Differential Geometry:

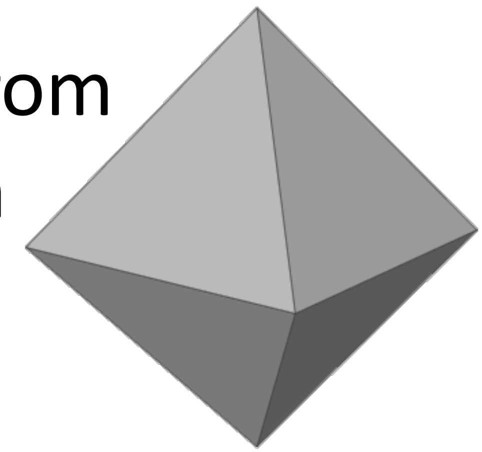
A Discrete Model of Thin Shells

[*Discrete Shells*, Grinspun *et al*., 2003]

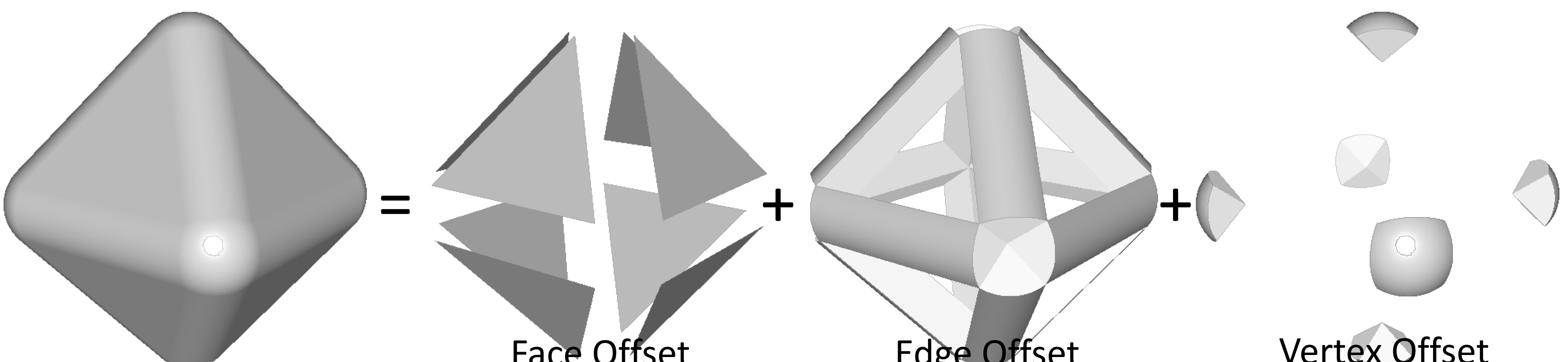
Mean Curvature

<u>Recall</u>:

We can derive the mean-curvature from the first-order term in the expression for the area of an offset surface:



$$A_\varepsilon = \sum_{t \in Tris.} A(t) + \varepsilon \sum_{e \in Edges.} |e| \theta_e + \varepsilon^2 \sum_{v \in Verts.} \theta_v$$



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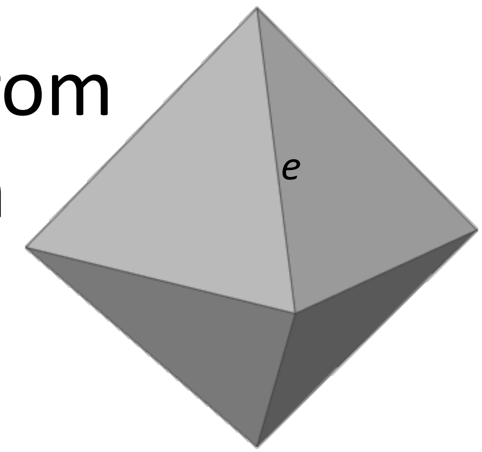

