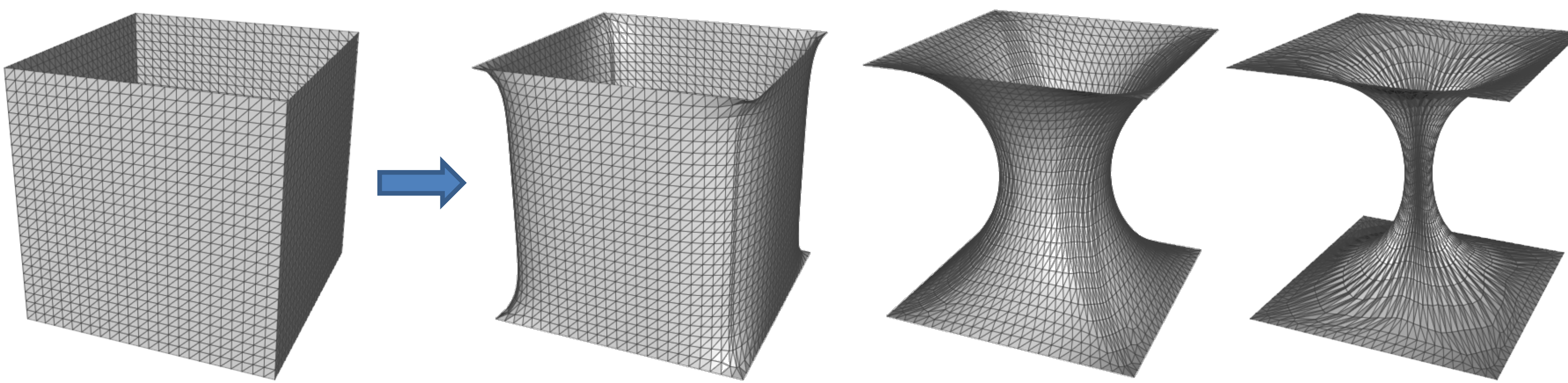
Differential Geometry:

Harmonics Maps and Energy Minimization



[*Multiresolution Analysis of Arbitrary Meshes*, Eck *et al*. 1995]

Dirichlet Energy

Recall:

Given a function $F=(f_1,\ldots,f_m):D\subset\mathbf{R}^n\rightarrow\mathbf{R}^m$, the *Dirichlet Energy* is a measure of how much the function F changes over D:

$$E(F)=\int_D \left|dF(p)\right|^2 dp=\sum_{i=1}^m \int_D \left|\nabla f_i(p)\right|^2 dp$$

In many applications, we seek functions that are "as smooth as possible", amounting to seeking functions F minimizing the Dirichlet Energy.

Dirichlet Energy

Recall:

Given a domain $D \subset \mathbf{R}^n$, the function $F: D \to \mathbf{R}^m$ that satisfies the boundary constraints:

$$F(p) = C(p) \quad \forall p \in \partial D$$

and minimizes the Dirichlet Energy is called a *harmonic function*.

Dirichlet Energy

<u>Recall</u>:

The condition that F is a harmonic function implies that for all functions G, with $G(p)=0$ for $p \in \partial D$, offsetting F in the direction of G will not decrease/increase the energy:

$$\lim_{\varepsilon \to 0} \frac{E(F+\varepsilon G) - E(F)}{\varepsilon} = 0$$

Dirichlet Energy

Recall:

The condition that *F* is a harmonic function implies that for all functions *G*, with *G*(*p*)=0 for $p \in \partial D$, offsetting *F* in the direction of *G* will not decrease/increase the energy:

$$\lim_{\varepsilon \to 0} \frac{E(F+\varepsilon G) - E(F)}{\varepsilon} = 0$$

$\Leftrightarrow$ The Laplacian of *F* is equal to zero.

Dirichlet Energy

Recall:

The Laplacian can be thought of as the gradient of the energy at the function *F* so the energy can be minimized by repeatedly offsetting *F* in the direction opposite the Laplacian:

$$F \leftarrow F - \varepsilon \Delta F$$

Dirichlet Energy

Recall:

In the case of smooth surfaces, if we consider the embedding function:

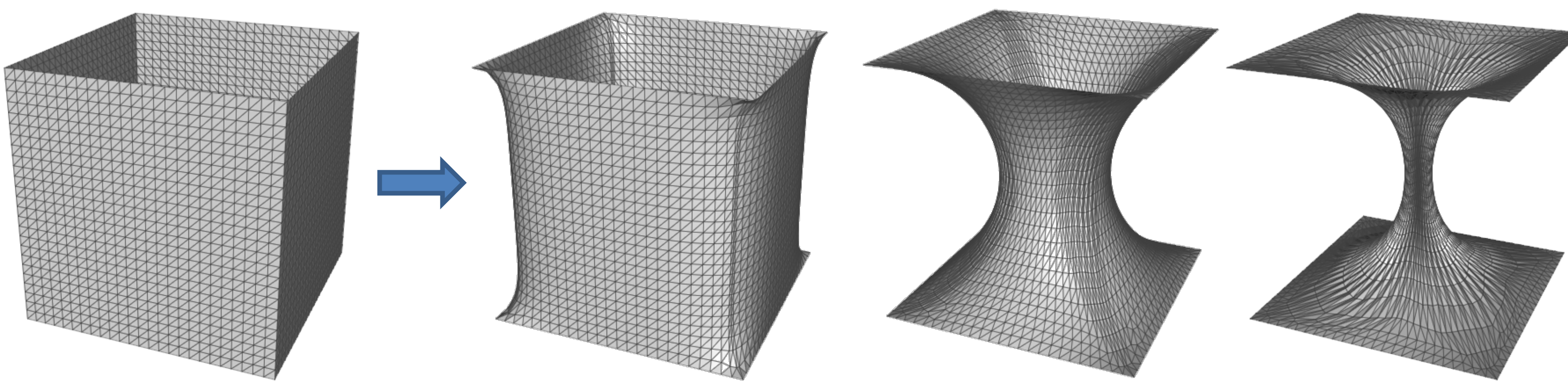
$$F(x,y,z) = (x,y,z)$$

- The Dirichlet energy is twice the area.
- The Laplacian is equal to the normal vector, scaled by the mean curvature.

Dirichlet Energy

<u>Recall</u>:

If we offset points on the surface in the direction of the negative mean curvature, we evolve the surface towards a smoother surface with smaller surface area.

