Differential Geometry:

Delaunay Triangulations and the Laplace-Beltrami Operator

[*A Discrete Laplace-Beltrami Operator for Simplicial Surfaces*, Bobenko and Springborn, 2006]

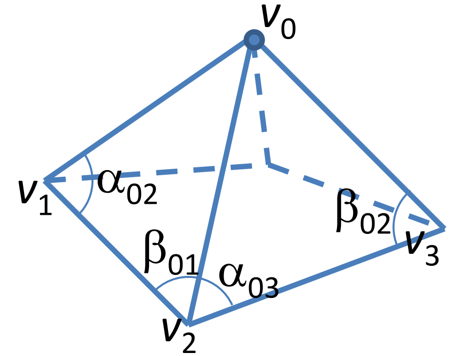
[*An Algorithm for the Construction of Intrinsic Delaunay Triangulations with Applications to Digital Geometry Processing*, Fisher *et al*., 2007]

Harmonics Maps

<u>Recall</u>:

Given a surface *S* with a (connected) boundary, we can define the cotangent-weight Laplacian:

$$L_{ij} = \begin{cases} \frac{1}{2}\left(\cot\left(\alpha_{ij}\right)p + \cot\left(\beta_{ij}\right)\right) & \text{if } i \neq j \text{ and } v_j \in \mathrm{Nbr}(v_j) \\ -\sum_{v_k \in \mathrm{Nbr}(v_i)} L_{ik} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$



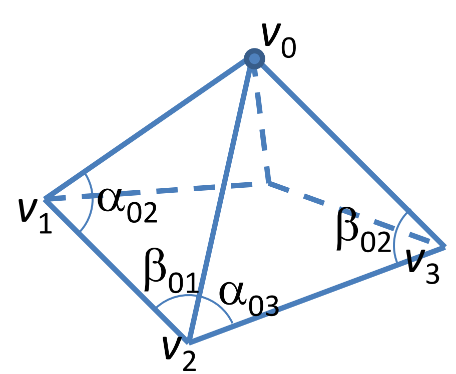
Harmonics Maps

<u>Recall</u>:

Given a surface *S* with a (connected) boundary, we can define the cotangent-weight Laplacian:

Given the Laplacian, we can define the Dirichlet energy of a function defined on the vertices:

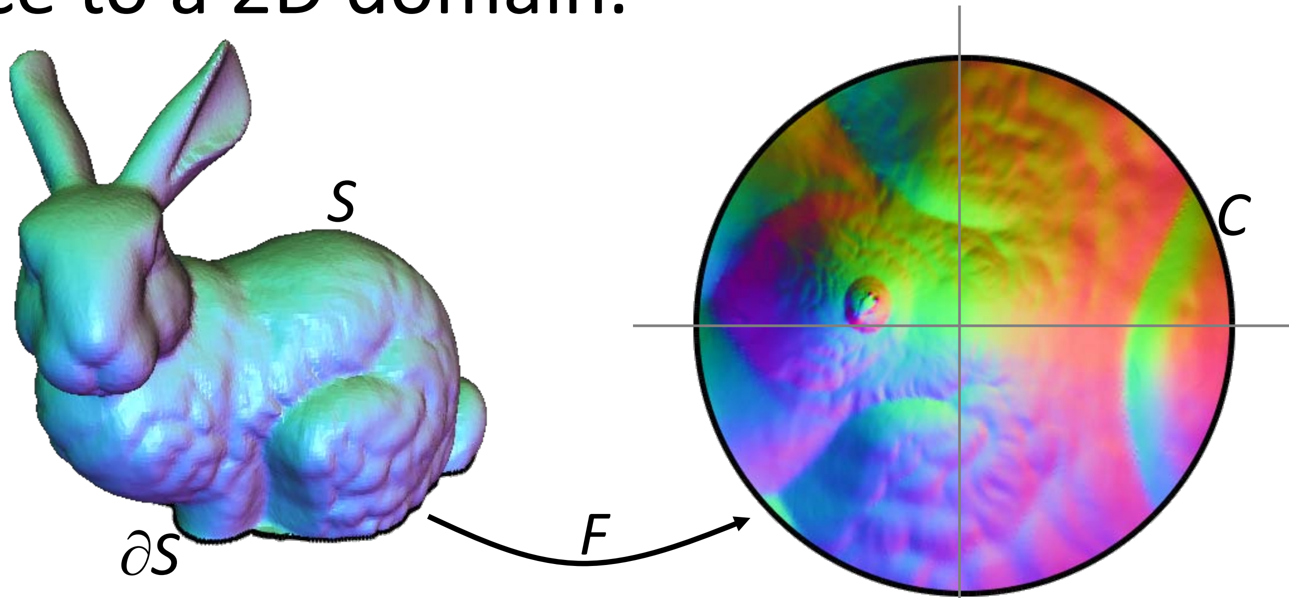
$$E(F) = \sum_{(i,j)\in Edges} L_{ij}\left(f_i - f_j\right)^2$$



Harmonics Maps

<u>Recall</u>:

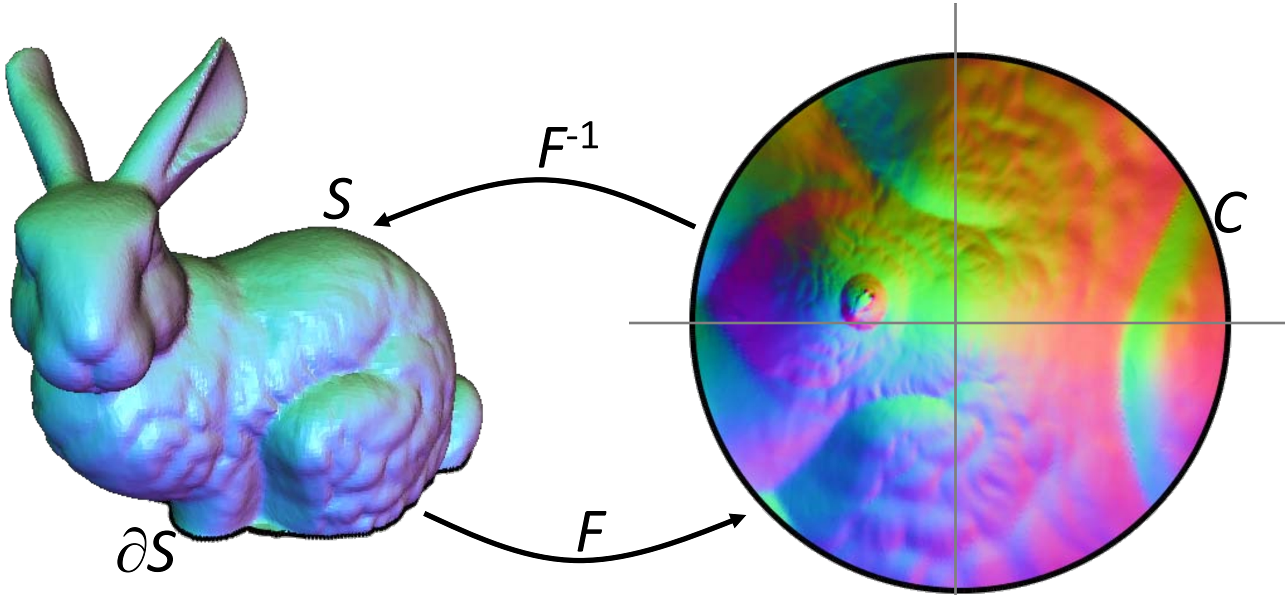
Associating the boundary with a convex curve in 2D and solving for the harmonic function that minimizes the Dirichlet energy, we get a map from the surface to a 2D domain.



