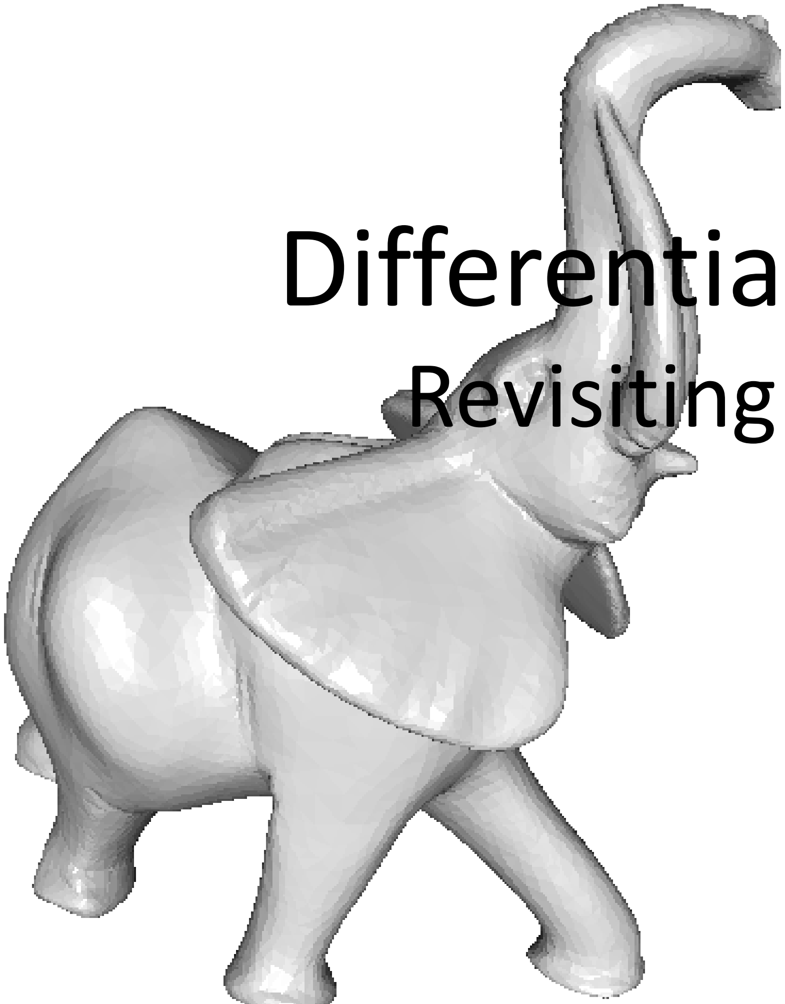
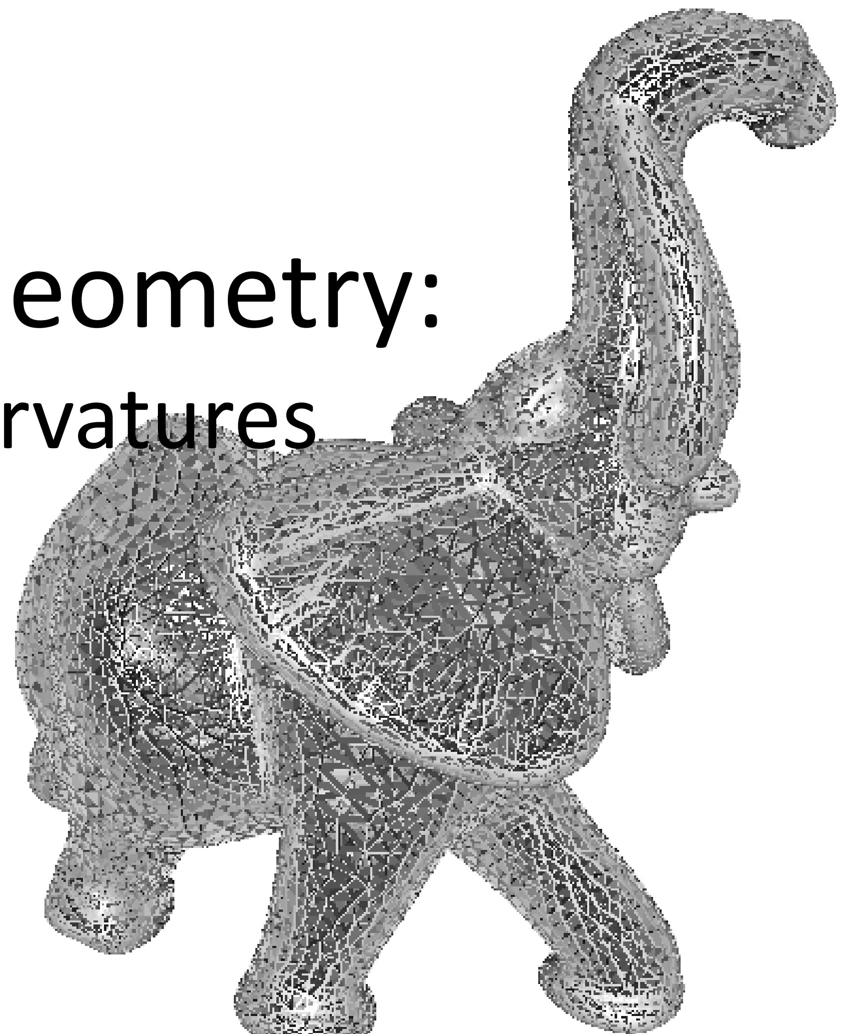
Differential Geometry:

Revisiting Curvatures





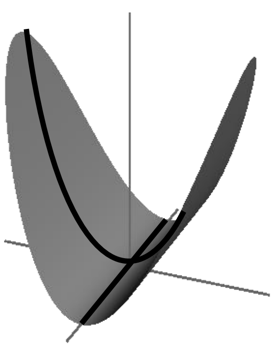
Curvature and Graphs

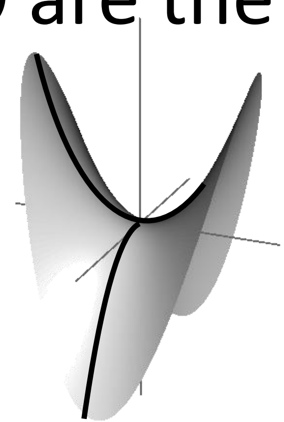
<u>Recall</u>:

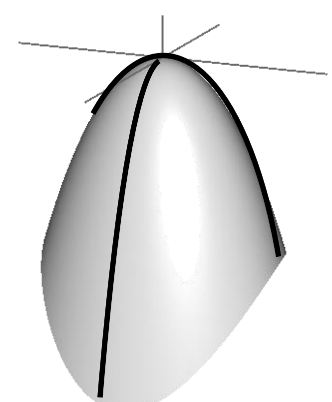
Thus, up to a rotation in the *x*-*y* plane, we have:

$$f(x,y) \approx \frac{\lambda_1}{2}x^2 + \frac{\lambda_2}{2}y^2$$

The values λ_1 and λ_2 are the *principal curvatures* at *p* and the corresponding directions of the curves at the point *p* are the *principal directions*.