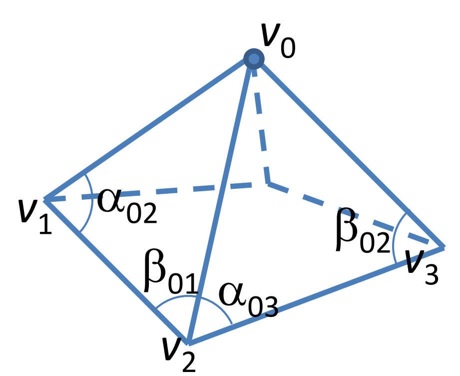


Dirichlet Energy

<u>Recall</u>:

In the case of discrete surfaces, if we consider functions as having values on the vertices, we get the cotangent-weight Laplacian:

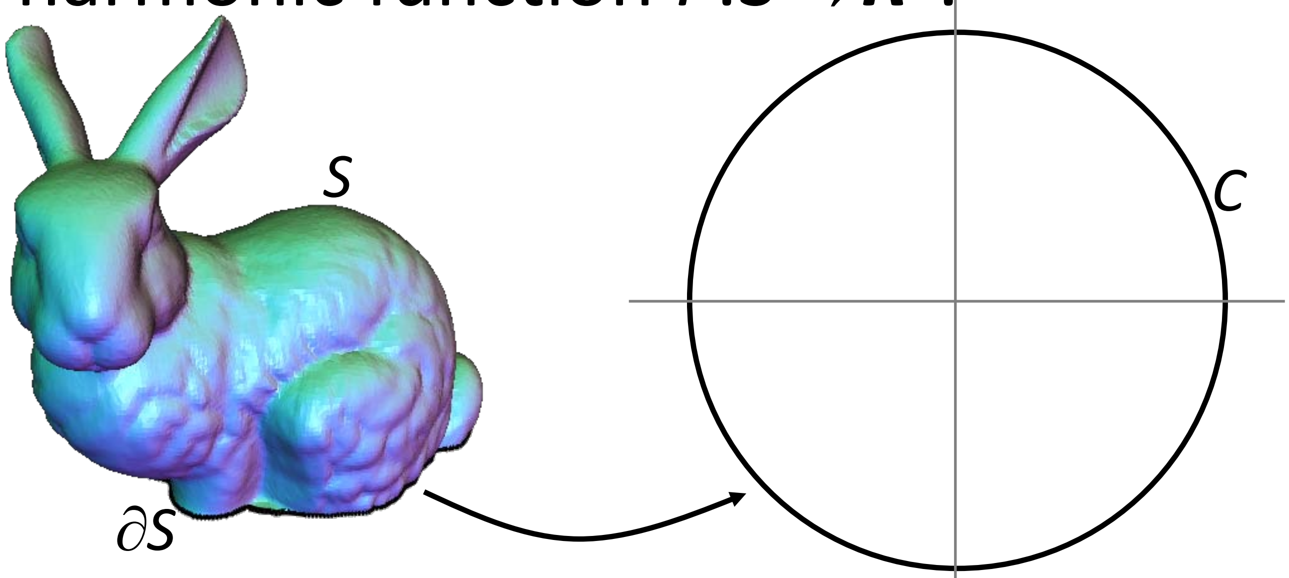
$$L_{ij} = \begin{cases} \frac{1}{2}\left(\cot(\alpha_{ij})p + \cot(\beta_{ij})\right) & \text{if } i \neq j \text{ and } v_j \in \text{Nbr}(v_j) \\ -\sum_{v_k \in \text{Nbr}(v_i)} L_{ik} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$



Parameterization

<u>Example</u>:

Given a surface *S* with a (connected) boundary, we can map *S* to the plane by mapping the boundary to a closed planar curve *C* and solving for the harmonic function $F:S\rightarrow\mathbf{R}^2$.



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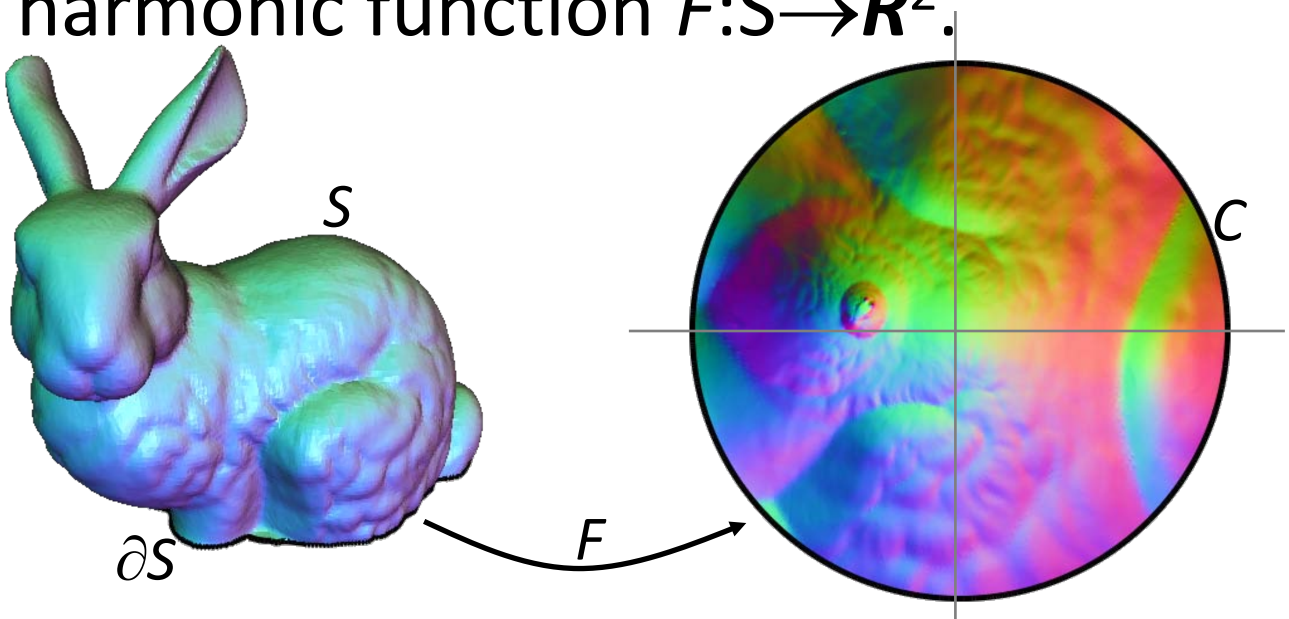