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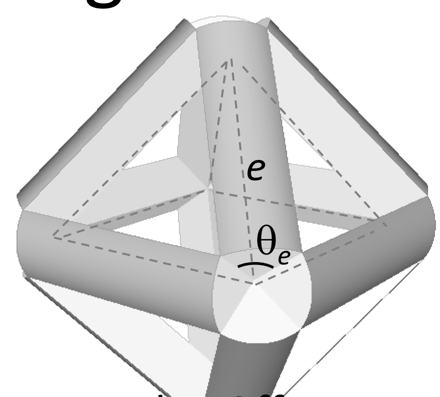
The mean-curvature is defined at the edge and has value equal to the area of the cylindrical wedge:

$$H_e = |e|\theta_e$$



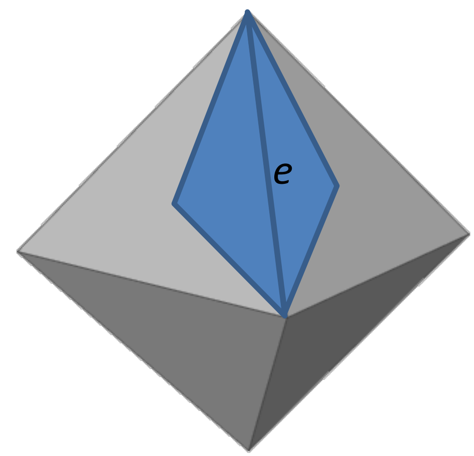
Edge Offset

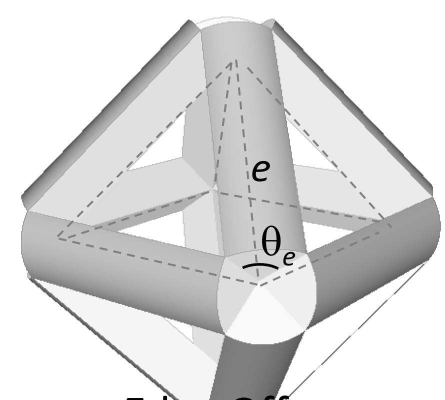
Mean Curvature

$$H_e = |e|\theta_e$$

<u>Recall</u>:

Note that this is not the value of the mean curvature at the edge, but an integral quantity corresponding to the integral of the mean curvature around the edge, (e.g. the diamond about the edge).





Edge Offset

Thin Shells

Goal:

To model the behavior of thin shells:

- Flexible structures
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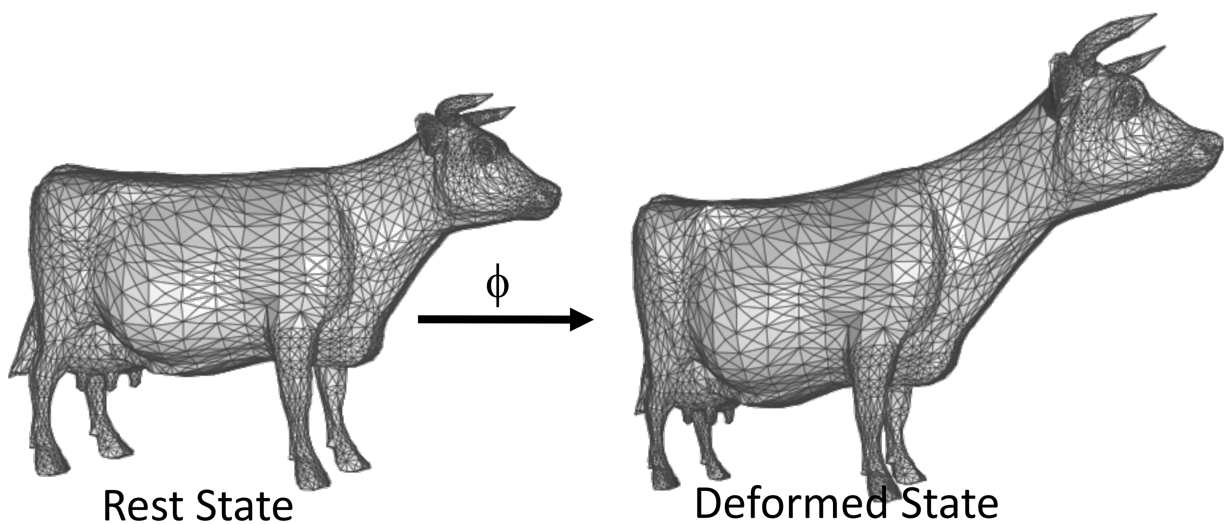
<u>Examples</u>:

- Creased Paper
- Clothing
- Etc.