Harmonics Maps

<u>Recall</u>:

In the continuous case, the map is bijective so the map F^{-1} is a well-defined parameterization.

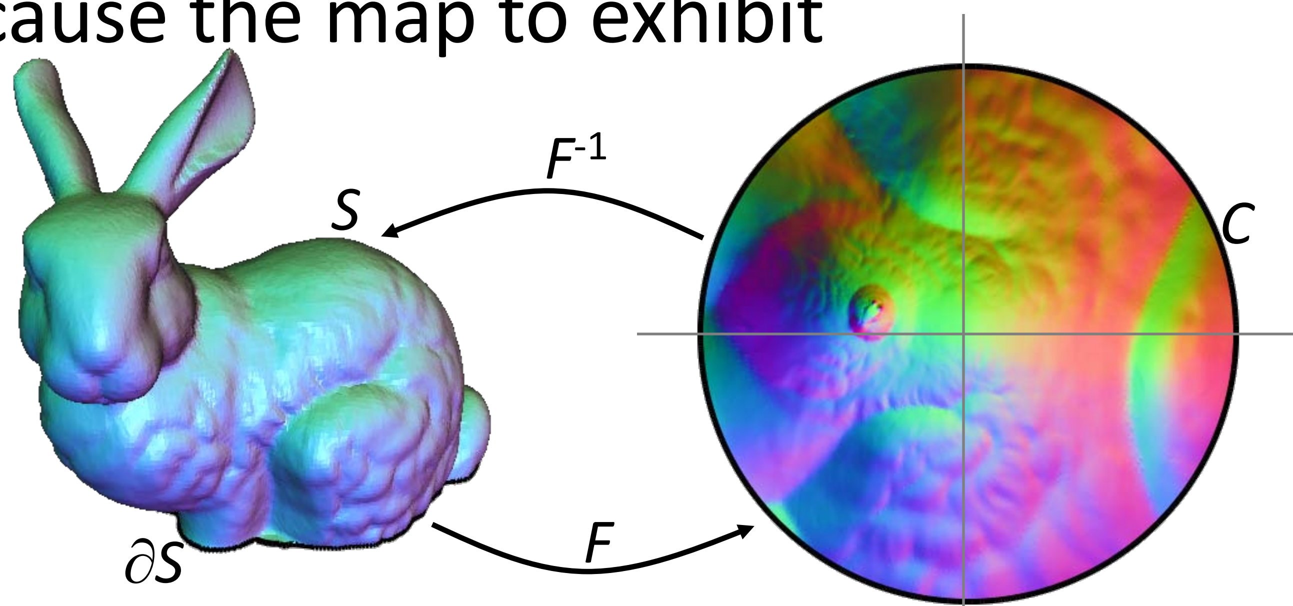


Harmonics Maps

Recall:

In the continuous case, the map is bijective so the map F^{-1} is a well-defined parameterization.

In the discrete case, the negative weights of the Laplacian can cause the map to exhibit edge-flips.

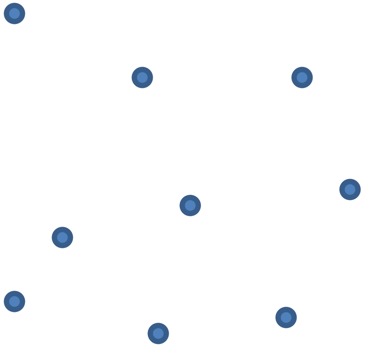


Convex Hulls

<u>Definition</u>:

Given a finite set of points $P=\{p_1,\ldots,p_n\}\subset\boldsymbol{R}^n$, the *convex hull* the set of points consisting of the convex combinations of points in P:

$$\mathrm{Convex}(P)=\left\{\sum_{p\in P}\alpha_p p \,\middle|\, \alpha_p \geq 0 \ \text{ and } \ \sum_{p\in P}\alpha_p\right\}$$



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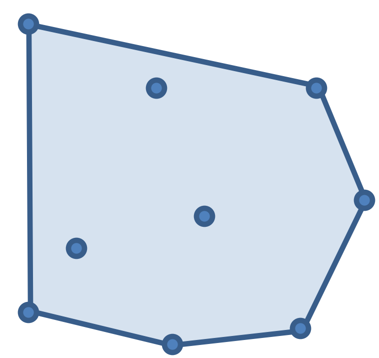
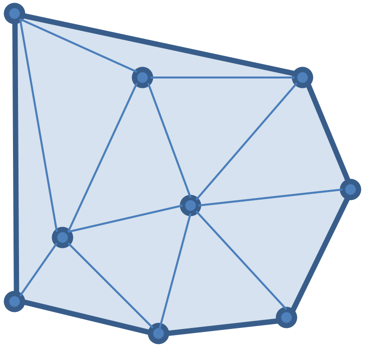


Planar Triangulations

<u>Definition</u>:

A *triangulation* of a finite set of points $P=\{p_1,\ldots,p_n\}$ is a decomposition of the convex hull of P into triangles with the property that:

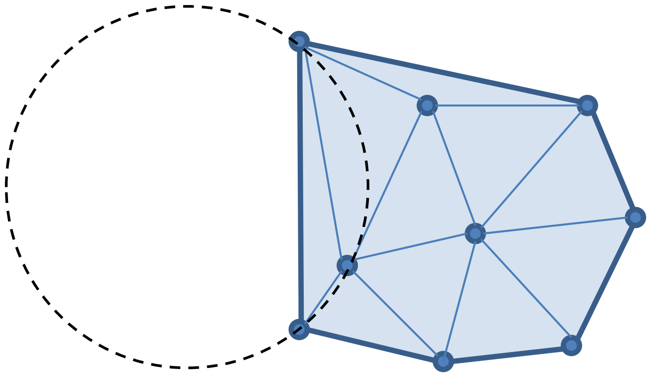
- The set of triangle vertices equals P
- The intersections of two triangles is either empty or is a common edge or vertex.