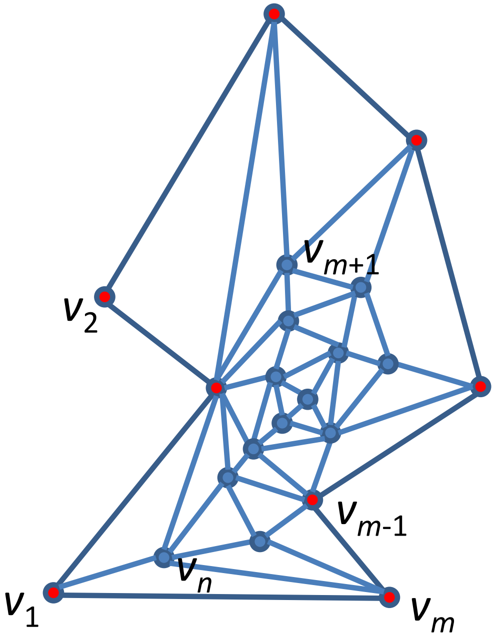
Solving for the Harmonic Function

Solve the linear system that gives the function *F* with the prescribed boundary values and has zero Laplacian at all (interior) vertices.

Solving for the Harmonic Function

We are given a mesh with vertices $\{v_1,..,v_m,v_{m+1},...,v_n\}\subset\boldsymbol{R}^3$:

- $\{v_1,..,v_m\}$ are boundary vertices
- $\{v_{m+1},...,v_n\}$ are interior vertices.



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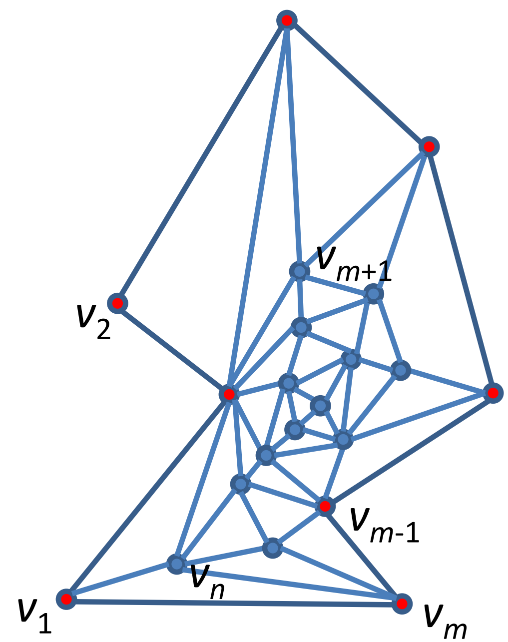


We have boundary constraints:

- $v_i\rightarrow c_i$, for all $1\le i\le m$, with $c_i\in\boldsymbol{R}^2$.

And interior constraints:

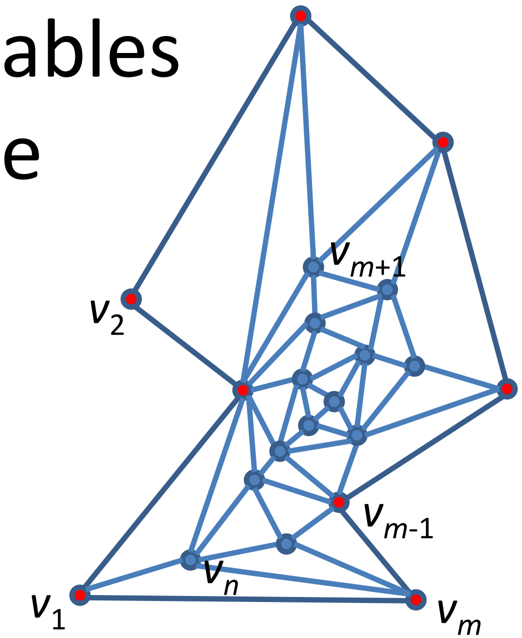
- $(Lv)_i=0$, for all $m+1\le i\le n$.

Solving for the Harmonic Function

<u>Defining the Linear System (1)</u>:

Treat the positions $\{p_1,\ldots,p_n\}$ as variables and explicitly add constraints on the boundary vertices:

$$\left(\begin{array}{ccc|c} 1 & \cdots & 0 & \\ \vdots & \ddots & \vdots & 0 \\ 0 & \cdots & 1 & \\ \hline \multicolumn{3}{c|}{L_{[1,m]x[m+1,n]}} & L_{[m+1,n]x[m+1,n]} \end{array}\right)\begin{pmatrix} p_1 \\ \vdots \\ p_m \\ p_{m+1} \\ \vdots \\ p_n \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_m \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$



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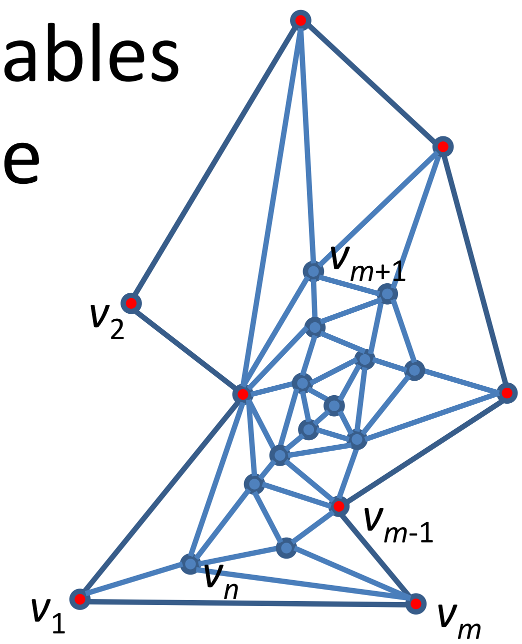


<u>Disadvantages</u>:

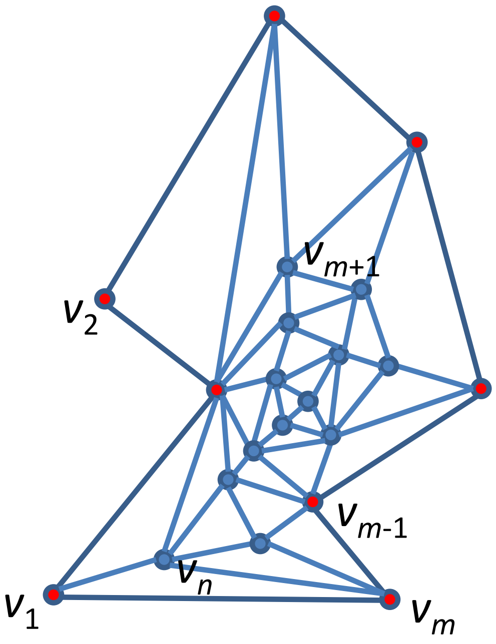
– The system is not symmetric.