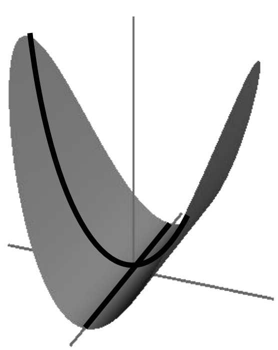


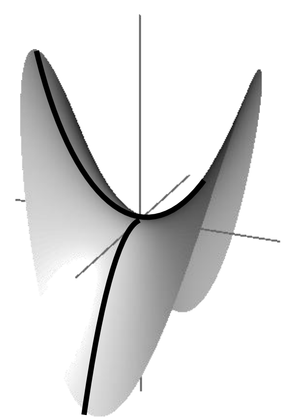
Curvature and Graphs

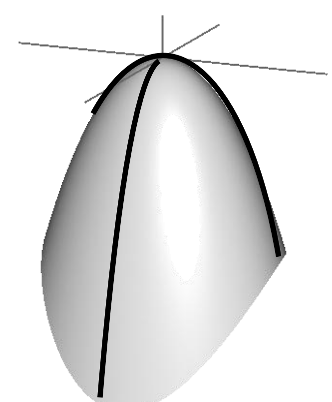
<u>Recall</u>:

The product of the principal curvatures, $\lambda_1 \cdot \lambda_2$, is the *Gaussian Curvature*.

The sum of the principal curvatures, $\lambda_1 + \lambda_2$, is the *Mean Curvature*.

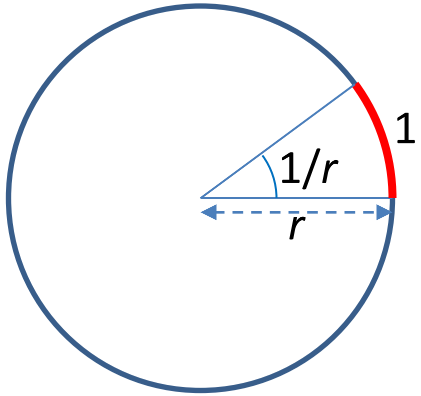






Smooth Curvature (Curves)

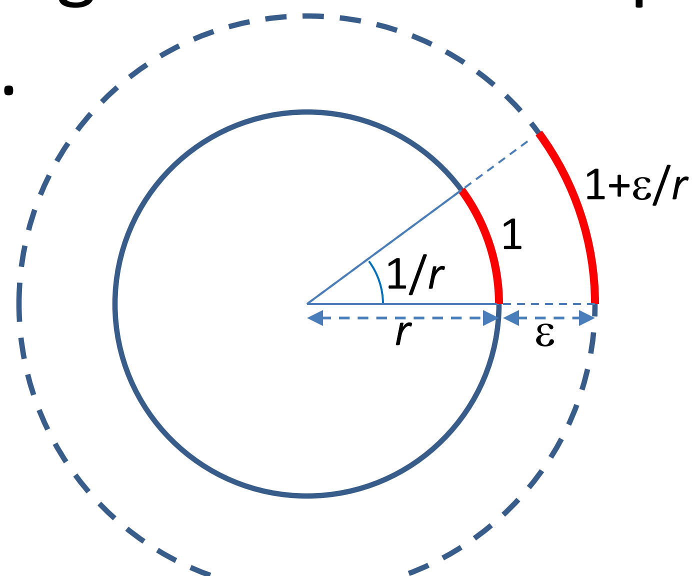
On a circle of radius *r*, an arc of unit-length will have angle 1/*r*.



Smooth Curvature (Curves)

On a circle of radius r, an arc of unit-length will have angle $1/r$.

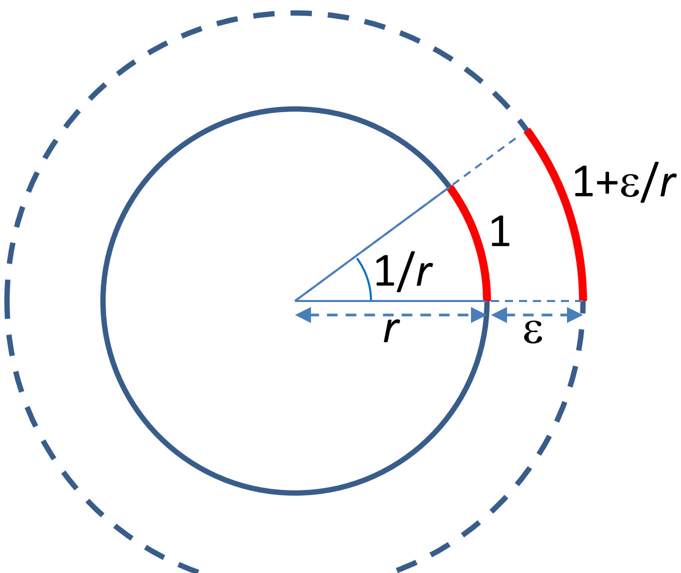
Offsetting the circle by a distance of ε in the normal direction, we get a circle with radius $(r+\varepsilon)$, and the length of the corresponding arc becomes $1+\varepsilon/r$.