Delaunay Triangulations

Definition:

A triangulation is said to be *Delaunay* if the interior of the triangles' circum-circles are empty.



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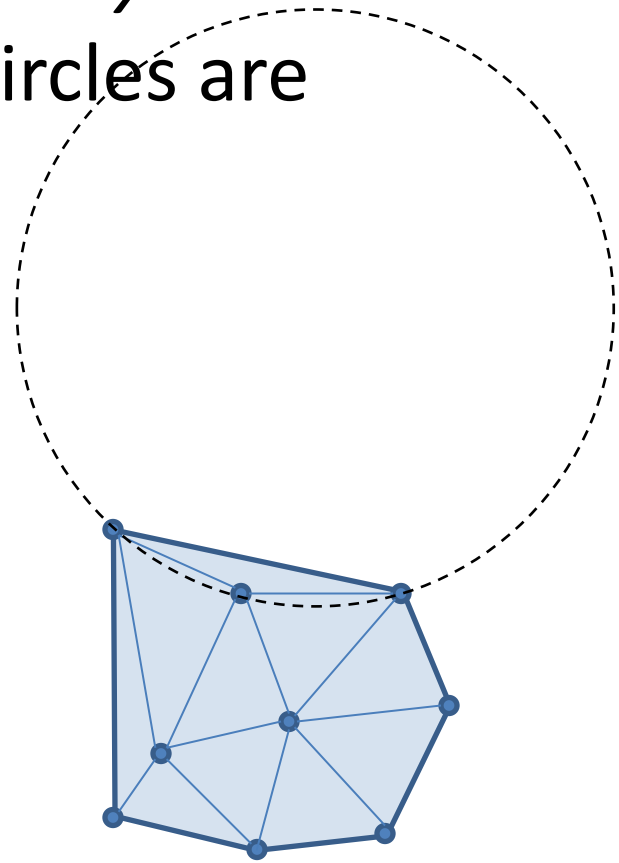

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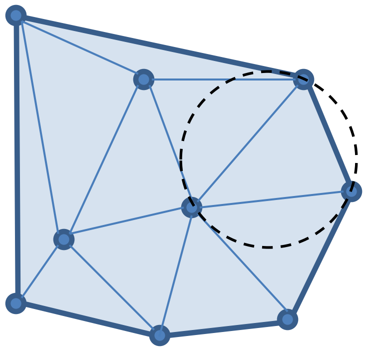

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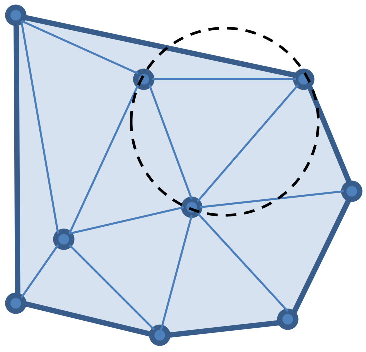
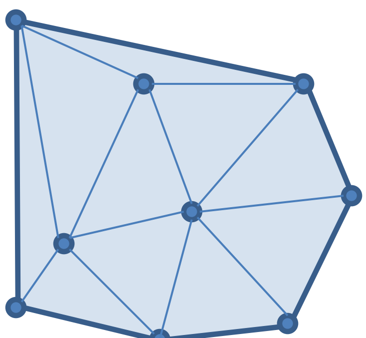


Delaunay Triangulations

<u>Computing the Delaunay Triangulation</u>:

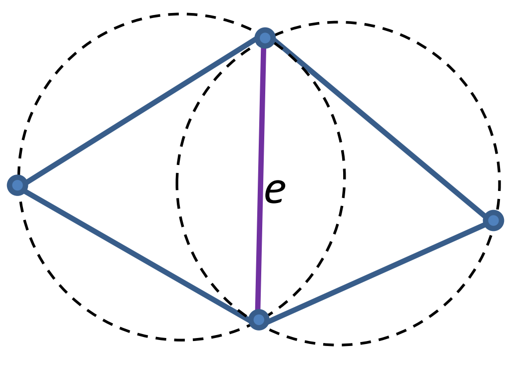
- Incremental
- Divide and Conquer
- Sweepline (planar)
- Convex hulls of paraboloids



Delaunay Edges

Definition:

An interior edge *e* is *locally Delaunay* if the interiors of the circum-circles of the two triangles do not contain the triangles' vertices.



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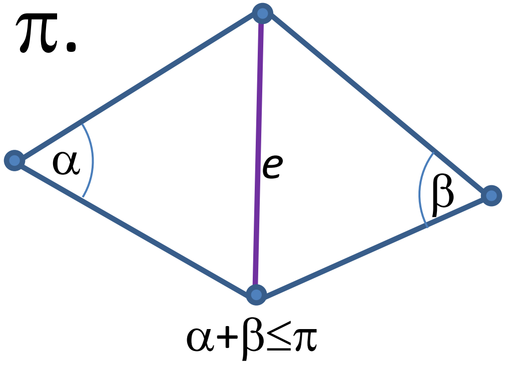
An interior edge e is *locally Delaunay* if the interiors of the circum-circles of the two triangles do not contain the triangles' vertices.

