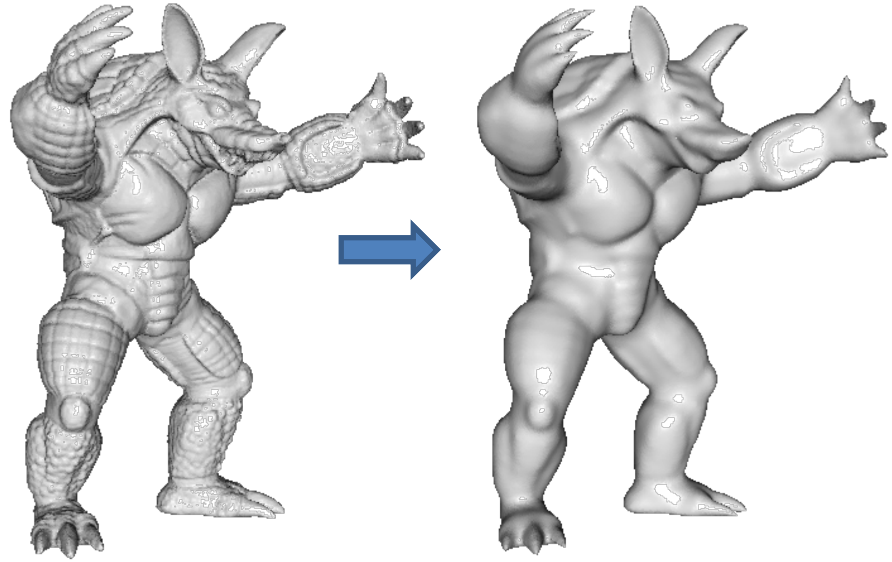
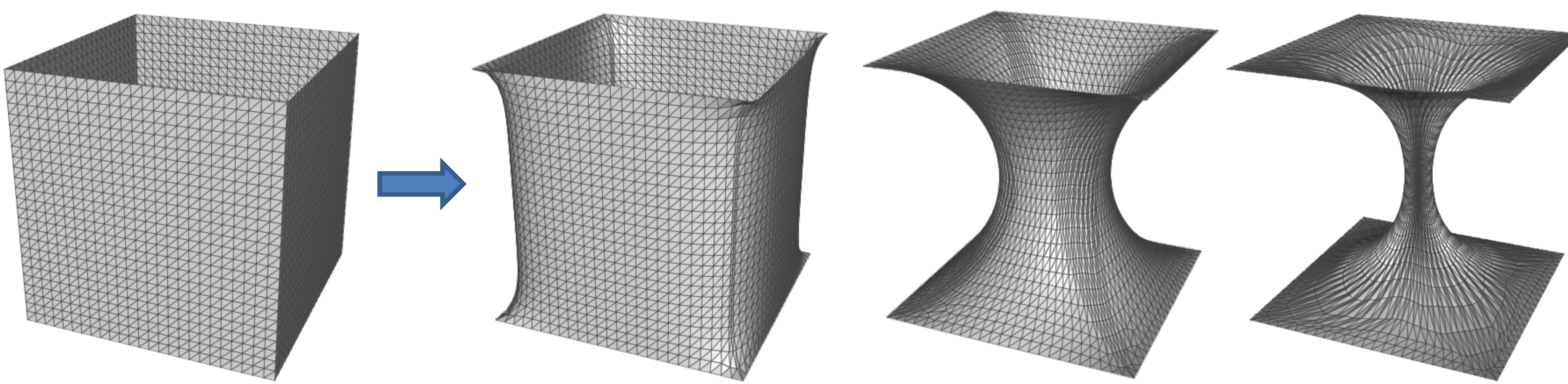


Differential Geometry:

Mean Curvature Flow



Math Review

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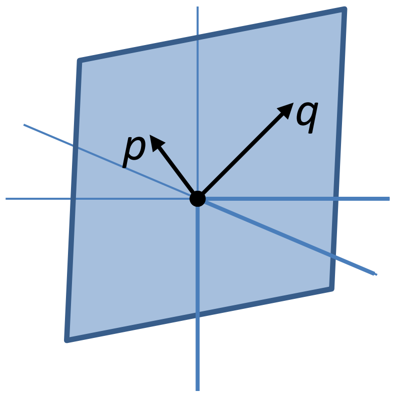

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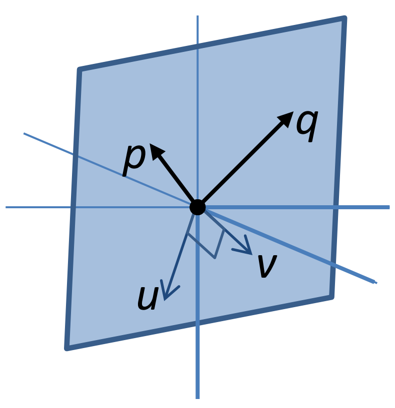
Given two (lin. ind.) vectors p, q in $\boldsymbol{R}^n$, we know that these vectors span a 2D subspace.

Given a vector $u=ap+bq$ in this subspace:

- How can we find a vector v in the subspace that is perpendicular to u?



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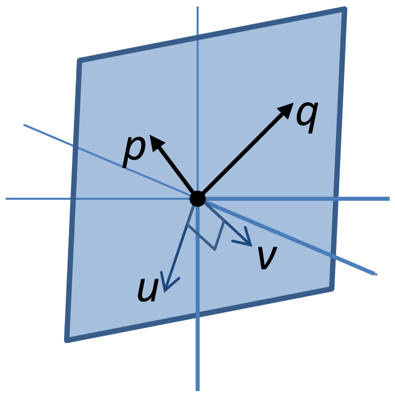

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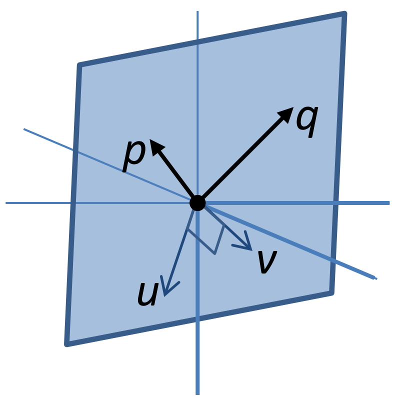
If *u* and *v* are orthonormal, this is easy:

$$v = \lambda \frac{-bp + aq}{\sqrt{a^2 + b^2}}$$



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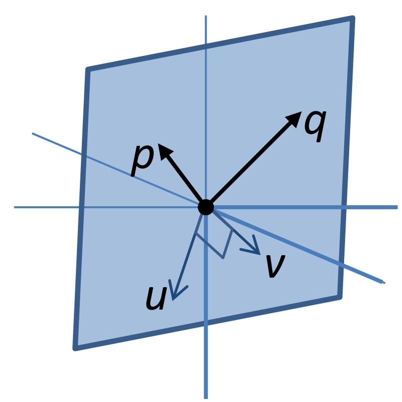
If *u* and *v* are orthogonal, it's still easy:

$$v = \lambda \frac{-bp + aq}{\|-bp + aq\|} = \lambda \frac{-bp + aq}{\sqrt{b^2\|p\|^2 + a^2\|q\|^2}}$$



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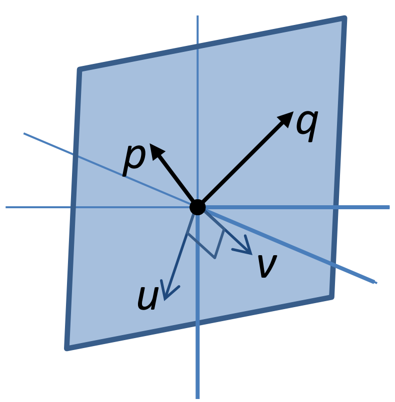
If u and v are orthogonal, it's still easy.

What about for arbitrary p and q?

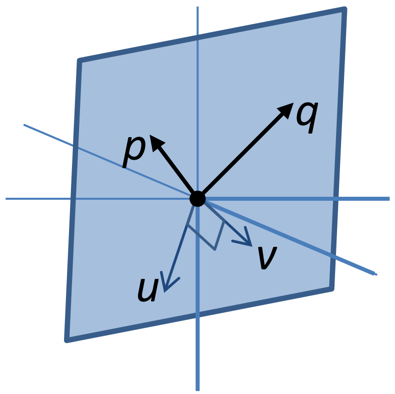


Math Review

Given two (lin. ind.) vectors p, q in $\boldsymbol{R}^n$, we know that these vectors span a 2D subspace.