– The system is larger than the number of unknown

Solving for the Harmonic Function

<u>Defining the Linear System (2)</u>:

$$\left(\begin{array}{ccc|c} 1 & \cdots & 0 & \\ \vdots & \ddots & \vdots & 0 \\ 0 & \cdots & 1 & \\ \hline \multicolumn{3}{c|}{L_{[1,m]x[m+1,n]}} & L_{[m+1,n]x[m+1,n]} \end{array}\right)\begin{pmatrix} p_1 \\ \vdots \\ p_m \\ p_{m+1} \\ \vdots \\ p_n \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_m \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$



Since the boundary constraints are fixed, we can re-write the above system as:

$$L_{[1,m]x[m+1,n]}\begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} + L_{[m+1,n]x[m+1,n]}\begin{pmatrix} p_{m+1} \\ \vdots \\ p_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

Solving for the Harmonic Function

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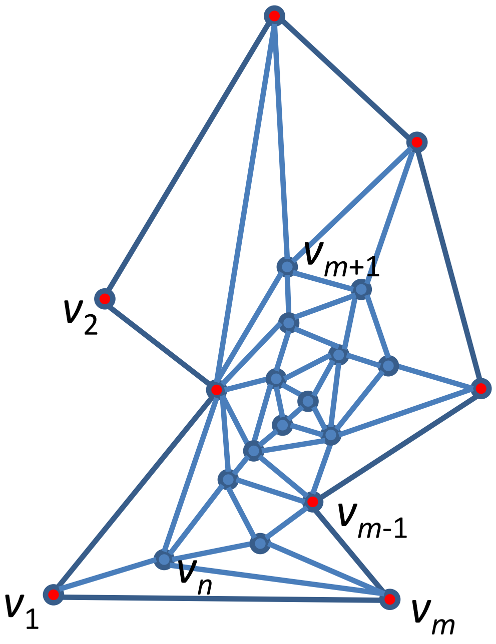
Or equivalently:

$$L_{[m+1,n]x[m+1,n]}\begin{pmatrix} p_{m+1} \\ \vdots \\ p_n \end{pmatrix} = -L_{[1,m]x[m+1,n]}\begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}$$



Solving for the Harmonic Function

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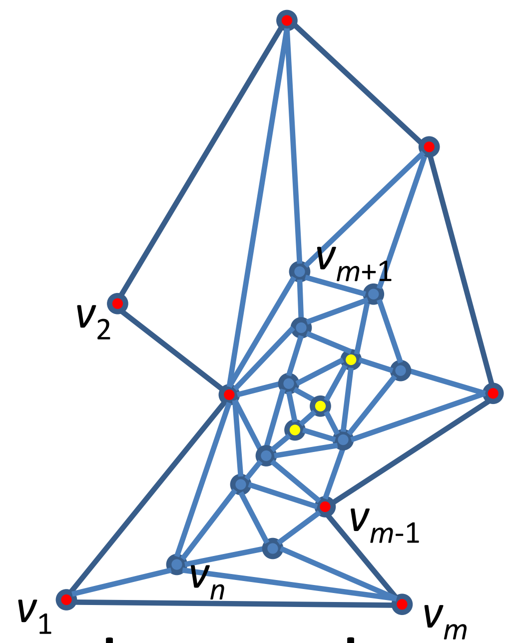
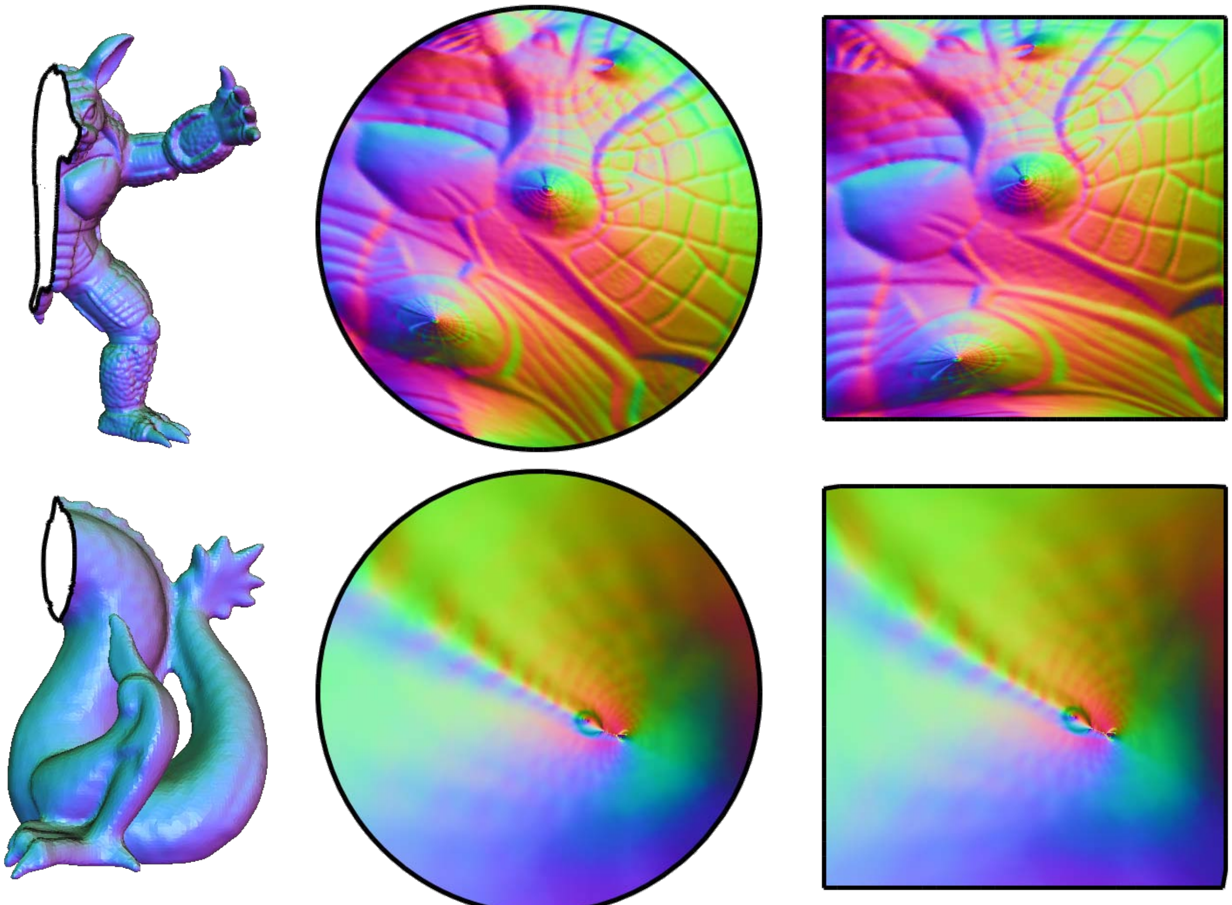


This gives a symmetric system, defined over the interior vertices, with non-zero Laplacian constraints at vertices adjacent to the boundary.