<u>Property</u>:

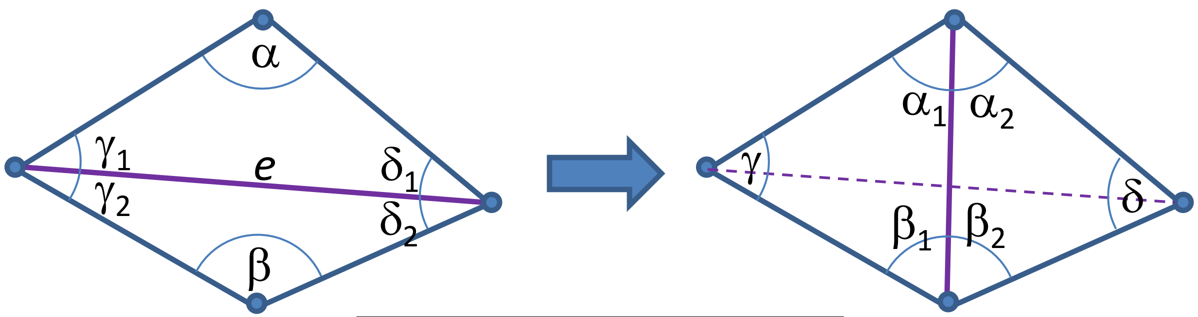
An interior edge is Delaunay iff. the sum of the opposite angles is not greater than π.



Delaunay Edges

<u>Note</u>:

If the sum of the opposite angles is greater than π, then flipping the edge will give a sum that is less than π.



$$\boxed{\gamma + \delta = 2\pi - (\alpha + \beta)}$$

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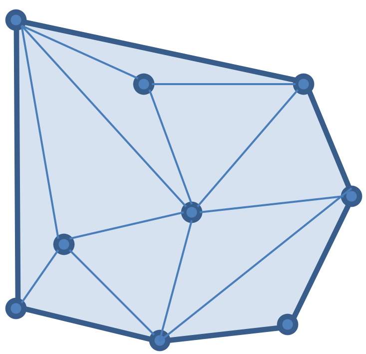
Edge Flipping Algorithm:

Starting with an arbitrary triangulation, flip edges until each edge is locally Delaunay.



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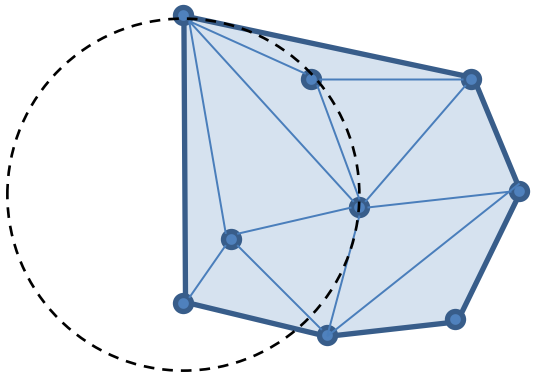

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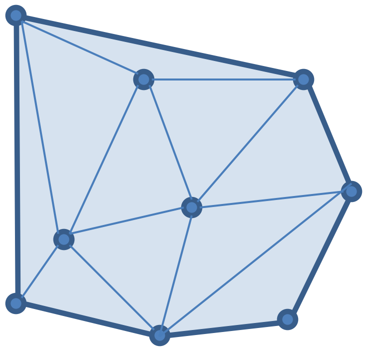

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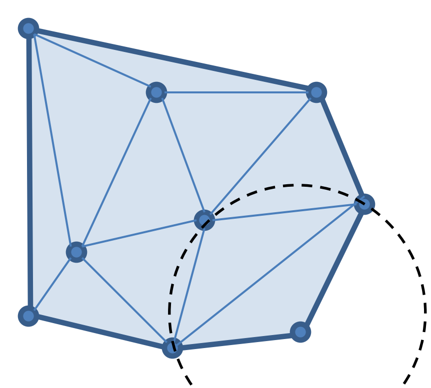

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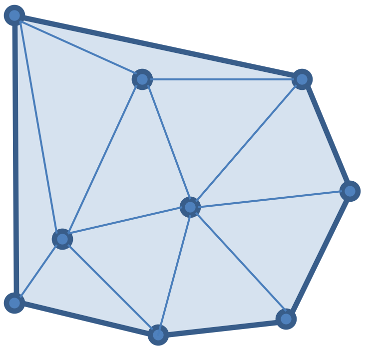

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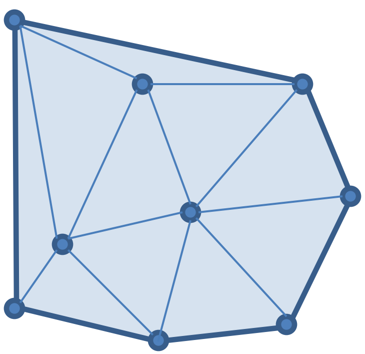
Is this algorithm guaranteed to terminate?



Delaunay Triangulations

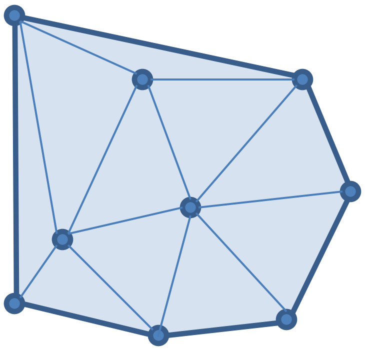
<u>Edge Flipping Algorithm</u>:

Starting with an arbitrary triangulation, flip edges until each edge is locally Delaunay.

<u>Is this algorithm guaranteed to terminate</u>?

Termination is proved by:

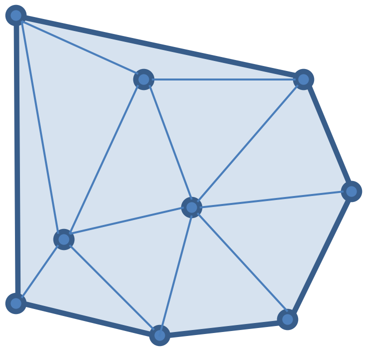
- Showing that there finitely many different triangulations.
- Defining a global "energy" that is reduced with each flip (e.g. sum of squared circum-radii.)



Delaunay Triangulations

Why Should we Care:

Of all possible triangulations, the Delaunay triangulation maximizes the minimal angle.



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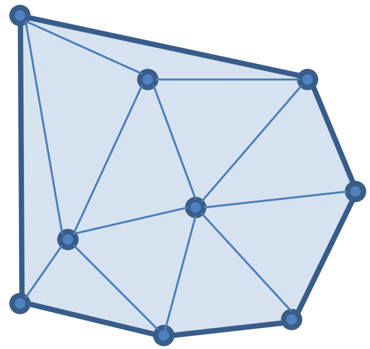
This results in a triangulation with “well-formed” triangles, facilitating numerical processing over the triangulation.