We can represent vectors in the 2D subspace by their coordinates with respect to p and q:

$$(a, b) \mapsto ap + bq$$



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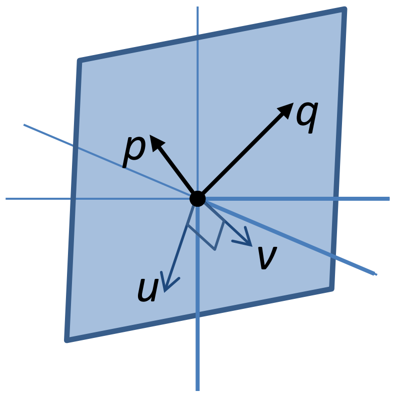
This induces a new inner product on $\mathbf{R}^2$ as:

$$\begin{aligned}\left\langle (a,b),(c,d) \right\rangle_{\{p,q\}} &= \left\langle ap+bq, cp+dq \right\rangle \\ &= ac\left\langle p,p \right\rangle + bd\left\langle q,q \right\rangle + (ad+bc)\left\langle p,q \right\rangle\end{aligned}$$



Math Review

Given two (lin. ind.) vectors *p, q* in ***R***^{n}, ….

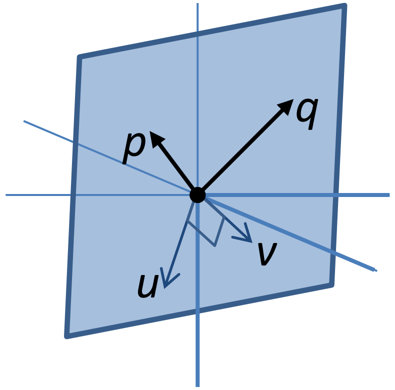
$$\left\langle (a,b),(c,d) \right\rangle_{\{p,q\}} = ac\left\langle p,p \right\rangle + bd\left\langle q,q \right\rangle + (ad+bc)\left\langle p,q \right\rangle$$

We can represent the induced inner-product by the symmetric (positive definite) matrix *M*:

$$M = \begin{pmatrix} \left\langle p,p \right\rangle & \left\langle p,q \right\rangle \\ \left\langle p,q \right\rangle & \left\langle q,q \right\rangle \end{pmatrix}$$

allowing us to represent the induced inner product as:

$$\left\langle (a,b),(c,d) \right\rangle_{\{p,q\}} = \begin{pmatrix} a & b \end{pmatrix} M \begin{pmatrix} c \\ d \end{pmatrix}$$



Math Review

$$M=\begin{pmatrix}\langle p,p\rangle & \langle p,q\rangle\\ \langle p,q\rangle & \langle q,q\rangle\end{pmatrix}$$

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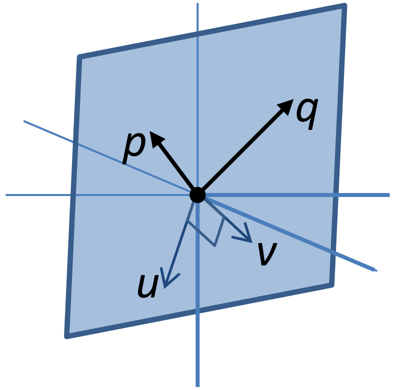
$$\langle (a,b),(c,d)\rangle_{\{p,q\}}=\begin{pmatrix}a & b\end{pmatrix}M\begin{pmatrix}c\\ d\end{pmatrix}$$

So, the vector represented by (*a*,*b*) will be orthogonal to the vector represented by (*c*,*d*) if:

$$0=\begin{pmatrix}a & b\end{pmatrix}M\begin{pmatrix}c\\ d\end{pmatrix}$$



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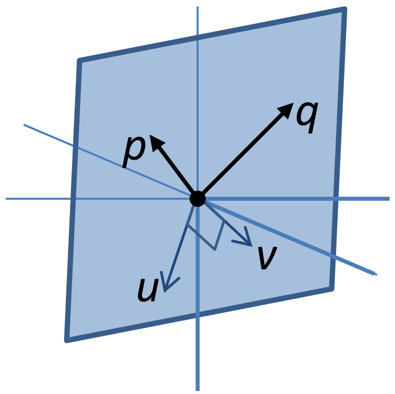
Thus, if we set $(c,d)=M^{-1}(-b,a)$, we get:

$$\begin{aligned}\langle (a,b),(c,d)\rangle_{\{p,q\}} &= \begin{pmatrix}a & b\end{pmatrix} M \begin{pmatrix}c\\ d\end{pmatrix}\\ &= \begin{pmatrix}a & b\end{pmatrix} M M^{-1} \begin{pmatrix}-b\\ a\end{pmatrix}\\ &= 0\end{aligned}$$



Math Review

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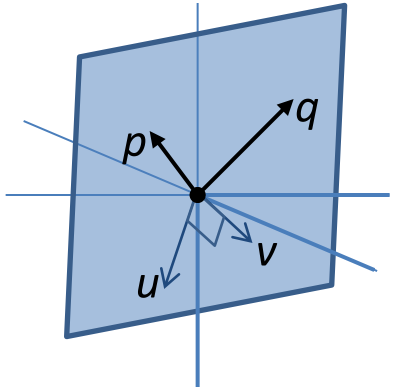
Thus, if we set $(c,d)=M^{-1}(-b,a)$, we get:

$$\langle (a,b),(c,d)\rangle_{\{p,q\}} = 0$$

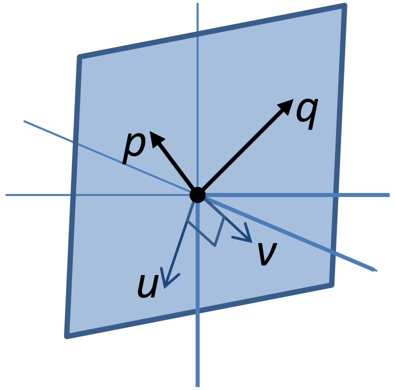
And, any scalar multiple $M^{-1}(-\varepsilon b, \varepsilon a)$ will also be orthogonal.



Math Review

So, if we would also like the vector (*c*,*d*) to have length λ, we can set:

$$\begin{pmatrix} c \\ d \end{pmatrix} = \lambda \frac{M^{-1}\begin{pmatrix} -b \\ a \end{pmatrix}}{\left\| M^{-1}\begin{pmatrix} -b \\ a \end{pmatrix} \right\|_{\{p,q\}}}$$



Dirichlet Energy

Given a function $F=(f_1,\ldots,f_m):D\subset \mathbf{R}^n \rightarrow \mathbf{R}^m$, the *Dirichlet Energy* is a measure of how much the function F changes over D:

$$E(F)=\int_D \left|dF(p)\right|^2 dp = \sum_{i=1}^{m} \int_D \left|\nabla f_i(p)\right|^2 dp$$

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Without constraints, the solution is trivially F=0.

Dirichlet Energy

<u>Challenge</u>:

Given a domain $D \subset \mathbf{R}^n$, solve for the function $F: D \rightarrow \mathbf{R}^m$ that satisfies the boundary constraints:

$$F(p) = C(p) \quad \forall p \in \partial D$$

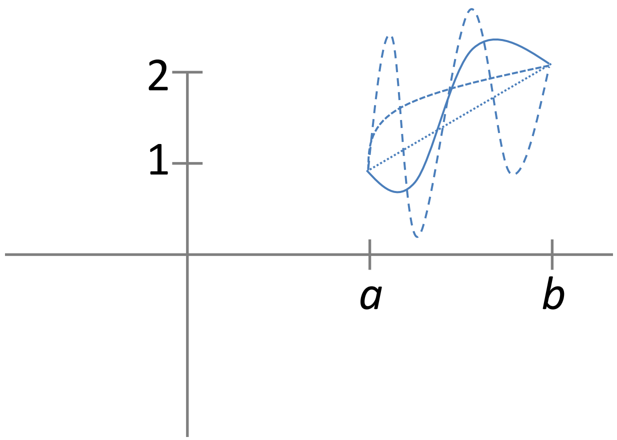
and minimizes the Dirichlet Energy.

Dirichlet Energy

$$E(F)=\int_D |dF(p)|^2\,dp=\sum_{i=1}^{m}\int_D |\nabla f_i(p)|^2\,dp$$

Example:

The domain is $D=[a,b]$ and the constraints are $C(a)=1$ and $C(b)=2$.