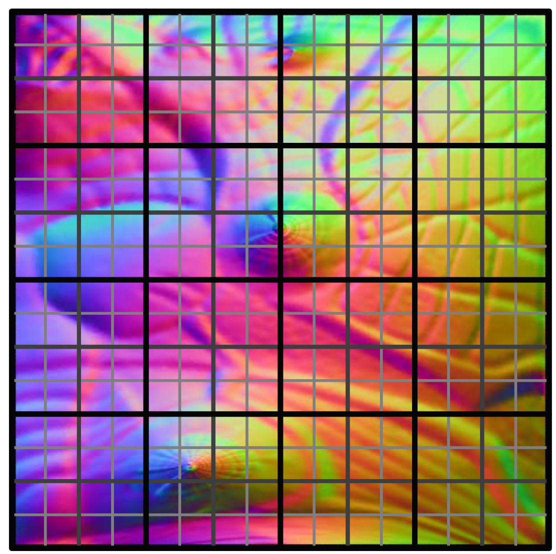
Harmonic Mapping

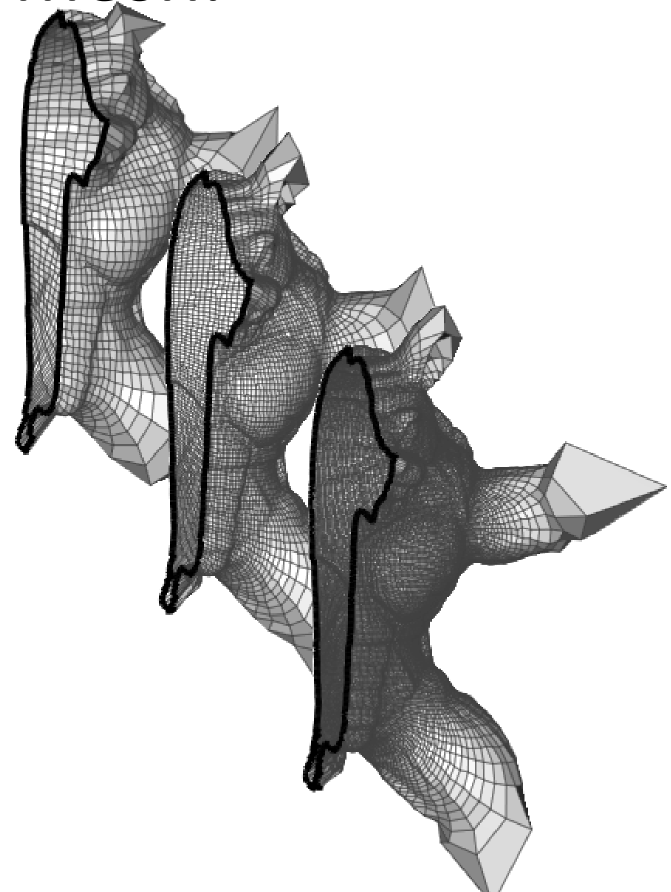
Different boundaries give different mappings.



Harmonic Mapping

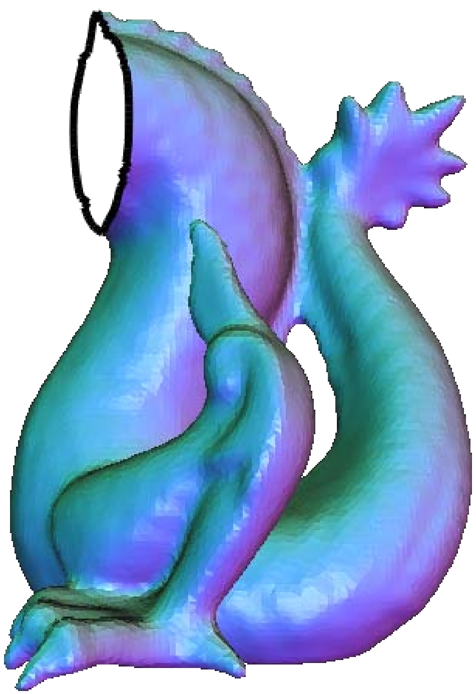
Sampling over a regular-grid in the 2D domain, we can get a (hierarchy of) quad-mesh.

