f \cdot d}$$

Möbius Transformations

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Note:

1. Scaling the four coefficients does not change the transformation:
$$\frac{az+b}{cz+d} = \frac{f \cdot az + f \cdot b}{f \cdot cz + f \cdot d}$$
2. If we represent the map by the 2x2 matrix:
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
then composing maps is the same as multiplying matrices.

Möbius Transformations

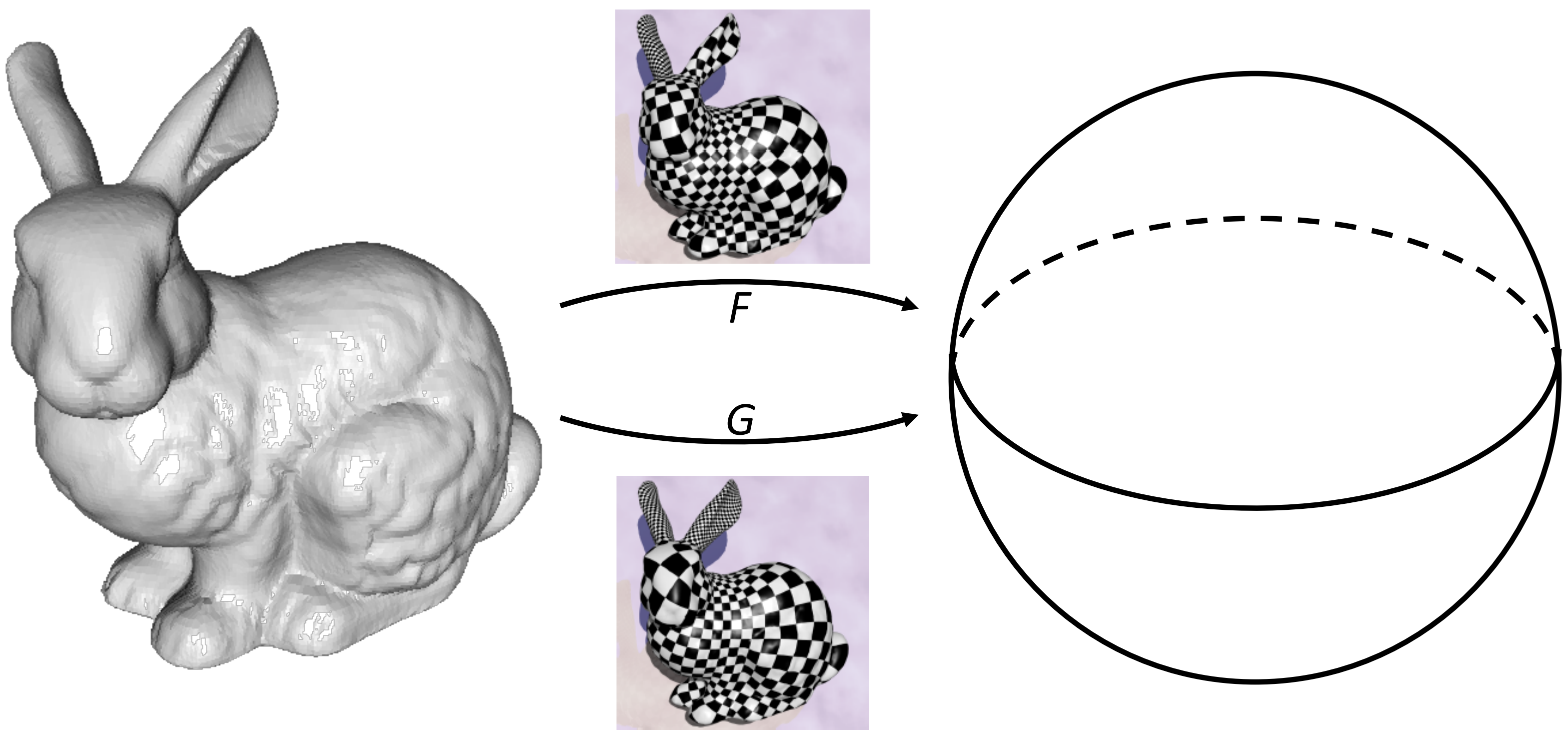
$$f(z) = \frac{az+b}{cz+d}$$

Note:

As a result, it is not uncommon to talk about the Möbius group as the group of 2x2 matrices, with complex entries and determinant 1.

Möbius Transformations

If we have conformal maps $F,G:S\rightarrow S^2$ from a surface S onto the sphere, then the composition $F\circ G^{-1}:S^2\rightarrow S^2$ must be conformal as well.



Möbius Transformations

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Since the only conformal maps from the sphere/plane into itself are the Möbius transformations, this means that, up to a Möbius transformation, the map $F:S\rightarrow S^2$ is unique.