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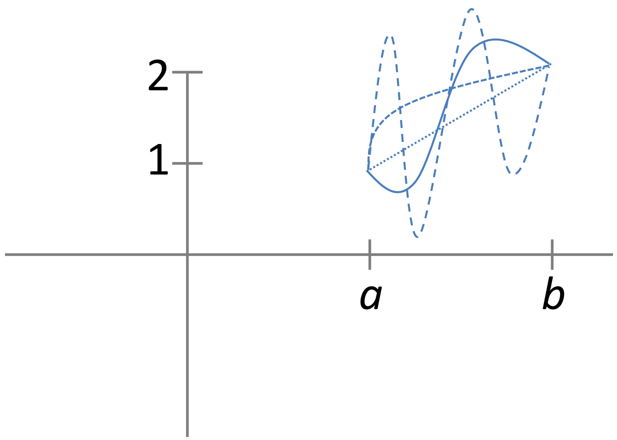
Since $C(a)=1$ and $C(b)=2$, we know that the total change from a to b has to be 1.



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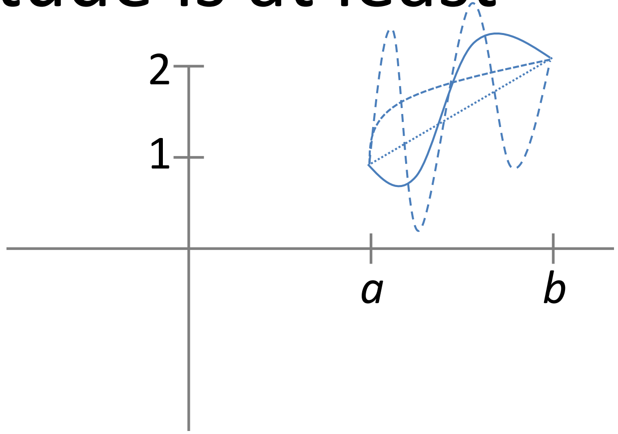
Thus, the average gradient magnitude is at least $1/(b-a)$ and the total integral is minimized if the magnitude is everywhere equal to the average.



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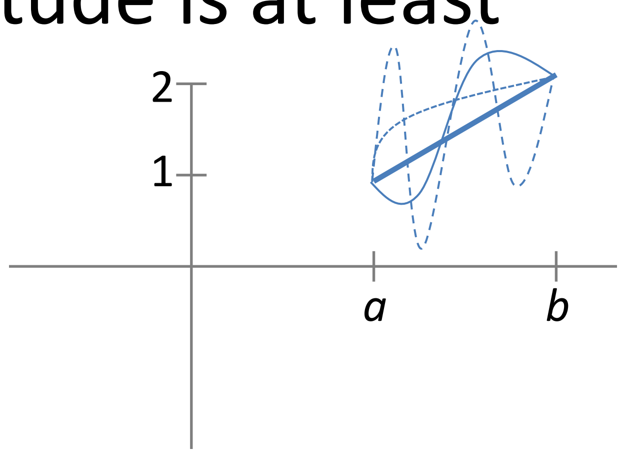

That is, if F is linear.

Harmonic Functions

In higher dimensions, the function F that satisfies the boundary constraints and minimizes the energy is called a *harmonic function* and is (generally) not linear.

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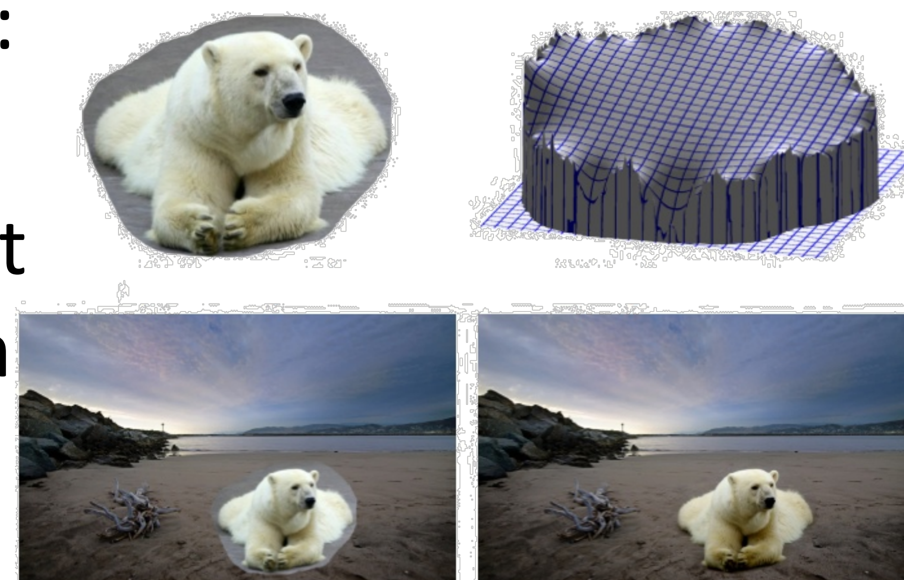
<u>Example (Image Cloning)</u>:

We copy the pattern into the target and then offset by the harmonic function *f* that interpolates the boundary differences.



Farbman *et al*. SIGGRAPH 2009

Dirichlet Energy

To solve for the (locally) optimal function *F*, we want to ensure that if we change the values of *F* slightly, the energy cannot decrease.

Dirichlet Energy

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Given a function *G* with zero values along the boundary, we can think of *G* as the offset values.

Dirichlet Energy

The condition that *F* is a local extremum of the energy implies that:

$$0 = \lim_{\varepsilon \to 0} \frac{E(F + \varepsilon G) - E(F)}{\varepsilon}$$

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$$= \lim_{\varepsilon \to 0} \frac{\int_D \left(\left| dF(p) \right|^2 + 2\varepsilon \left\langle dF(p), dG(p) \right\rangle + \varepsilon^2 \left| dG(p) \right|^2 \right) dp - \int_D \left| dF(p) \right|^2 dp}{\varepsilon}$$

Dirichlet Energy

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$$
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&= \lim_{\varepsilon \to 0} \frac{\int_D \left(\left| dF(p) \right|^2 + 2\varepsilon \left\langle dF(p), dG(p) \right\rangle + \varepsilon^2 \left| dG(p) \right|^2 \right) dp - \int_D \left| dF(p) \right|^2 dp}{\varepsilon} \\
&= 2\int_D \left\langle dF(p), dG(p) \right\rangle dp + \lim_{\varepsilon \to 0} \varepsilon \int_D \left| dG(p) \right|^2 dp
\end{aligned}
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&= 2\int_D \left\langle dF(p), dG(p) \right\rangle dp + \lim_{\varepsilon \to 0} \varepsilon \int_D \left| dG(p) \right|^2 dp \\
&= 2\sum_{i=1}^{m} \int_D \left\langle \nabla f_i(p), \nabla g_i(p) \right\rangle dp = -2\sum_{i=1}^{m} \int_D \Delta f_i(p) g_i(p) dp
\end{aligned}
$$

Dirichlet Energy

Thus, the function is a local extremum of the Dirichlet energy if:

$$0 = \int_D \Delta f_i(p) g_i(p) dp$$

for every coordinate offset function g_i.

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That is, F is optimal if it satisfies the boundary condition and has the property that the Laplacian of the coordinate functions are 0.

Dirichlet Energy

Additionally, since the "change in direction *G*" is the inner-product of *G* with the Laplacian of *F*:

$$\lim_{\varepsilon \to 0} \frac{E(F+\varepsilon G)-E(F)}{\varepsilon} = -2\sum_{i=1}^{m}\int_D \Delta f_i(p)\, g_i(p)\, dp$$

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we can think of the Laplacian of the coordinates of *F* as the "gradient" of the energy.

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we can think of the Laplacian of the coordinates of *F* as the "gradient" of the energy.

<u>Laplacian Smoothing</u>:

This means that if we start with an arbitrary function *F*, we can minimize the energy by stepping in the direction opposite the Laplacian:

$$f_i \leftarrow f_i - \varepsilon\Delta f_i$$

Dirichlet Energy and Laplacians

Formally, the Laplacian of *f* at a point *p* is defined as the sum of second partial derivatives:

$$\Delta f = \frac{\partial^2 f}{\partial x_1^2} + \ldots + \frac{\partial^2 f}{\partial x_n^2}$$

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Informally, the Laplacian measures how different the value of *f* at *p* is from the average value of the neighbors.