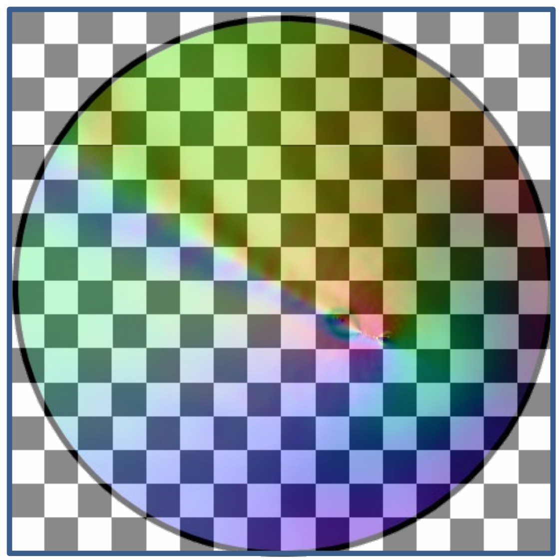




Harmonic Mapping

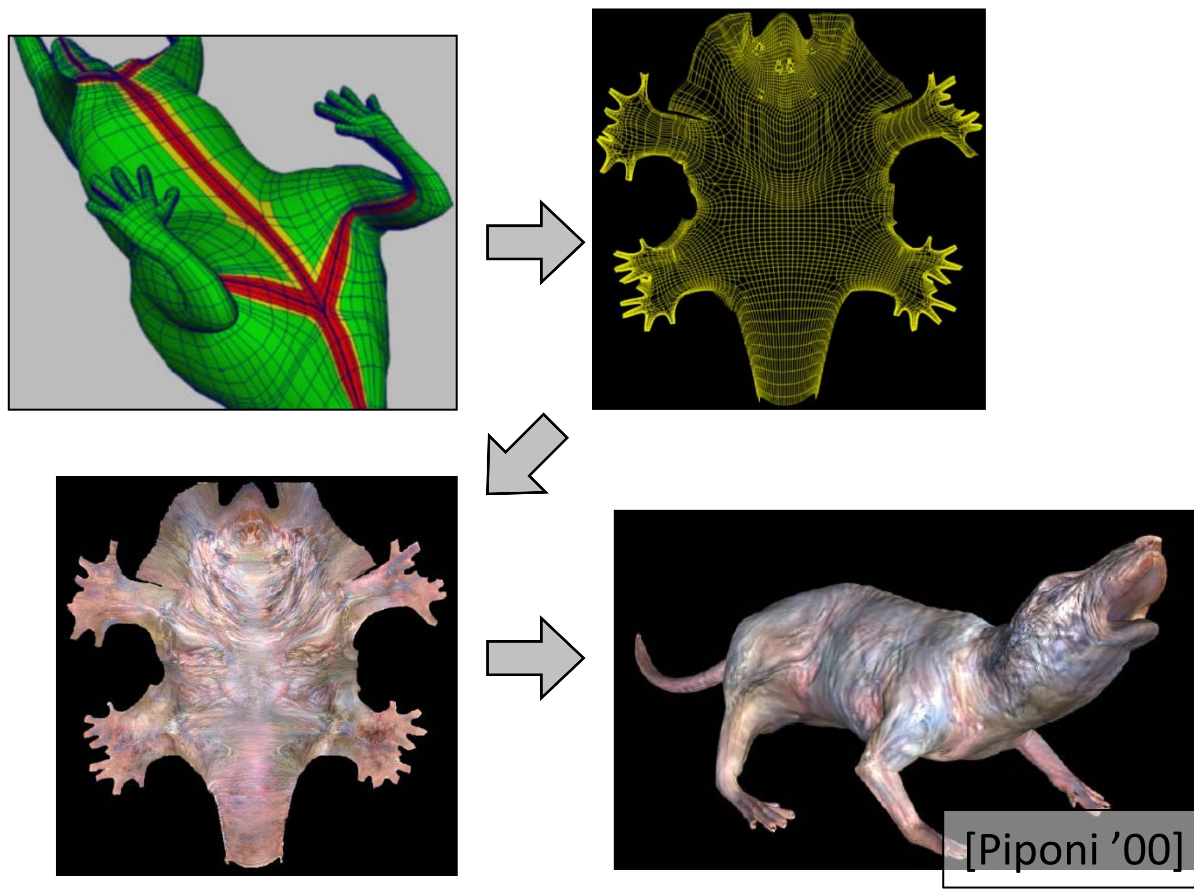
Using the mapping we can texture the surface.





Harmonic Mapping

Note that getting a good texture-map requires choosing a "good" cut on the surface and a correspondingly "good" planar curve defining the constraints.

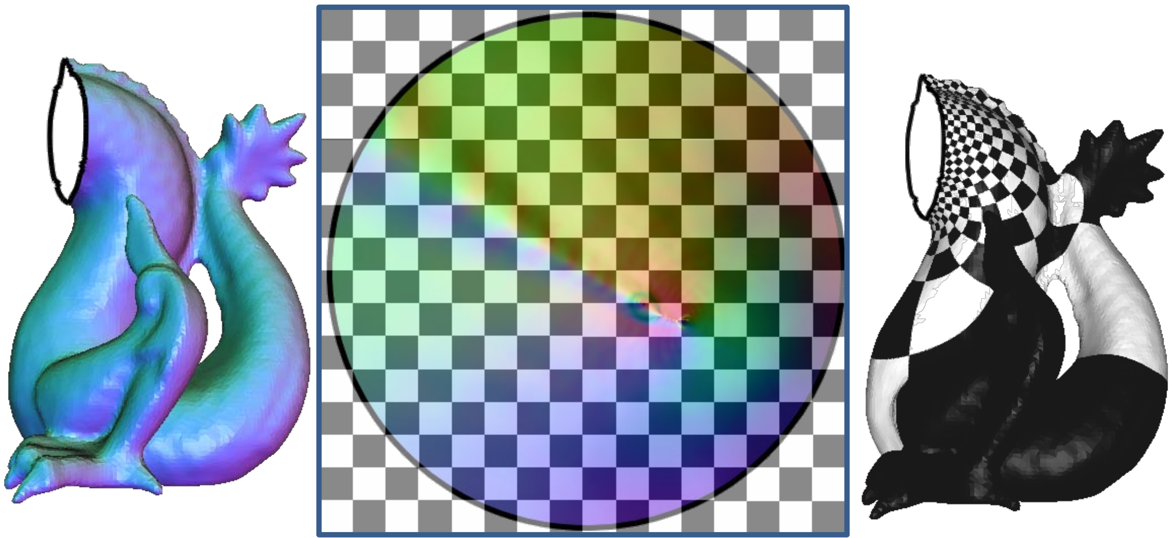


[Piponi '00]

Harmonic Mapping

<u>Limitations</u>:

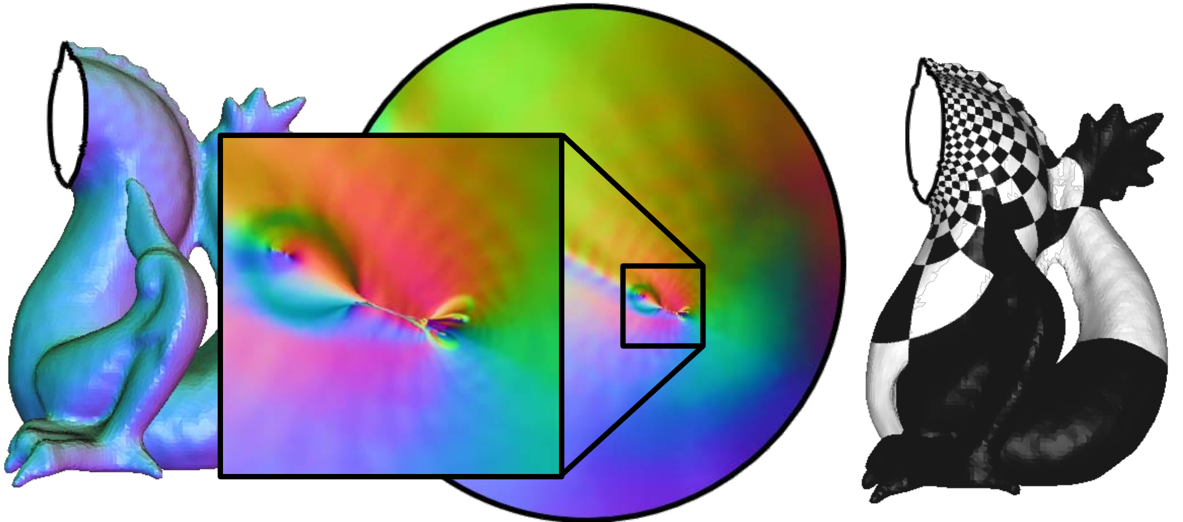
- The mapping does not preserve areas.
- The mapping does not preserve angles.