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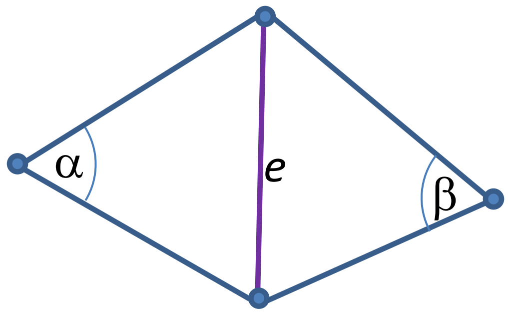
Given two triangles sharing an edge, the Laplacian weight associated to the edge is:

$$w_e = \frac{1}{2}\left(\cot\alpha + \cot\beta\right)$$



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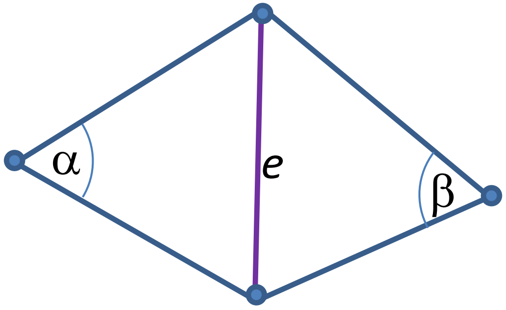

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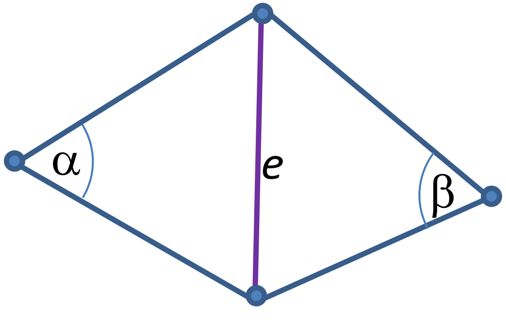

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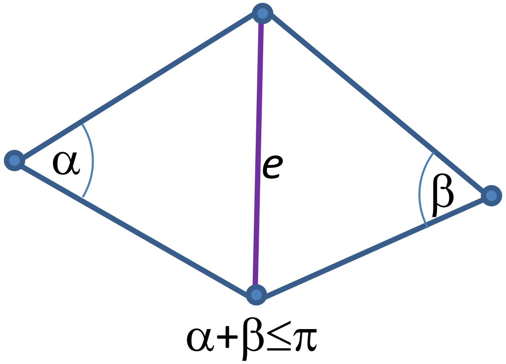
Thus, the cotangent weight is non-negative iff. the sum of the angles is less than or equal to π.



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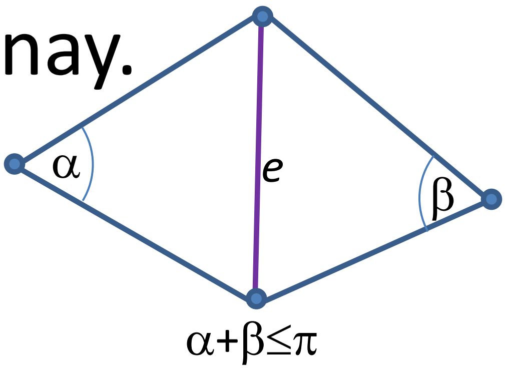

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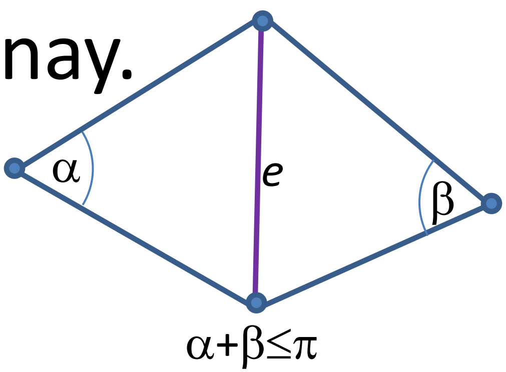
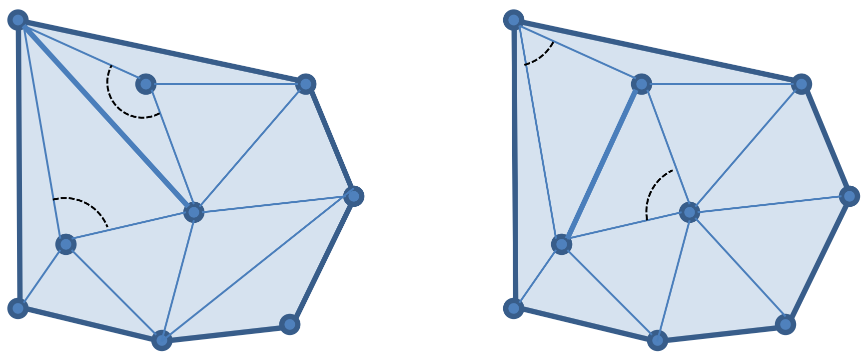
So the cotangent weights are ≥ 0 iff. the triangulation is Delaunay.



Delaunay Triangulations and Laplacians

Given different triangulations of the point-set $P=\{p_1,\ldots,p_n\}\subset\boldsymbol{R}^2$, the angles of the triangulation define different cotangent-Laplacian.



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So, if we have a function $F=\{f_1,\ldots,f_n\}$ defined on the points, the different triangulations define different Dirichlet energies:

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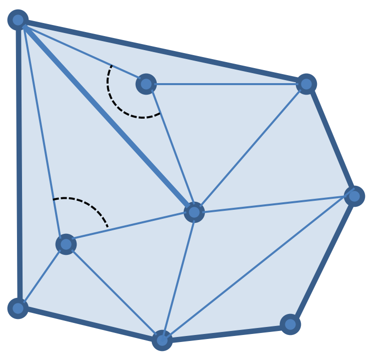