Dirichlet Energy

One can extend the notion of Dirichlet Energy to a regular surface *S*, to obtain an anologous formulation:

$$\lim_{\varepsilon \to 0} \frac{E(F+\varepsilon G)-E(F)}{\varepsilon} = -2\sum_{i=1}^{m} \int_S \Delta_S f_i(p) g_i(p) dp$$

Where Δ_S is the analog of the Laplacian defined over *S*, called the *Laplace-Beltrami* operator.

Dirichlet Energy

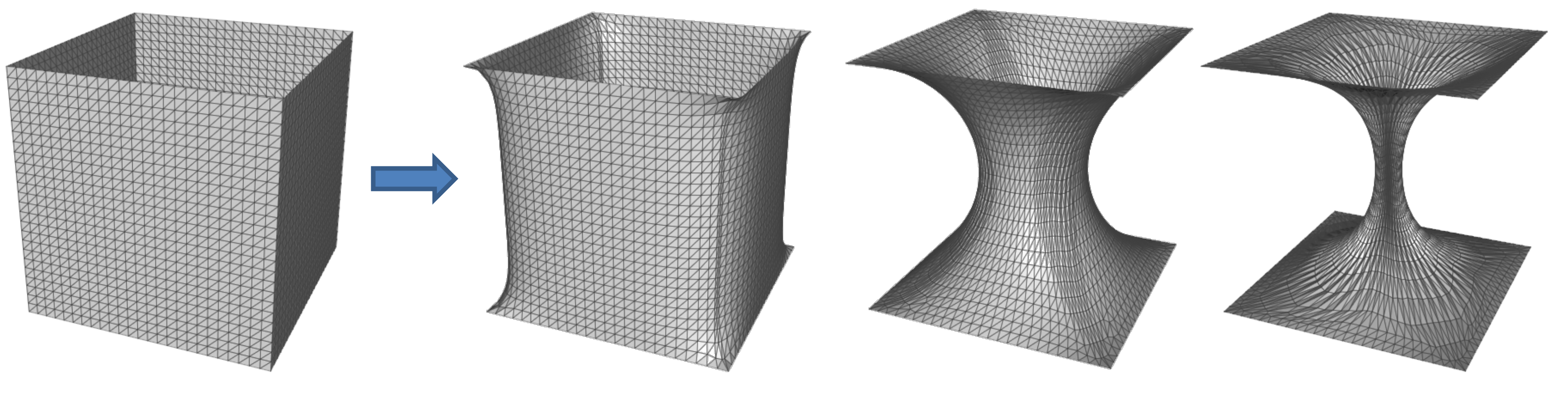
On a surface *S*, if one considers the embedding function:

$$F(x, y, z) = (x, y, z)$$

the Dirichlet Energy becomes twice the area, and the Laplacian is equal to the normal, scaled by the mean curvature.

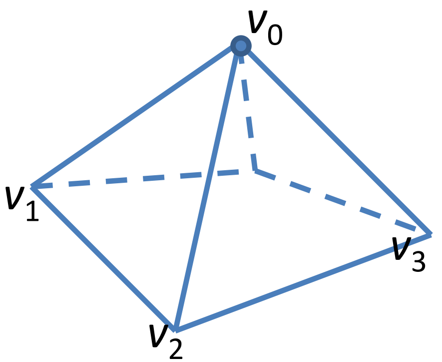
Mean Curvature Flow

Thus, if we offset points on the surface in the direction of the negative mean curvature, we evolve the surface towards a smoother surface with smaller surface area.



Discrete Laplacians

Given a triangle mesh (*V*,*E*,*F*), how should we define the Laplacian?

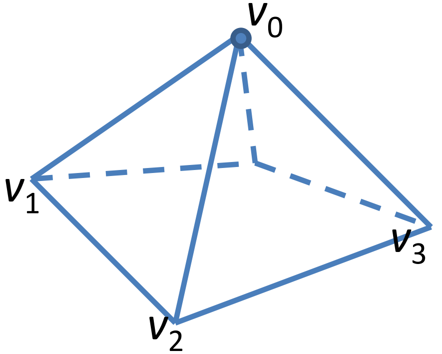


Discrete Laplacians

The Laplacian should be a linear operator that takes a function defined on the mesh vertices and returns a function defined on the vertices.

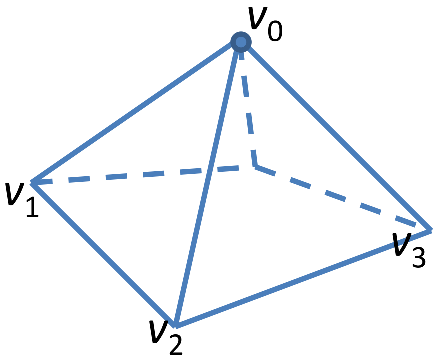
So if there are $|V|$ vertices on the mesh, it can be represented by a $|V| \mathrm{x} |V|$ matrix.

$$g_i = \sum_{v_j \in V} L_{ij} f_j$$



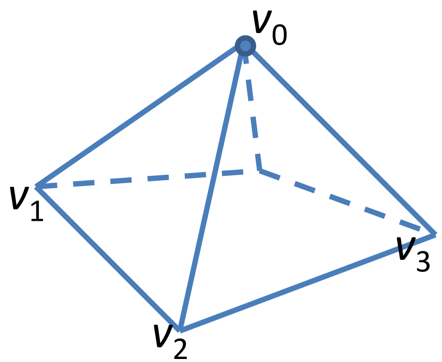
Discrete Laplacians

- Sparse:
  Since we are only interested in the average value at neighboring vertices, distant vertices should not effect the value.



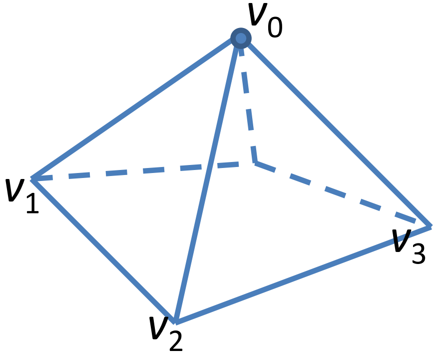
Discrete Laplacians

- Sparse
- Positivity:
  When averaging of the neighbors' values, we want to use non-negative linear combinations.



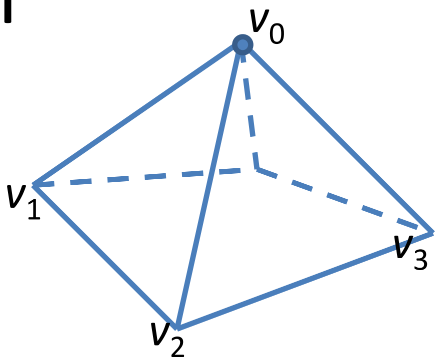
Discrete Laplacians

- <u>Sparse</u>
- <u>Positivity</u>
- <u>Symmetry</u>:
  As in the continuous case, we want the matrix to be symmetric PSD.



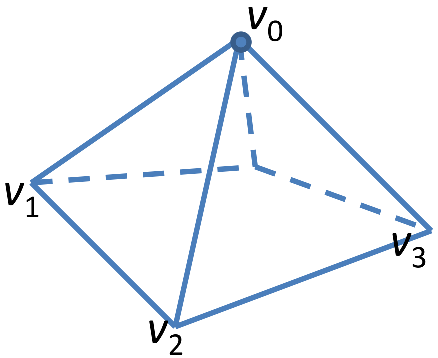
Discrete Laplacians

- Sparse
- Positivity
- Symmetry
- Linear Precision:
  If the mesh lives in a plane and the function values are obtained by sampling a linear function, the Laplacian of the function should be zero.



Discrete Laplacians

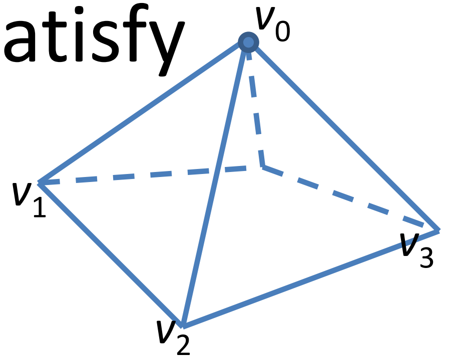
- Sparse
- Positivity
- Symmetry
- Linear Precision
- Convergence:
  The discrete Laplacian should converge to the smooth Laplacian under mesh refinement.