Harmonic Mapping

Limitations:

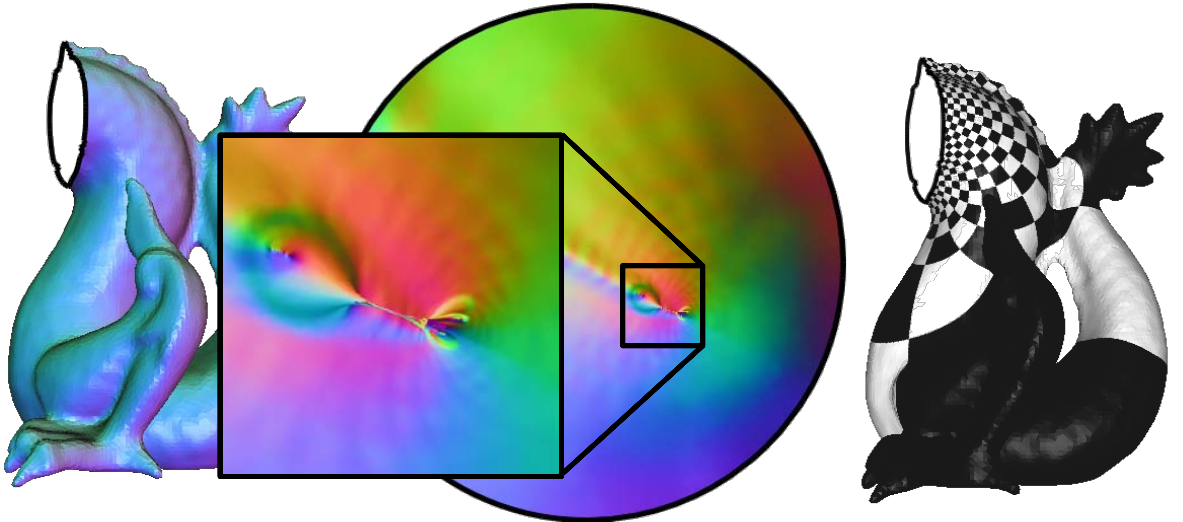
- The mapping does not preserve areas.
- The mapping does not preserve angles.
- The mapping cannot be a parameterization if the surface is not topologically a disk.



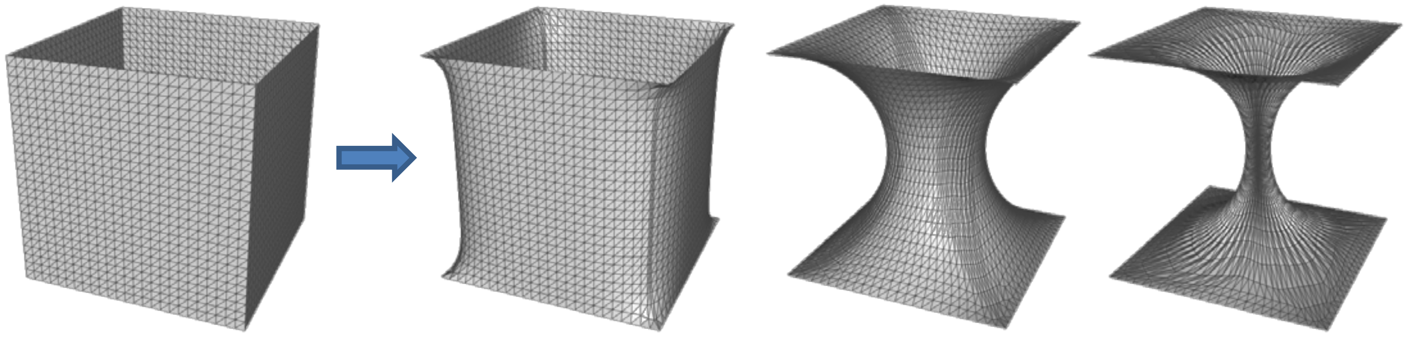
Harmonic Mapping

<u>Limitations</u>:

- The mapping does not preserve areas.
- The mapping does not preserve angles.
- The mapping cannot be a parameterization if the surface is not topologically a disk.
- Even if it is, the mapping may have triangle-flips [Floater '97].

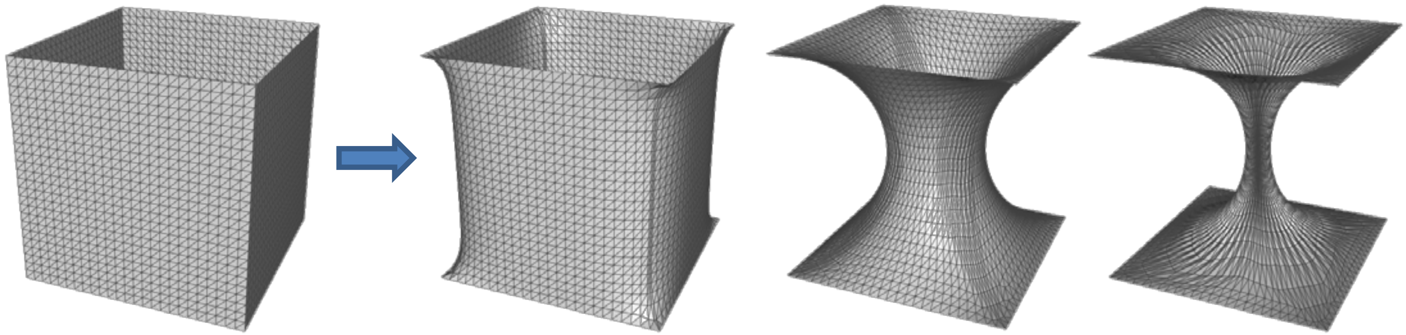


Mean-Curvature Flow



In performing mean-curvature flow, we sought to minimize the Dirichlet energy by stepping in the direction opposite the gradient.

Mean-Curvature Flow



Q: Can we just solve the linear system directly to get the minimal surface?