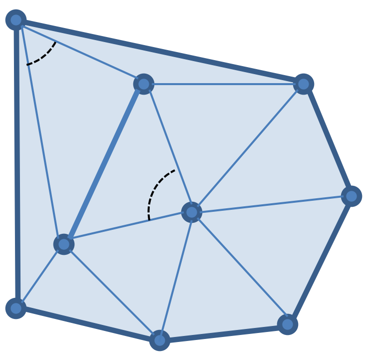

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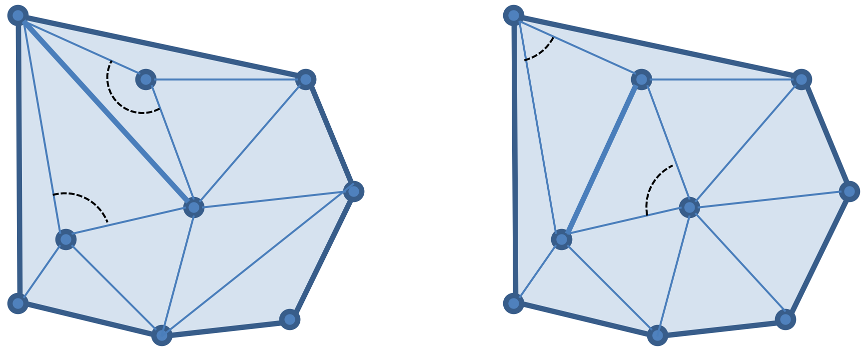
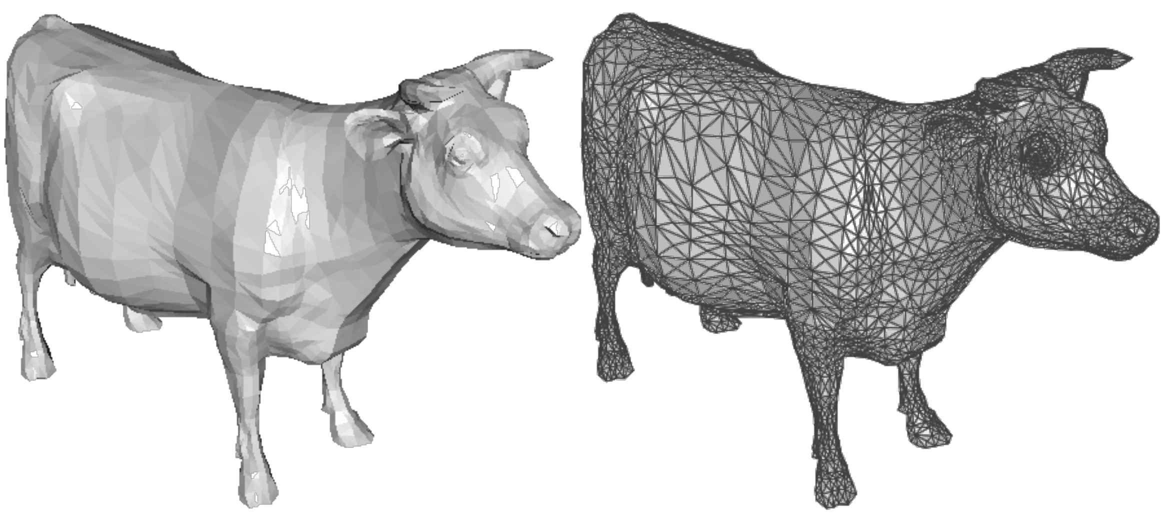
Turns out that a Delaunay triangulation always gives the smallest energy.



From Planar Meshes to Surfaces in 3D

When given a triangle mesh, we are given two distinct pieces of information:

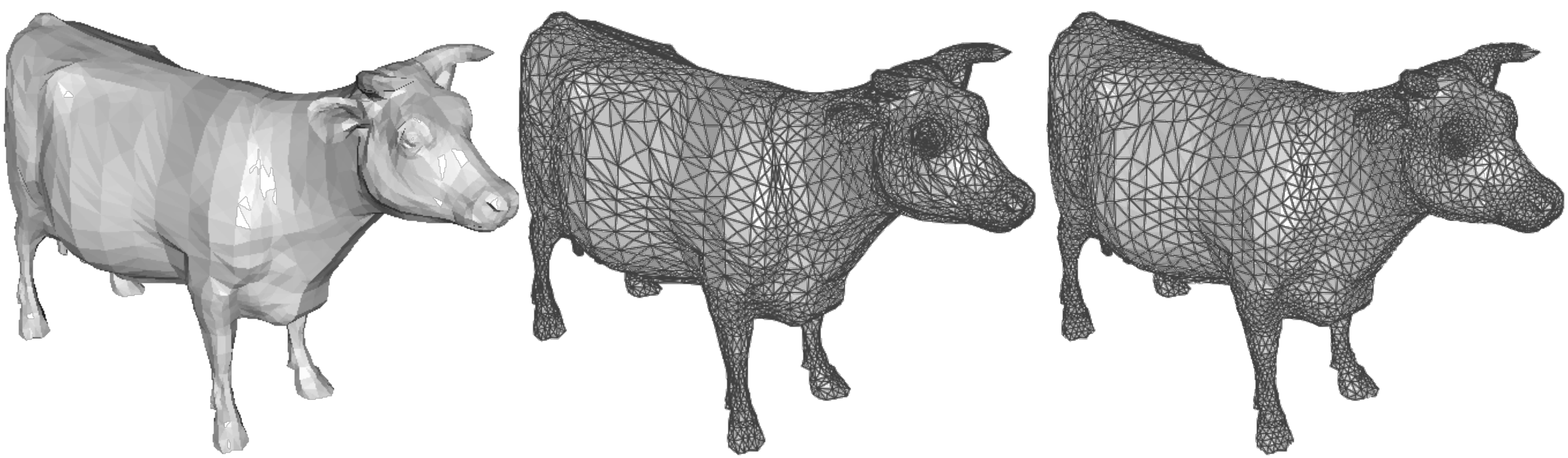
- Surface: The points inside the triangles
- Triangulation: A decomposition of the surface



From Planar Meshes to Surfaces in 3D

<u>Key Observation</u>:

It is possible to triangulate the same surface in different ways.