Thin Shells

Approach:

Define an energy over the surface that characterizes how the surface deviates from its rest state.

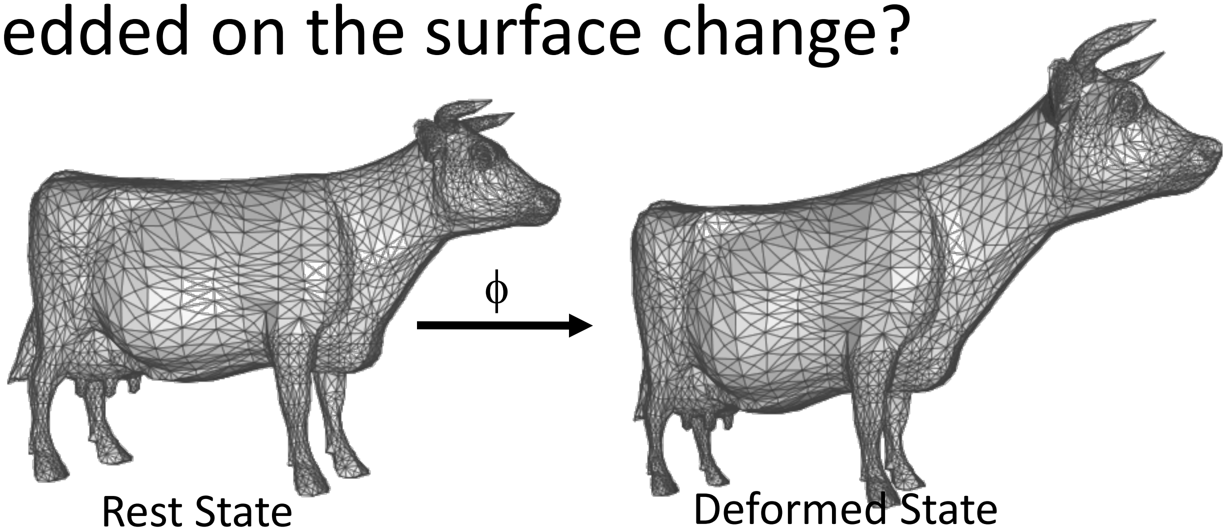


Thin Shells

Approach:

Two components to the energy:

- **Intrinsic**: How much are distances between points on the surface changed?
- **Extrinsic**: How much does the curvature of curves embedded on the surface change?

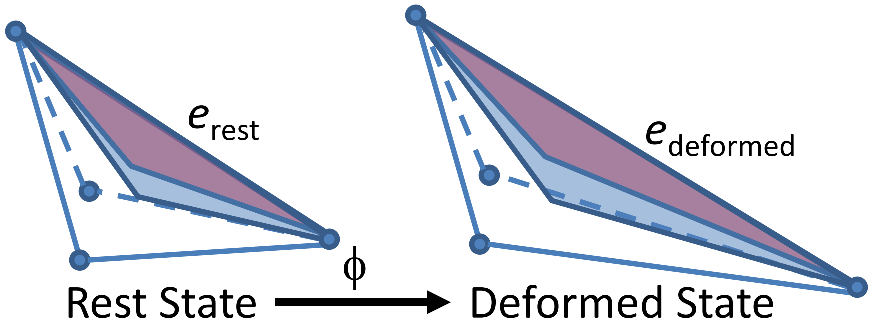


Thin Shells

<u>Intrinsic Energy</u>:

At a point on the edge, the stretch energy is:

$$S(p) = \left(1 - \frac{|e_{deformed}|}{|e_{rest}|}\right)^2$$



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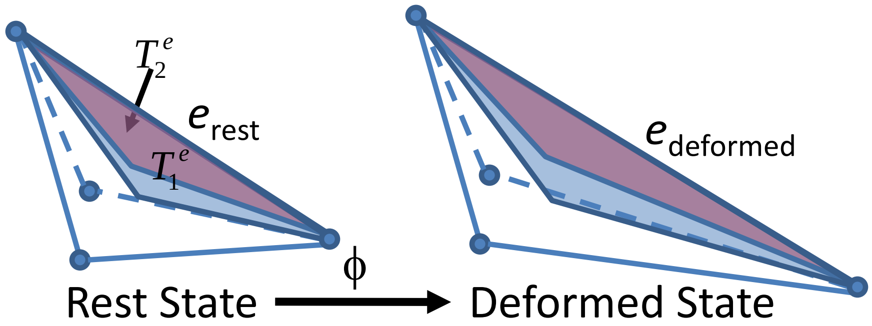
Around the edge, it can be approximated by:

$$E_S(e) = \int_{T_1^e \cup T_2^e} S(p)\,dp$$



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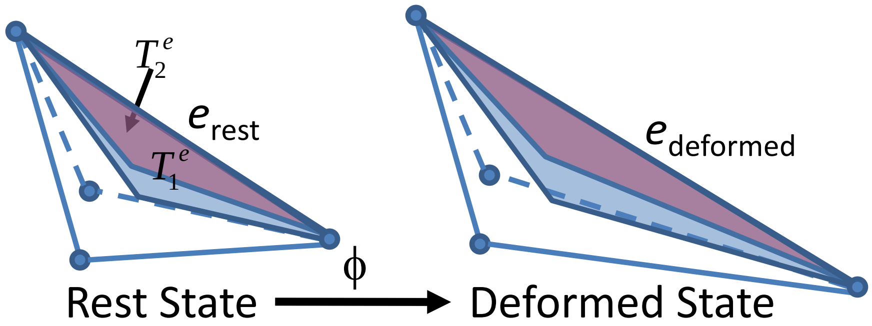

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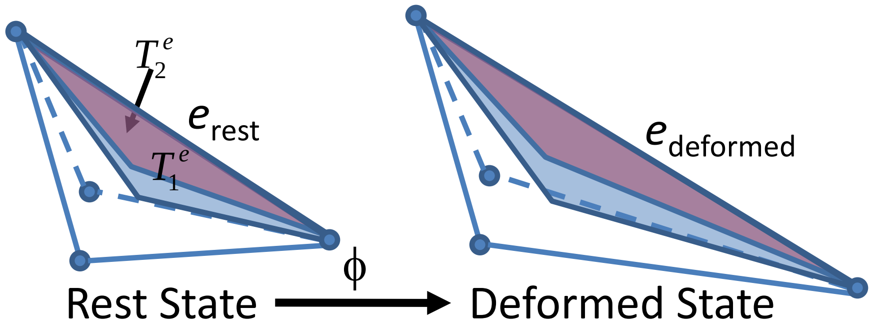
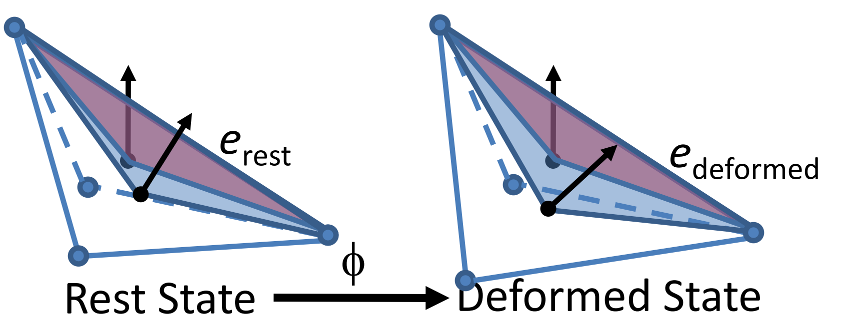