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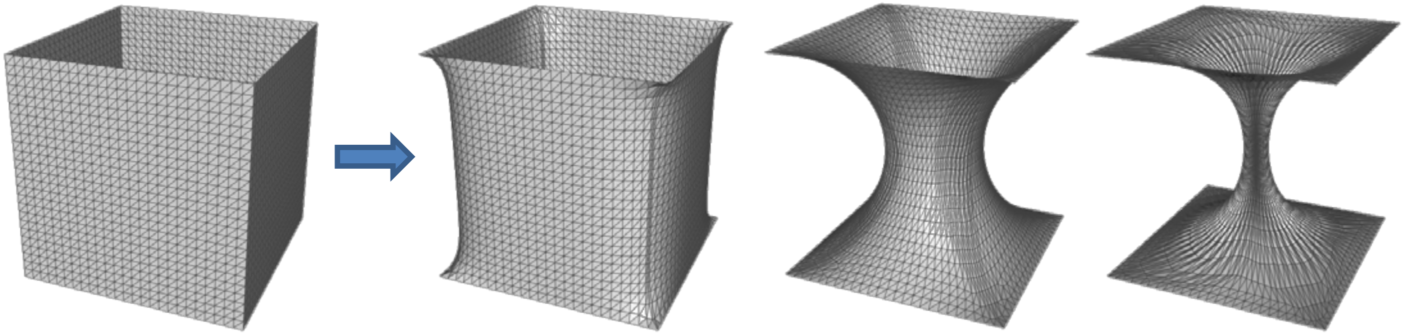

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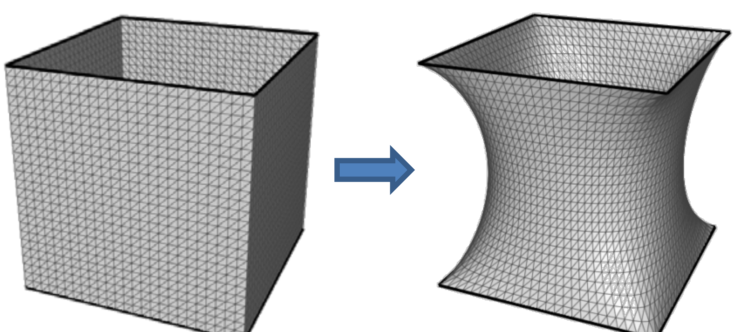

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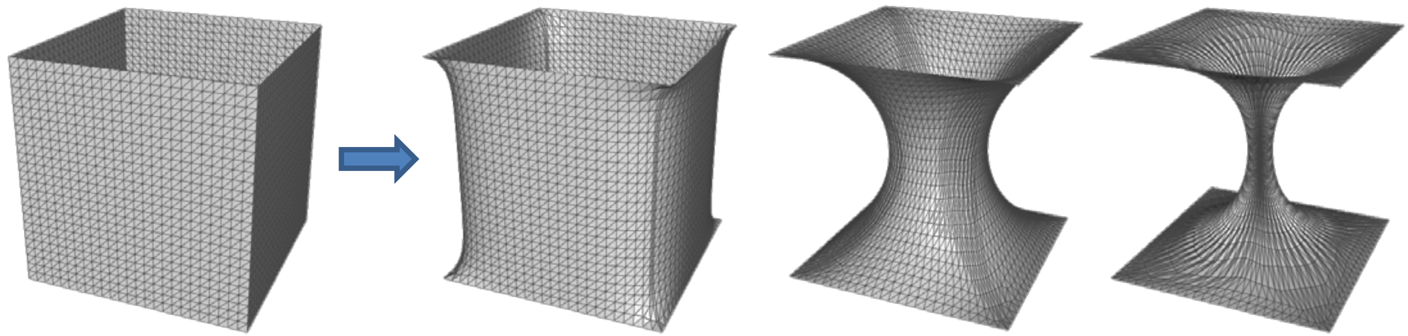
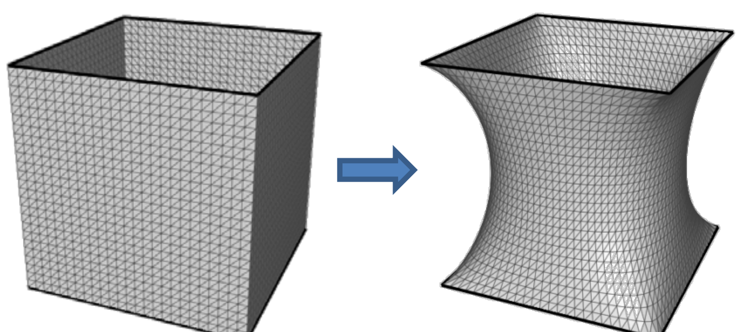


Q: Can we just solve the linear system directly to get the minimal surface?

A: No. As the surface evolves along the mean-curvature direction, the Laplacian also changes.



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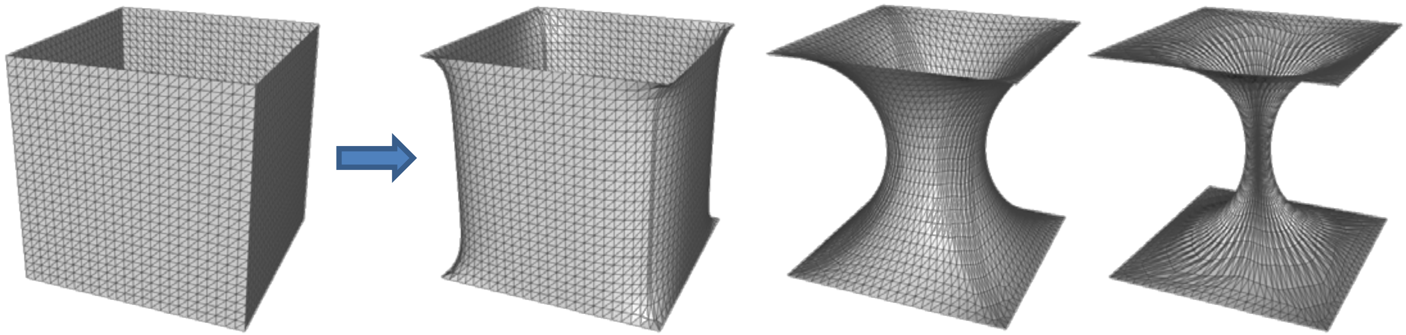

Q: Can we just solve the linear system directly to get the minimal surface?

A: No. As the surface evolves along the mean-curvature direction, the Laplacian also changes. Iterating, we get to the minimal surface.