Discrete Laplacians

- Sparse
- Positivity
- Symmetry
- Linear Precision
- Convergence

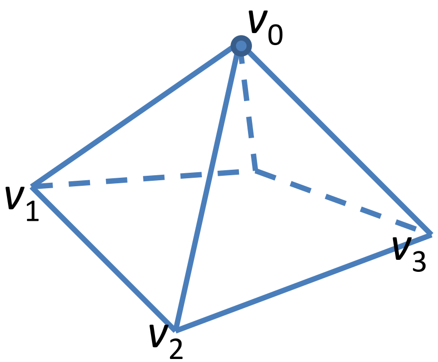
[Wardetzky *et al*. 2007] Show that though it is desirable, it is not actually possible to satisfy all of these properties simultaneously.



Discrete Laplacians

Note that since the Laplacian measures the difference between a vertex and the average values of its neighbors, we expect that is equal to zero on constant functions, so:

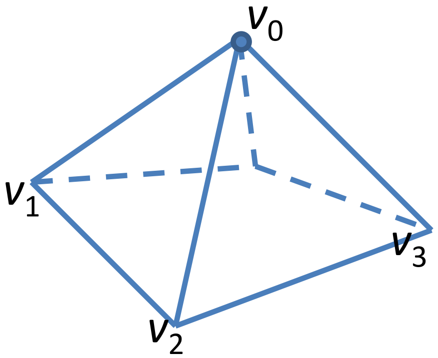
$$L_{ii} = -\sum_{v_j \in V | i \neq j} L_{ij}$$



Discrete Laplacians

Additionally, if the Laplacian is supported in the 1-ring of a vertex, so that averaging is only performed over immediate neighbors:

$$g_i = L_{ii} f_i + \sum_{v_j \in \mathrm{Nbr}(v_i)} L_{ij} f_j$$



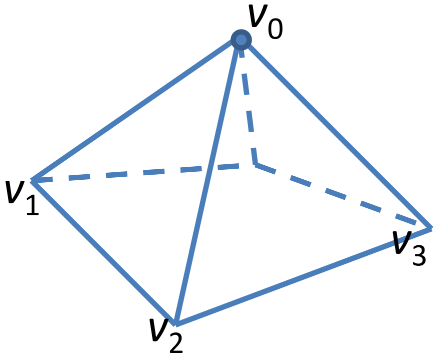
Discrete Laplacians

Additionally, if the Laplacian is supported in the 1-ring of a vertex, so that averaging is only performed over immediate neighbors:

$$g_i = L_{ii} f_i + \sum_{v_j \in \mathrm{Nbr}(v_i)} L_{ij} f_j$$

then we can think of the Laplacian as weights associated to the (directed) edges of a graph:

$$g_i = \sum_{v_j \in \mathrm{Nbr}(v_i)} L_{ij} \left(f_j - f_i \right)$$

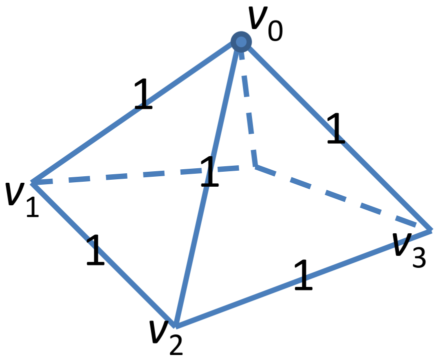


Discrete Laplacians

$$g_i = \sum_{v_j \in \mathrm{Nbr}(v_i)} L_{ij}\left(f_j - f_i\right)$$

Some Possibilities

- Tutte Laplacian: L_{ij}=1

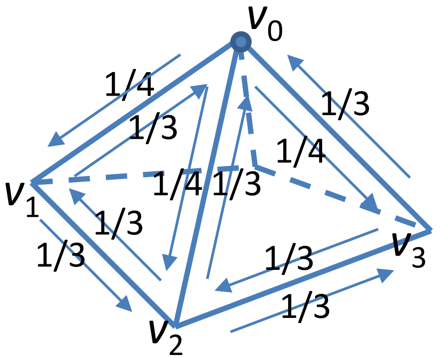


Discrete Laplacians

$$g_i = \sum_{v_j \in \mathrm{Nbr}(v_i)} L_{ij}\left(f_j - f_i\right)$$

Some Possibilities

- Tutte Laplacian: L_{ij}=1
- Graph Laplacian: L_{ij}=1/valence(v_i)

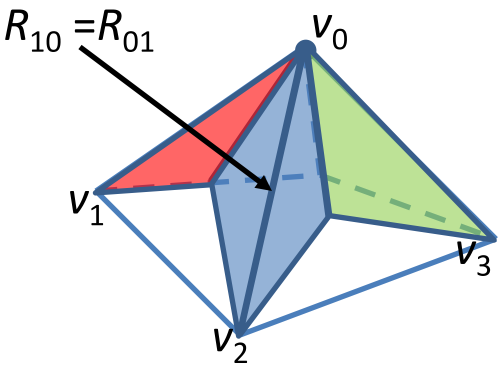


Discrete Laplacians

$$g_i = \sum_{v_j \in \mathrm{Nbr}(v_i)} L_{ij}\left(f_j - f_i\right)$$

Some Possibilities

- Tutte Laplacian: L_{ij}=1
- Graph Laplacian: L_{ij}=1/valence(v_i)
- Area-Weighted Laplacian: L_{ij}= R_{ij}



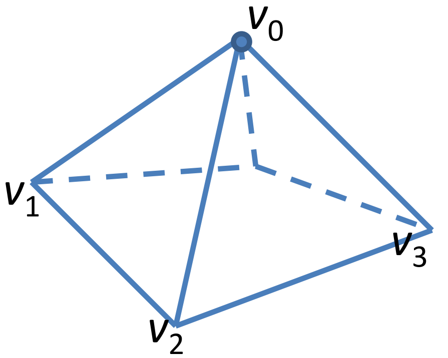
Discrete Laplacians

$$g_i = \sum_{v_j \in \mathrm{Nbr}(v_i)} L_{ij}\left(f_j - f_i\right)$$

Some Possibilities

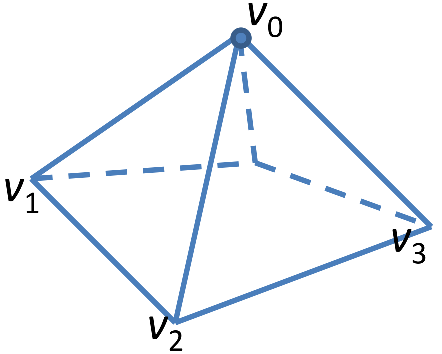
- Tutte Laplacian: L_{ij}=1
- Graph Laplacian: L_{ij}=1/valence(v_i)
- Area-Weighted Laplacian: L_{ij}= R_{ij}

Which one should we choose?



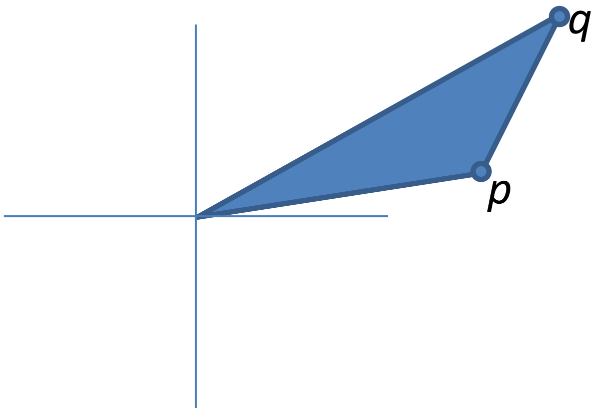
Discrete Laplacians

In the smooth case, we know that if the value of the function at vertex v is the position of v, then the Laplacian of the function at v should be the area gradient.