Thin Shells

<u>Extrinsic Energy</u>:

At a point on the edge, the bending energy can be measured as the difference between the mean-curvatures:

$$B(p) = \left(H_{\text{rest}}(p) - H_{\text{deformed}} \circ \phi(p)\right)^2$$



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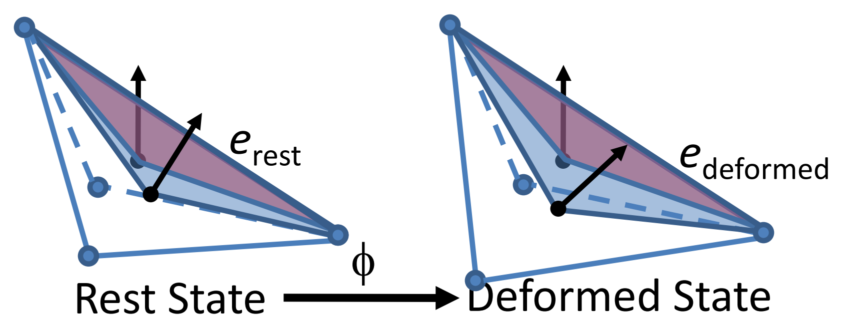

Thin Shells

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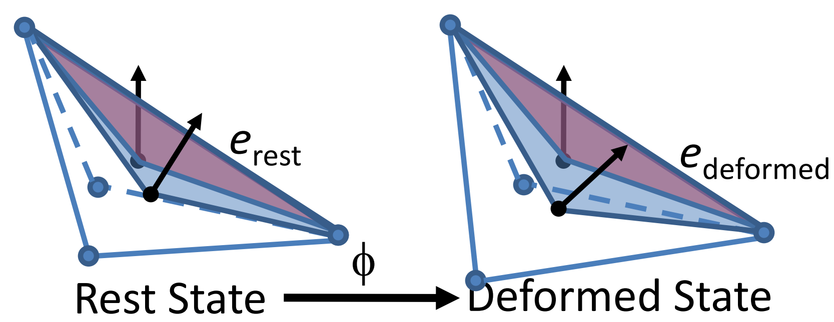
The problem is that we can't really talk about the mean-curvature at a point, as the discrete mean-curvature value is an integral quantity representing the curvature around the edge.



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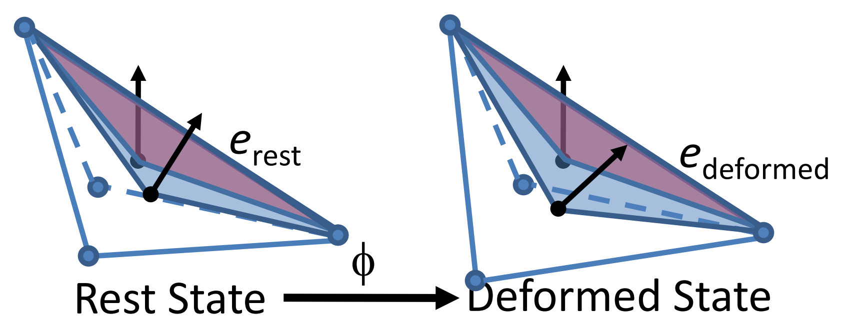
We get around this by using the fact that if a function *f* does not change very much, we can approximate:

$$\int_D [f(p)]^2\,dp \approx \frac{1}{\text{Area}(D)}\left[\int_{p\in D} f(p)\,dp\right]^2$$