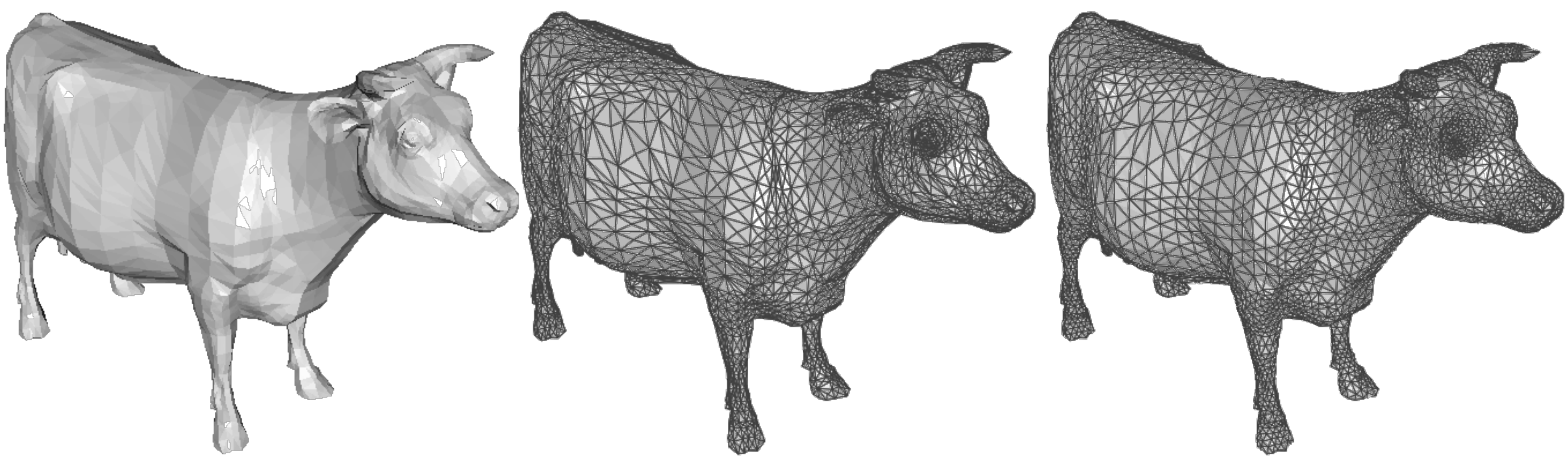


From Planar Meshes to Surfaces in 3D

<u>Key Observation</u>:

It is possible to triangulate the same surface in different ways.

$\Rightarrow$ Use the original surface geometry, but do geometry processing with the best triangulation.



Surface Triangulations

For a planar domain, a triangulation is a decomposition of a domain into patches where:

- The boundary of the patches are made up of straight line segments, and
- The end-points of the segments are the points

Surface Triangulations

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Surface Triangulations

For a ~~planar domain~~ triangle mesh, a triangulation is a decomposition of a domain into patches where:

- The boundary of the patches are made up of ~~straight line segments~~ geodesics, and
- The end-points of the segments are the ~~points~~ vertices

The edges of the triangulation live on the surface of the mesh and go through the mesh vertices, but are not comprised of the old edges.

Surface Delaunay Triangulations

As in the planar case, we can define a Delaunay triangulation as the triangulation which satisfies the empty circum-circle property.

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As in the planar case, we can define a Delaunay triangulation as the triangulation which satisfies the empty circum-circle property.

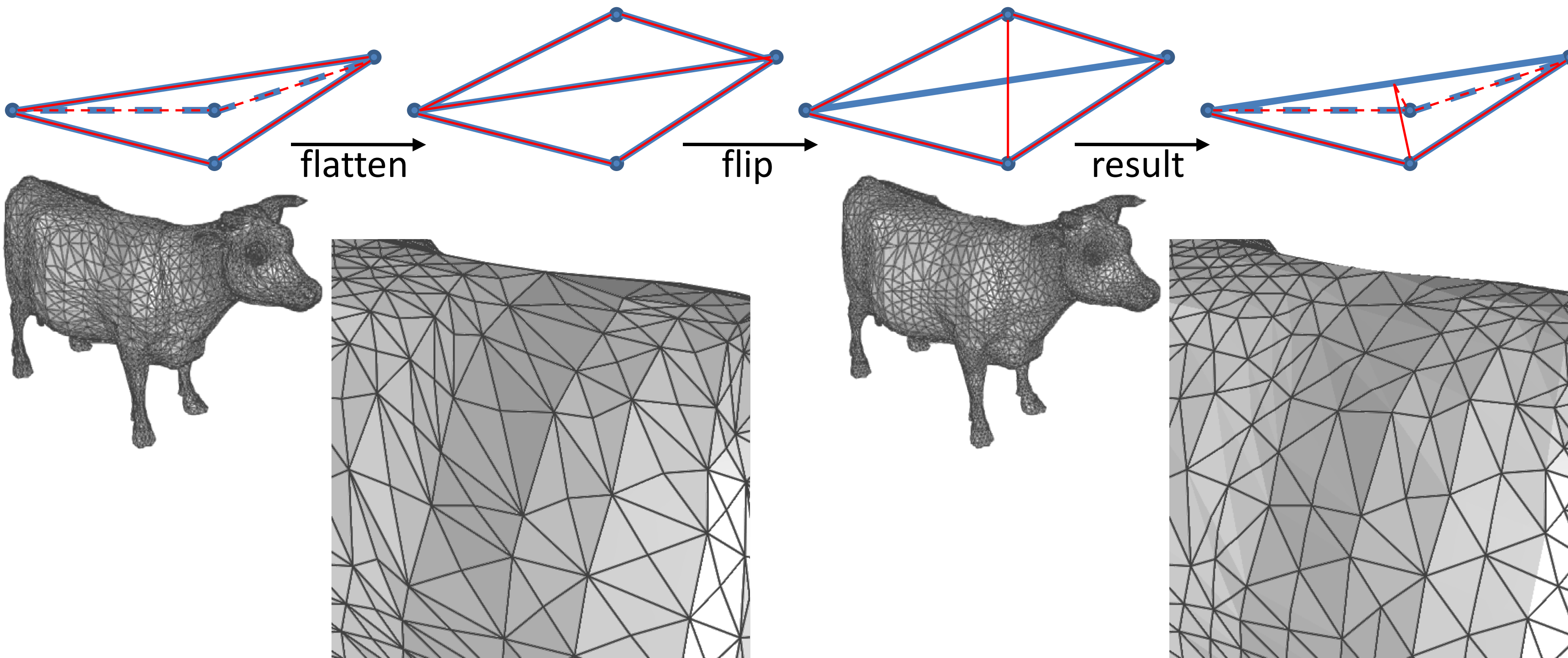
And, as with the planar case, we can get the Delaunay triangulation by performing a sequence of intrinsic edge-flips.

Note:

As before, we do this by showing that an energy is minimized with each flip, but it's trickier because there are infinitely many triangulations.

Edge-Flipping on a Mesh

When we perform an edge-flip on a surface triangulation, we need to ensure that the new edge to still reside on the surface.