Discrete Laplacians

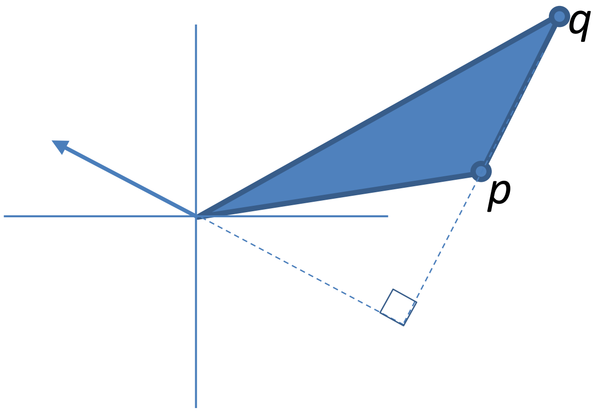
Given a triangle $(0,p,q)$ what direction should we move the vertex at the origin in order to maximally increase the area?



Discrete Laplacians

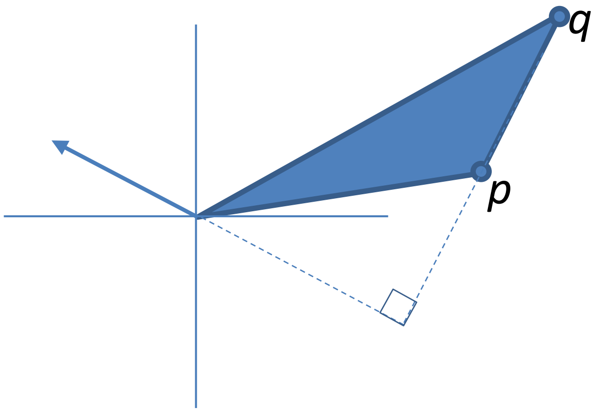
Given a triangle $(0,p,q)$ what direction should we move the vertex at the origin in order to maximally increase the area?

The area of the triangle is half the base times the height. So if we set pq to be the base, we want to move in the perpendicular direction.



Discrete Laplacians

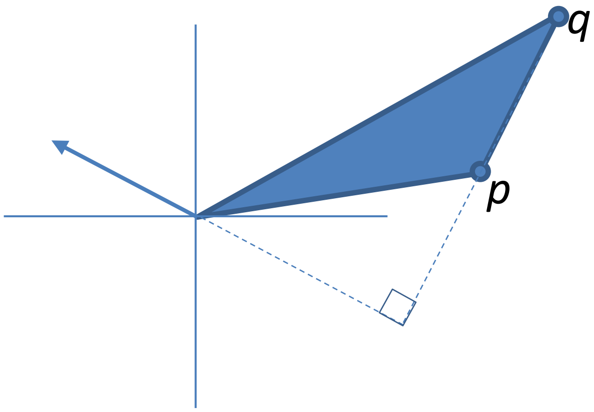
If we take a step of size ε in this direction, how will the area change?



Discrete Laplacians

If we take a step of size ε in this direction, how will the area change?

The base remains $|p\text{-}q|$ and the height becomes ($height+\varepsilon$), so the change is $(\varepsilon \times |p\text{-}q|)/2$.

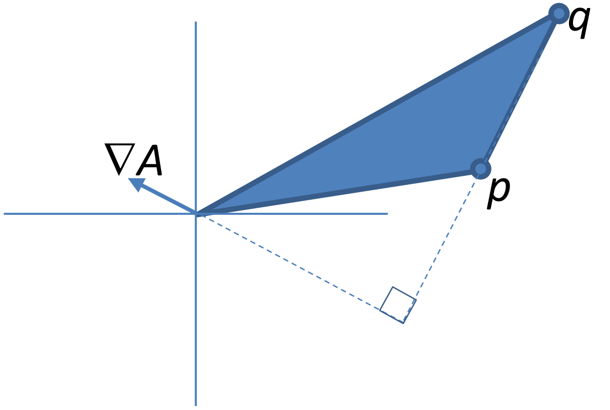


Discrete Laplacians

If we take a step of size ε in this direction, how will the area change?

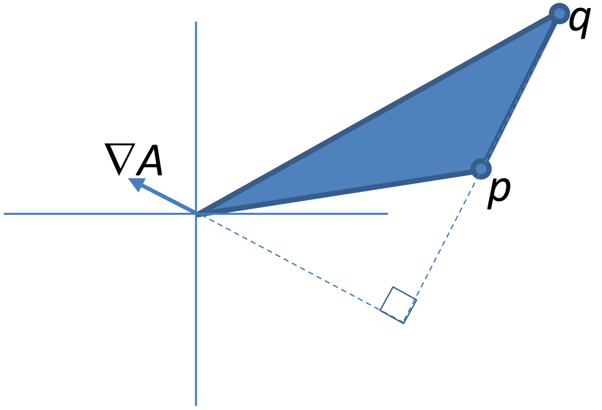
The base remains $|p\text{-}q|$ and the height becomes ($height+\varepsilon$), so the change is $(\varepsilon \times |p\text{-}q|)/2$.

Thus, the gradient is the vector perpendicular to $p\text{-}q$ with length equal to $|p\text{-}q|/2$.



Discrete Laplacians

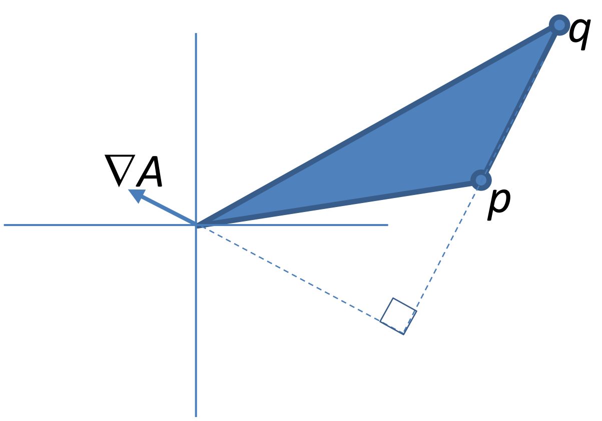
Given vectors p and q, what is the vector that is perpendicular to p-q and with length $|p\text{-}q|/2$?



Discrete Laplacians

Given vectors *p* and *q*, what is the vector that is perpendicular to *p*-*q* and with length |*p*-*q*|/2?

With respect to the basis {*p*,*q*}, the coefficients of *p*-*q* are (1,-1).



Discrete Laplacians

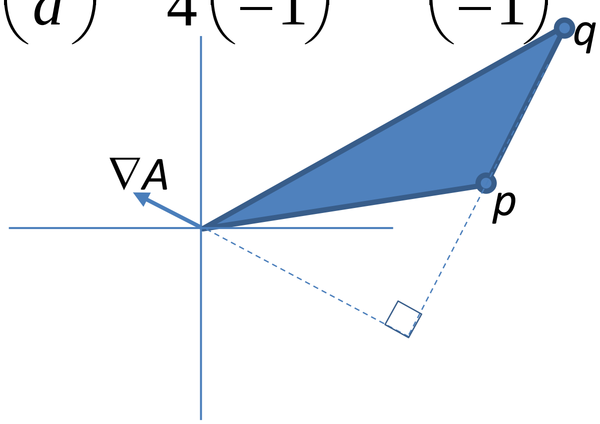
Given vectors *p* and *q*, what is the vector that is perpendicular to *p*-*q* and with length |*p*-*q*|/2?

With respect to the basis {*p*,*q*}, the coefficients of *p*-*q* are (1,-1). So we need the coefficients (*c*,*d*) such that:

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}^T M \begin{pmatrix} c \\ d \end{pmatrix} = 0 \quad \text{and} \quad \begin{pmatrix} c \\ d \end{pmatrix}^T M \begin{pmatrix} c \\ d \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 \\ -1 \end{pmatrix}^T M \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

where:

$$M = \begin{pmatrix} \langle p,p \rangle & \langle p,q \rangle \\ \langle p,q \rangle & \langle q,q \rangle \end{pmatrix}$$



Discrete Laplacians

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}^T M \begin{pmatrix} c \\ d \end{pmatrix} = 0$$

To satisfy the perpendicularity condition, the coefficients (*c*,*d*) must be:

$$\begin{pmatrix} c \\ d \end{pmatrix} = M^{-1} \begin{pmatrix} \varepsilon \\ \varepsilon \end{pmatrix} = \frac{\varepsilon}{\langle p,p \rangle \langle q,q \rangle - \langle p,q \rangle^2} \begin{pmatrix} \langle q-p,q \rangle \\ \langle p-q,p \rangle \end{pmatrix}$$

for some value of ε.