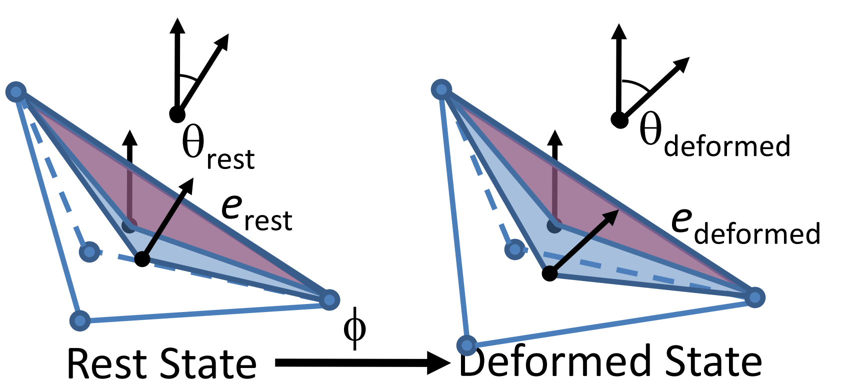


Thin Shells

Extrinsic Energy:

Thus, we can approximate the integral of the square mean-curvature differences as:

$$E_B(e) = \int_{T_1^e \cup T_2^e} \left[H_{\text{rest}}(p) - H_{\text{deformed}} \circ \phi(p)\right]^2 dp$$



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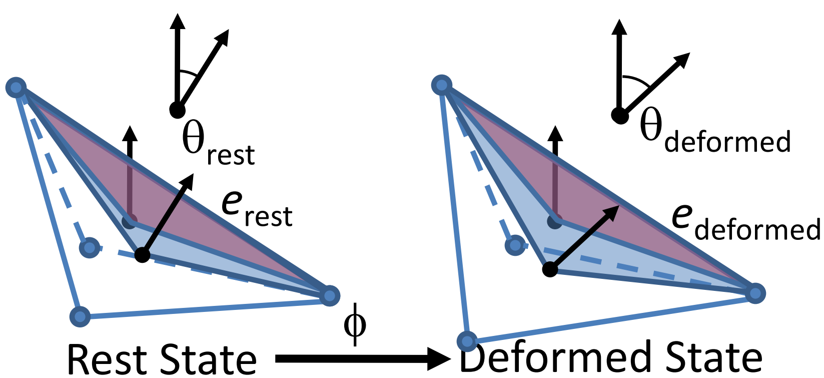

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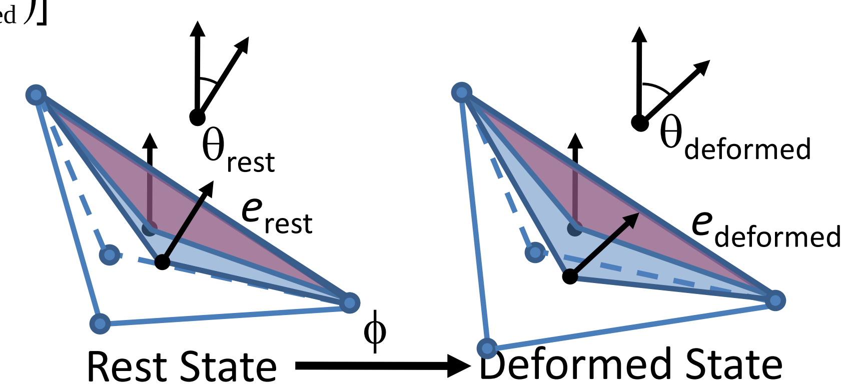
$$= \frac{1}{\text{Area}(T_1^e \cup T_2^e)} \left[\|e_{\text{rest}}\| \left(\theta_{\text{rest}} - \theta_{\text{deformed}} \right) \right]^2$$



Thin Shells

Applying the Dynamics:

To solve for the new positions of the surface, we use the fact that the negative gradient of the energy defines the force acting on the surface acting to distort the vertices.

