Differential Geometry:
Harmonics Maps and Energy Minimization

[Multiresolution Analysis of Arbitrary Meshes, Eck et al. 1995]
Dirichlet Energy

Recall:
Given a function $F = (f_1, \ldots, f_m): D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$, the Dirichlet Energy is a measure of how much the function $F$ changes over $D$:

$$E(F) = \int_D |dF(p)|^2 \, dp = \sum_{i=1}^m \int_D |\nabla f_i(p)|^2 \, dp$$

In many applications, we seek functions that are “as smooth as possible”, amounting to seeking functions $F$ minimizing the Dirichlet Energy.
Recall:

Given a domain $D \subset \mathbb{R}^n$, the function $F: D \rightarrow \mathbb{R}^m$ that satisfies the boundary constraints:

$$F(p) = C(p) \quad \forall p \in \partial D$$

and minimizes the Dirichlet Energy is called a \textit{harmonic function}. 

\textbf{Dirichlet Energy}
Dirichlet Energy

Recall:
The condition that $F$ is a harmonic function implies that for all functions $G$, with $G(p)=0$ for $p \in \partial D$, offsetting $F$ in the direction of $G$ will not decrease/increase the energy:

$$
\lim_{\varepsilon \to 0} \frac{E(F + \varepsilon G) - E(F)}{\varepsilon} = 0
$$
Dirichlet Energy

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$\iff$ The Laplacian of $F$ is equal to zero.
Dirichlet Energy

Recall:
The Laplacian can be thought of as the gradient of the energy at the function $F$ so the energy can be minimized by repeatedly offsetting $F$ in the direction opposite the Laplacian:

$$F \leftarrow F - \varepsilon \Delta F$$
Dirichlet Energy

Recall:
In the case of smooth surfaces, if we consider the embedding function:

\[ F(x, y, z) = (x, y, z) \]

• The Dirichlet energy is twice the area.
• The Laplacian is equal to the normal vector, scaled by the mean curvature.
Recall:
If we offset points on the surface in the direction of the negative mean curvature, we evolve the surface towards a smoother surface with smaller surface area.
Dirichlet Energy

Recall:
In the case of discrete surfaces, if we consider functions as having values on the vertices, we get the cotangent-weight Laplacian:

\[
L_{ij} = \begin{cases} 
\frac{1}{2} \left( \cot(\alpha_{ij}) p + \cot(\beta_{ij}) \right) & \text{if } i \neq j \text{ and } v_j \in \text{Nbr}(v_j) \\
- \sum_{v_k \in \text{Nbr}(v_i)} L_{ik} & \text{if } i = j \\
0 & \text{otherwise}
\end{cases}
\]
Example:
Given a surface $S$ with a (connected) boundary, we can map $S$ to the plane by mapping the boundary to a closed planar curve $C$ and solving for the harmonic function $F:S \to \mathbb{R}^2$. 
Parameterization

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Solving for the Harmonic Function

Solve the linear system that gives the function $F$ with the prescribed boundary values and has zero Laplacian at all (interior) vertices.
Solving for the Harmonic Function

We are given a mesh with vertices
\( \{v_1, \ldots, v_m, v_{m+1}, \ldots, v_n\} \subset \mathbb{R}^3: \)
- \( \{v_1, \ldots, v_m\} \) are boundary vertices
- \( \{v_{m+1}, \ldots, v_n\} \) are interior vertices.
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We have boundary constraints:
- \(v_i \to c_i\), for all \(1 \leq i \leq m\), with \(c_i \in \mathbb{R}^2\).

And interior constraints:
- \((\nabla v)_i = 0\), for all \(m+1 \leq i \leq n\).
Solving for the Harmonic Function

Defining the Linear System (1):
Treat the positions \( \{p_1, \ldots, p_n\} \) as variables and explicitly add constraints on the boundary vertices:

\[
\begin{pmatrix}
1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1
\end{pmatrix}
\begin{pmatrix}
\cdots \\
p_m \\
p_{m+1} \\
p_n
\end{pmatrix}
= 
\begin{pmatrix}
\cdots \\
c_m \\
0 \\
0
\end{pmatrix}
\]
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\begin{pmatrix}
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c_m \\
\vdots \\
0
\end{pmatrix}
\]

Disadvantages:

- The system is not symmetric.
- The system is larger than the number of unknown
Solving for the Harmonic Function

Defining the Linear System (2):

\[
\begin{bmatrix}
1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
P_1 \\
\vdots \\
P_m \\
P_{m+1} \\
\vdots \\
P_n
\end{bmatrix}
= 
\begin{bmatrix}
c_1 \\
\vdots \\
c_m \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

Since the boundary constraints are fixed, we can re-write the above system as:

\[
L_{[1,m]x[m+1,n]} \begin{bmatrix}
c_1 \\
\vdots \\
c_m
\end{bmatrix}
+ L_{[m+1,n]x[m+1,n]} \begin{bmatrix}
P_{m+1} \\
\vdots \\
P_n
\end{bmatrix}
= \begin{bmatrix}
0 \\
\vdots \\
0
\end{bmatrix}
\]
Solving for the Harmonic Function

Defining the Linear System (2):

\[ L_{[1,m] \times [m+1,n]} \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} + L_{[m+1,n] \times [m+1,n]} \begin{pmatrix} p_{m+1} \\ \vdots \\ p_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \]

Or equivalently:

\[ L_{[m+1,n] \times [m+1,n]} \begin{pmatrix} p_{m+1} \\ \vdots \\ p_n \end{pmatrix} = -L_{[1,m] \times [m+1,n]} \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \]
Solving for the Harmonic Function

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\]

This gives a symmetric system, defined over the interior vertices, with non-zero Laplacian constraints at vertices adjacent to the boundary.
Harmonic Mapping

Different boundaries give different mappings.
Harmonic Mapping

Sampling over a regular-grid in the 2D domain, we can get a (hierarchy of) quad-mesh.
Harmonic Mapping

Using the mapping we can texture the surface.
Harmonic Mapping

Note that getting a good texture-map requires choosing a “good” cut on the surface and a correspondingly “good” planar curve defining the constraints.

[Image: Showing the process of harmonic mapping from a 3D model to a 2D texture map, and then mapping the texture back to the 3D model.]

[Piponi ’00]
Harmonic Mapping

Limitations:
• The mapping does not preserve areas.
• The mapping does not preserve angles.
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• The mapping does not preserve areas.
• The mapping does not preserve angles.
• The mapping cannot be a parameterization if the surface is not topologically a disk.
• Even if it is, the mapping may have triangle-flips [Floater ’97].
In performing mean-curvature flow, we sought to minimize the Dirichlet energy by stepping in the direction opposite the gradient.
Q: Can we just solve the linear system directly to get the minimal surface?
Mean-Curvature Flow

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A: No. As the surface evolves along the mean-curvature direction, the Laplacian also changes.
Mean-Curvature Flow

Q: Can we just solve the linear system directly to get the minimal surface?

A: No. As the surface evolves along the mean-curvature direction, the Laplacian also changes. Iterating, we get to the minimal surface.