$$M = \begin{pmatrix} \langle p,p \rangle & \langle p,q \rangle \\ \langle p,q \rangle & \langle q,q \rangle \end{pmatrix}$$

$$M^{-1} = \frac{1}{\langle p,p \rangle \langle q,q \rangle - \langle p,q \rangle^2} \begin{pmatrix} \langle q,q \rangle & -\langle p,q \rangle \\ -\langle p,q \rangle & \langle p,p \rangle \end{pmatrix}$$

Discrete Laplacians

$$\boxed{\begin{pmatrix} c \\ d \end{pmatrix}^T M \begin{pmatrix} c \\ d \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 \\ -1 \end{pmatrix}^T M \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

To satisfy the condition on the length, note that:

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}^T M \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \langle p, p\rangle + \langle q, q\rangle - 2\langle p, q\rangle$$

and

$$\begin{pmatrix} c \\ d \end{pmatrix}^T M \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} \varepsilon \\ \varepsilon \end{pmatrix}^T M^{-1} \begin{pmatrix} \varepsilon \\ \varepsilon \end{pmatrix} = \varepsilon^2 \frac{\langle p, p\rangle + \langle q, q\rangle - 2\langle p, q\rangle}{\langle p, p\rangle\langle q, q\rangle - \langle p, q\rangle^2}$$

$$M = \begin{pmatrix} \langle p, p\rangle & \langle p, q\rangle \\ \langle p, q\rangle & \langle q, q\rangle \end{pmatrix}$$

$$M^{-1} = \frac{1}{\langle p, p\rangle\langle q, q\rangle - \langle p, q\rangle^2}\begin{pmatrix} \langle q, q\rangle & -\langle p, q\rangle \\ -\langle p, q\rangle & \langle p, p\rangle \end{pmatrix}$$

Discrete Laplacians

$$\begin{pmatrix} c \\ d \end{pmatrix}^T M \begin{pmatrix} c \\ d \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 \\ -1 \end{pmatrix}^T M \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

To satisfy the condition on the length, note that:

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}^T M \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \langle p,p\rangle + \langle q,q\rangle - 2\langle p,q\rangle \qquad \begin{pmatrix} c \\ d \end{pmatrix}^T M \begin{pmatrix} c \\ d \end{pmatrix} = \varepsilon^2 \frac{\langle p,p\rangle + \langle q,q\rangle - 2\langle p,q\rangle}{\langle p,p\rangle\langle q,q\rangle - \langle p,q\rangle^2}$$

which gives:

$$\varepsilon = \frac{\sqrt{\langle p,p\rangle\langle q,q\rangle - \langle p,q\rangle^2}}{2}$$

$$M = \begin{pmatrix} \langle p,p\rangle & \langle p,q\rangle \\ \langle p,q\rangle & \langle q,q\rangle \end{pmatrix} \qquad M^{-1} = \frac{1}{\langle p,p\rangle\langle q,q\rangle - \langle p,q\rangle^2}\begin{pmatrix} \langle q,q\rangle & -\langle p,q\rangle \\ -\langle p,q\rangle & \langle p,p\rangle \end{pmatrix}$$

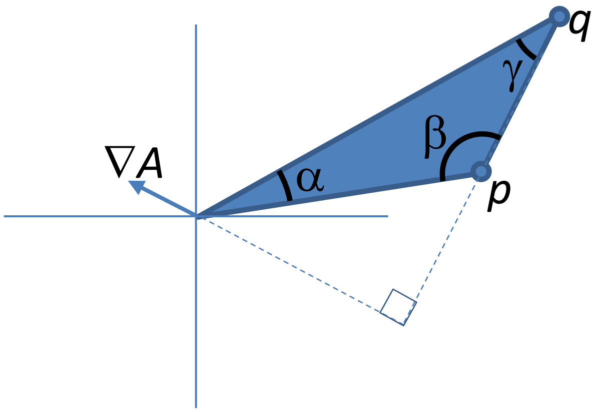
Discrete Laplacians

$$\begin{pmatrix} c \\ d \end{pmatrix} = \frac{\varepsilon}{\langle p,p\rangle\langle q,q\rangle - \langle p,q\rangle^2} \begin{pmatrix} \langle q-p,q\rangle \\ \langle p-q,q\rangle \end{pmatrix}$$

$$\varepsilon = \frac{\sqrt{\langle p,p\rangle\langle q,q\rangle - \langle p,q\rangle^2}}{2}$$

Putting everything together, we get:

$$\begin{pmatrix} c \\ d \end{pmatrix} = \frac{1}{2} \frac{1}{\sqrt{\langle p,p\rangle\langle q,q\rangle - \langle p,q\rangle^2}} \begin{pmatrix} \langle q-p,q\rangle \\ \langle p-q,p\rangle \end{pmatrix}$$



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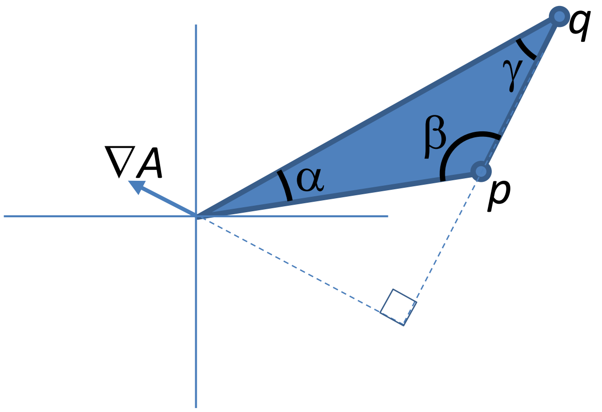
$$= \frac{1}{2}\frac{1}{\sqrt{\|p\|^2\|q\|^2 - \|p\|^2\|q\|^2\cos^2\alpha}}\begin{pmatrix} \langle q-p,q\rangle \\ \langle p-q,p\rangle \end{pmatrix}$$



Discrete Laplacians

$$\begin{pmatrix} c \\ d \end{pmatrix} = \frac{\varepsilon}{\langle p,p\rangle\langle q,q\rangle - \langle p,q\rangle^2}\begin{pmatrix} \langle q-p,q\rangle \\ \langle p-q,q\rangle \end{pmatrix}$$

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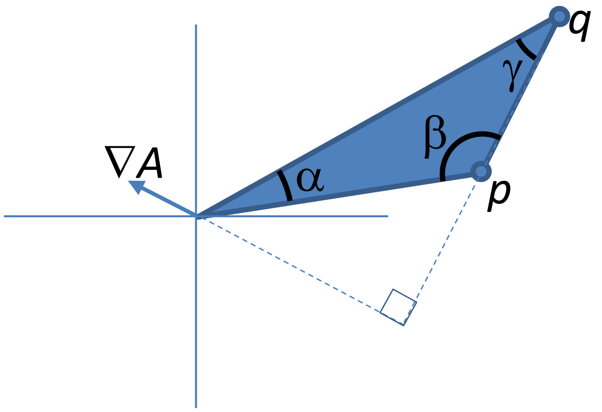
$$= \frac{1}{2}\frac{1}{\|p\|\|q\|\sin\alpha}\begin{pmatrix} \langle q-p,q\rangle \\ \langle p-q,p\rangle \end{pmatrix}$$



Discrete Laplacians

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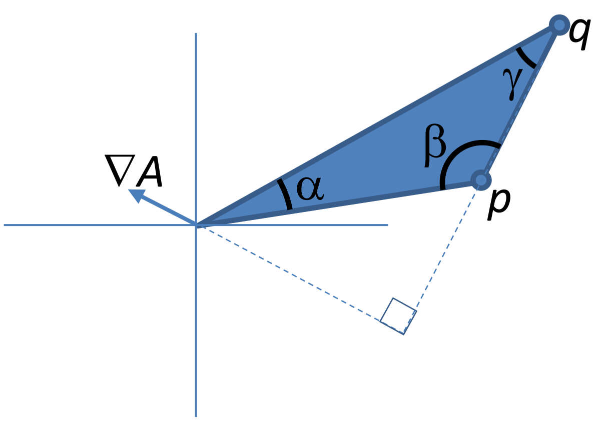
$$= \frac{1}{2}\frac{1}{\|p\|\|q\|\sin\alpha} \begin{pmatrix} \|q-p\|\|q\|\cos\gamma \\ \|q-p\|\|p\|\cos\beta \end{pmatrix}$$



Discrete Laplacians

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Putting everything together, we get:

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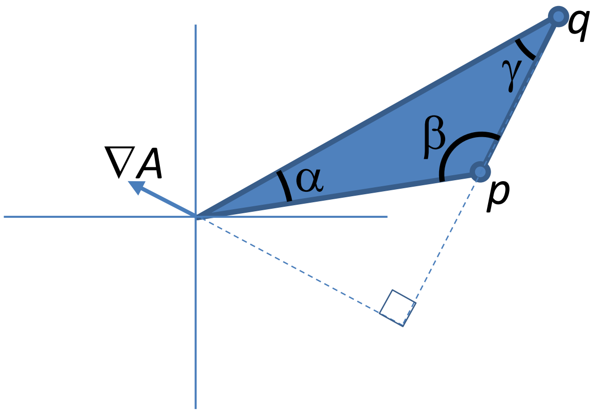
$$= \frac{1}{2}\begin{pmatrix} \frac{\|q-p\|\|q\|\cos\gamma}{\|p\|\|q\|\sin\alpha} & \frac{\|q-p\|\|p\|\cos\beta}{\|p\|\|q\|\sin\alpha} \end{pmatrix}^T$$



Discrete Laplacians

$$\begin{pmatrix} c \\ d \end{pmatrix} = \frac{\varepsilon}{\langle p,p\rangle\langle q,q\rangle - \langle p,q\rangle^2} \begin{pmatrix} \langle q-p,q\rangle \\ \langle p-q,q\rangle \end{pmatrix}$$

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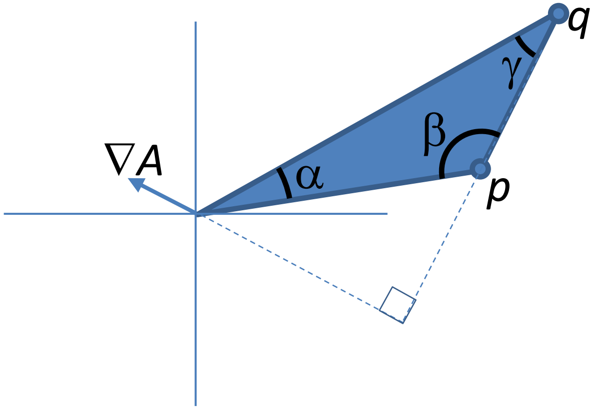
$$= \frac{1}{2}\begin{pmatrix} \frac{\|q-p\|\cos\gamma}{\|p\|\sin\alpha} & \frac{\|q-p\|\cos\beta}{\|q\|\sin\alpha} \end{pmatrix}^T$$



Discrete Laplacians

$$\begin{pmatrix} c \\ d \end{pmatrix} = \frac{\varepsilon}{\langle p,p\rangle\langle q,q\rangle - \langle p,q\rangle^2}\begin{pmatrix} \langle q-p,q\rangle \\ \langle p-q,q\rangle \end{pmatrix}$$

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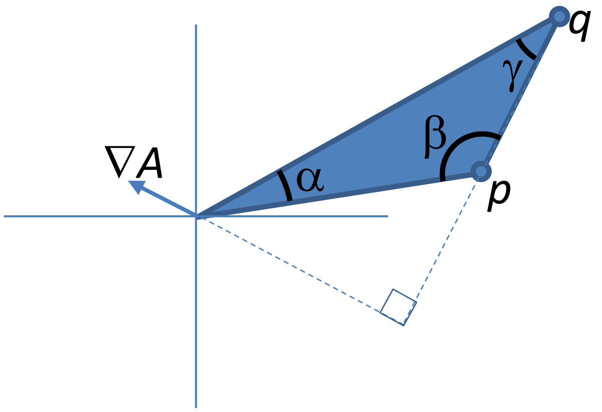
Putting everything together, we get:

$$\begin{pmatrix} c \\ d \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \frac{\|q-p\|\cos\gamma}{\|p\|\sin\alpha} & \frac{\|q-p\|\cos\beta}{\|q\|\sin\alpha} \end{pmatrix}^T$$

$$= \frac{1}{2}\begin{pmatrix} \frac{\|q-p\|\cos\gamma}{\|p\|}\frac{\|p\|}{\|q-p\|\sin\gamma} & \frac{\|q-p\|\cos\beta}{\|q\|}\frac{\|q\|}{\|q-p\|\sin\beta} \end{pmatrix}^T$$



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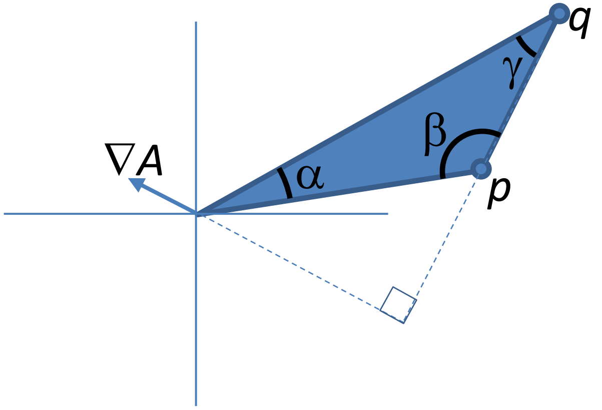
$$\varepsilon = \frac{\sqrt{\langle p,p\rangle\langle q,q\rangle - \langle p,q\rangle^2}}{2}$$

Putting everything together, we get:

$$\begin{pmatrix} c \\ d \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \frac{\|q-p\|\cos\gamma}{\|p\|}\frac{\|p\|}{\|q-p\|\sin\gamma} & \frac{\|q-p\|\cos\beta}{\|q\|}\frac{\|q\|}{\|q-p\|\sin\beta} \end{pmatrix}^T$$

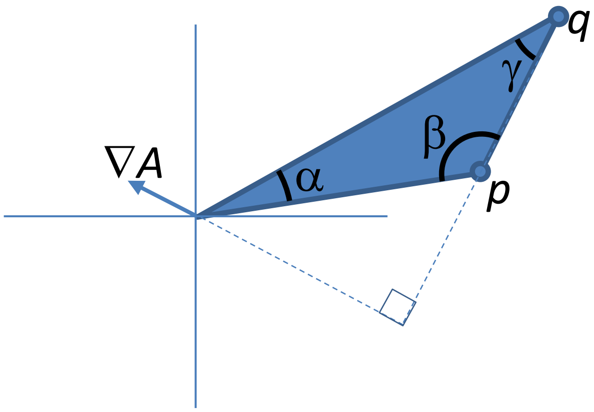
$$= \frac{1}{2}\begin{pmatrix} \frac{\cos\gamma}{\sin\gamma} & \frac{\cos\beta}{\sin\beta} \end{pmatrix}^T = \frac{1}{2}\begin{pmatrix} \cot\gamma \\ \cot\beta \end{pmatrix}$$



Discrete Laplacians

Thus, the vector that is perpendicular to *p*-*q* with length |*p*-*q*|/2 is the vector:

$$\frac{1}{2}\left(\cot(\beta)p+\cot(\gamma)q\right)$$



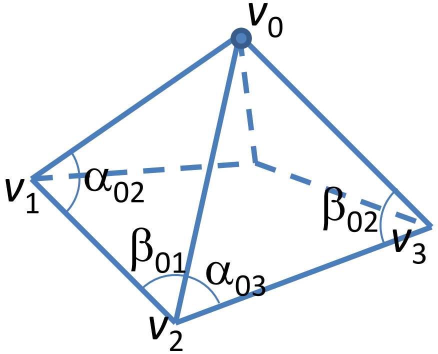
Discrete Laplacians

Thus, the vector that is perpendicular to *p*-*q* with length |*p*-*q*|/2 is the vector:

$$\frac{1}{2}\left(\cot(\beta)p+\cot(\gamma)q\right)$$

so the area gradient at vertex v_i can be obtained by iterating over the edges adjacent to v_i and summing using the cotans of opposite angles:

$$\frac{\partial A}{\partial v_i}=\frac{1}{2}\sum_{v_j\in \mathrm{Nbr}(v_i)}^{k}\left(v_i-v_j\right)\left(\cot\left(\alpha_{ij}\right)+\cot\left(\beta_{ij}\right)\right)$$



Discrete Laplacians

This leads to the cotangent-weight Laplacian:

$$L_{ij} = \begin{cases} \frac{1}{2}\left(\cot(\alpha_{ij}) + \cot(\beta_{ij})\right) & \text{if } i \neq j \text{ and } v_j \in \mathrm{Nbr}(v_i) \\ -\sum\limits_{v_k \in \mathrm{Nbr}(v_i)} L_{ik} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$