$$F(v) = -\nabla\left(E_S(v) + E_B(v)\right)$$

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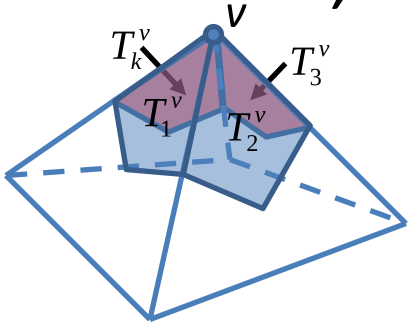
Thus, if *m*(*v*) is the area associated to a vertex, the acceleration of *v* is given by:

$$\ddot{v} = \frac{-\nabla\left(E_S(v) + E_B(v)\right)}{m(v) = \sum_j \text{Area}(T_j^v)}$$



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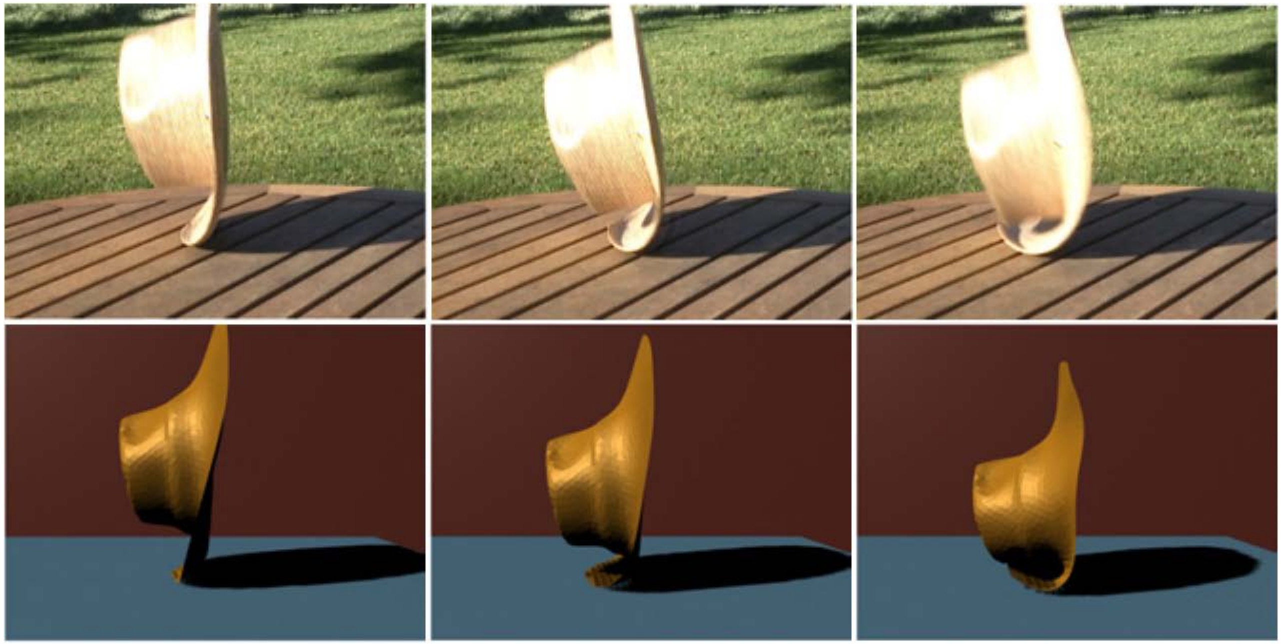

$$E_S(e) = \left(1 - \frac{|e_{deformed}|}{|e_{rest}|}\right)^2 \cdot \left(\text{Area}(T_1^e) + \text{Area}(T_2^e)\right)$$

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Implementation:

Actually generating the simulation requires addressing two challenges:

1. Computing the gradient of the energy:

$$\nabla\left(E_S(v)+E_B(v)\right)$$

2. Integrating the system, using the computed acceleration to modify the position and velocity of the vertices (e.g. [Newmark 1959]):

$$v^{t+1}=v^t+\Delta t\dot{v}^t+\frac{1}{2}\Delta t^2\left((1-\beta)\ddot{v}^t+\beta\ddot{v}^{t+1}\right)$$
$$\dot{v}^{t+1}=\dot{v}^t+\Delta t\left((1-\gamma)\ddot{v}^t+\gamma\ddot{v}^{t+1}\right)$$