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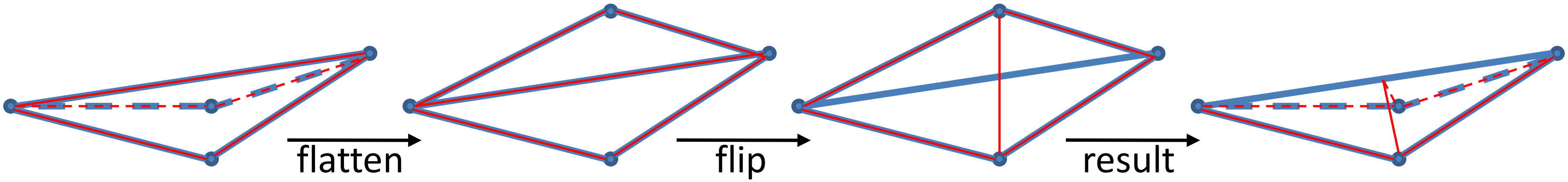
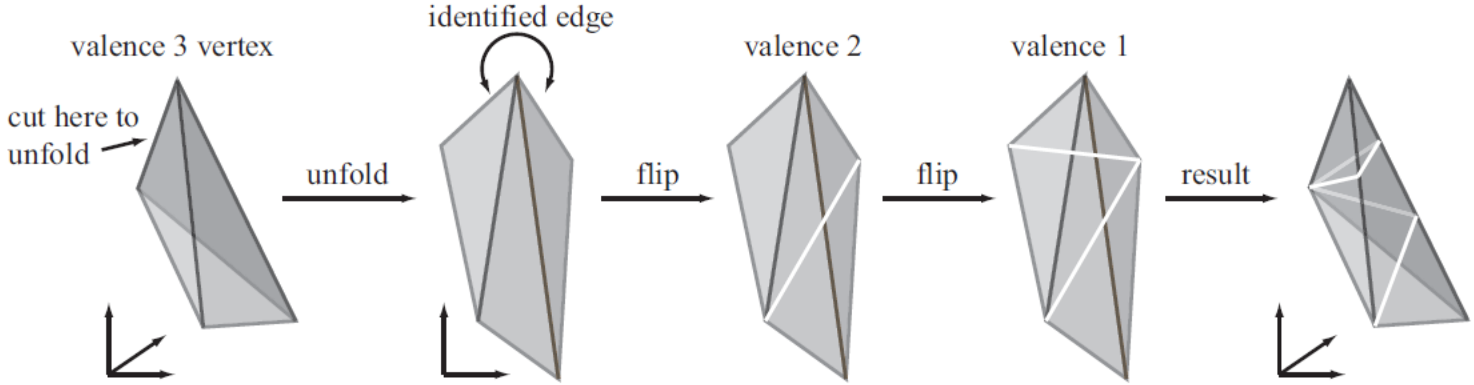


Note that this can result in non-regular triangles.



Intrinsic Laplacian

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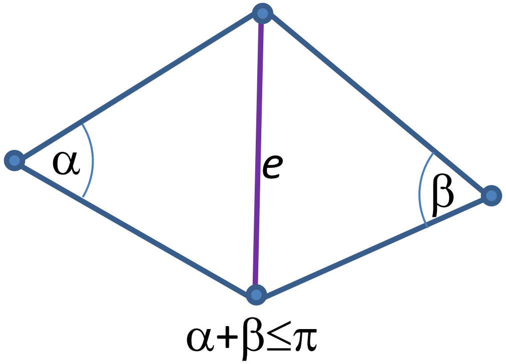

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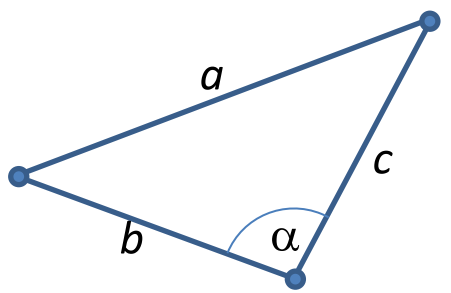
- The Laplacian must have non-negative weights since edge-flipping ensure that $\alpha+\beta\leq\pi$.
- The Laplacian is intrinsic since performing the edge-flips to get a Delaunay triangulation only requires knowledge of edge-lengths.



$$\tan\frac{\alpha}{2}=\sqrt{\frac{(a-b+c)(a+b-c)}{(a+b+c)(-a+b+c)}}$$

$$\boxed{\cot\alpha=\frac{1-\tan^2(\alpha/2)}{2\tan(\alpha/2)}}$$

Intrinsic Laplacian

Using the Delaunay triangulation, we can define a new cotangent Laplacian over the surface:

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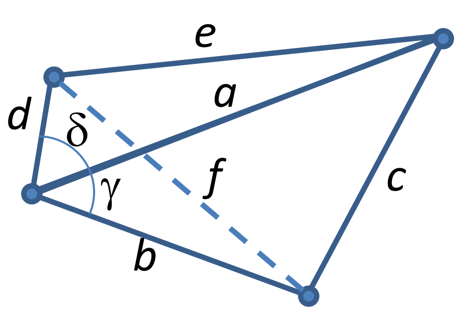

$$\tan\frac{\gamma+\delta}{2} = \frac{\tan(\gamma/2)+\tan(\gamma/2)}{\tan(\gamma/2)\tan(\gamma/2)} \qquad \cos(\gamma+\delta) = \frac{1-\tan^2\left((\gamma+\delta)/2\right)}{1+\tan^2\left((\gamma+\delta)/2\right)}$$

$$\boxed{f = \sqrt{b^2+d^2-2bd\cos(\gamma+\delta)}}$$

Intrinsic Laplacian

Using the Delaunay triangulation, we can define a new cotangent Laplacian over the surface:

- The Laplacian must have non-negative weights since edge-flipping ensure that $\alpha+\beta\leq\pi$.
- The Laplacian is intrinsic since performing the edge-flips to get a Delaunay triangulation only requires knowledge of edge-lengths.
- The triangles are well-formed.

Intrinsic Laplacian

<u>Positive Weights</u>:

With the extrinsic Laplacian, negative weights in the Laplacian would imply that harmonic functions could have extrema off the boundary.

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Positive Weights:

With the extrinsic Laplacian, negative weights in the Laplacian would imply that harmonic functions could have extrema off the boundary.

$\Rightarrow$ The derived mapping to the plane could have flipped triangles and is not invertible.

With the intrinsic Laplacian, all the weights are positive, the extrema are on the boundary, and there are no edge-flips, so the inverse is well-defined and gives a parameterization.

Intrinsic Laplacian

Well-Formed Triangles:

Having triangles with good aspect ratio implies that the linear system defined by the cotangent Laplacian is better-conditioned, making it easier to solve.