Smooth Curvature (Curves)

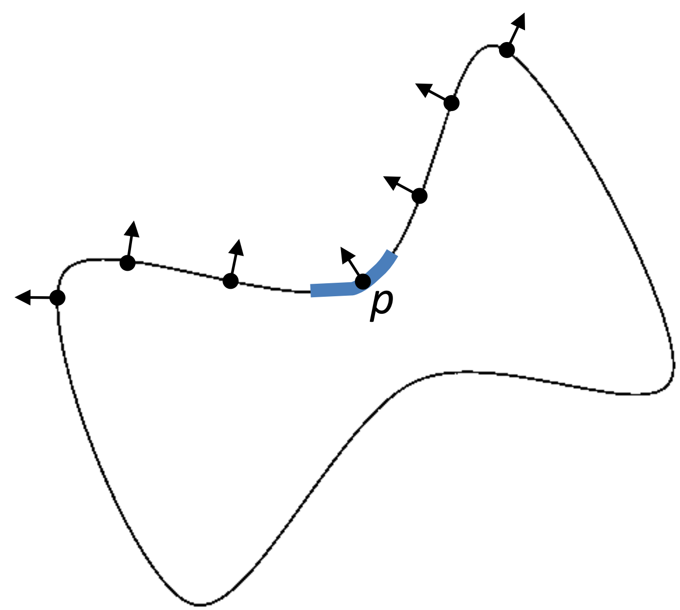
We measure the curvature as the rate of change in length as a function of offset distance ε:

$$\kappa = \frac{1}{r}$$



Smooth Curvature (Curves)

In a similar fashion, we can define the curvature at a point *p* on an arbitrary curve by considering the rate of change in arc-length as we offset in the normal direction by a distance of ε.



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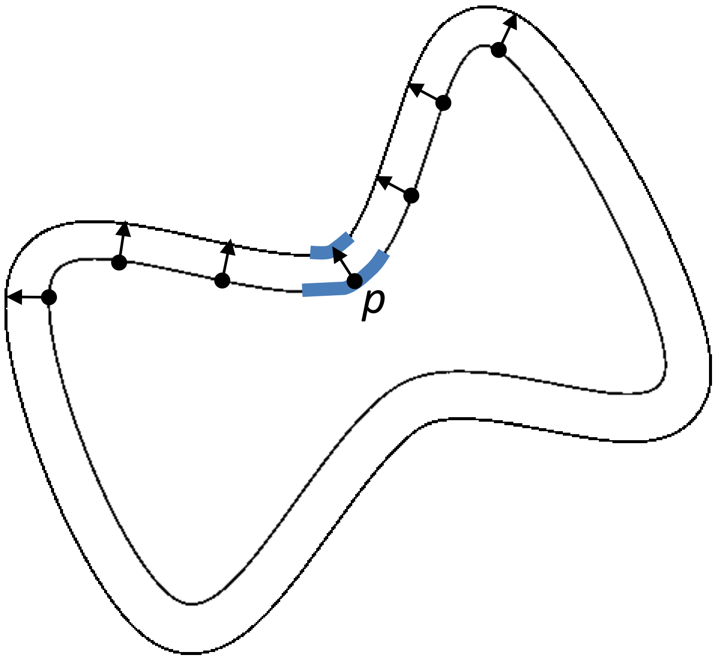
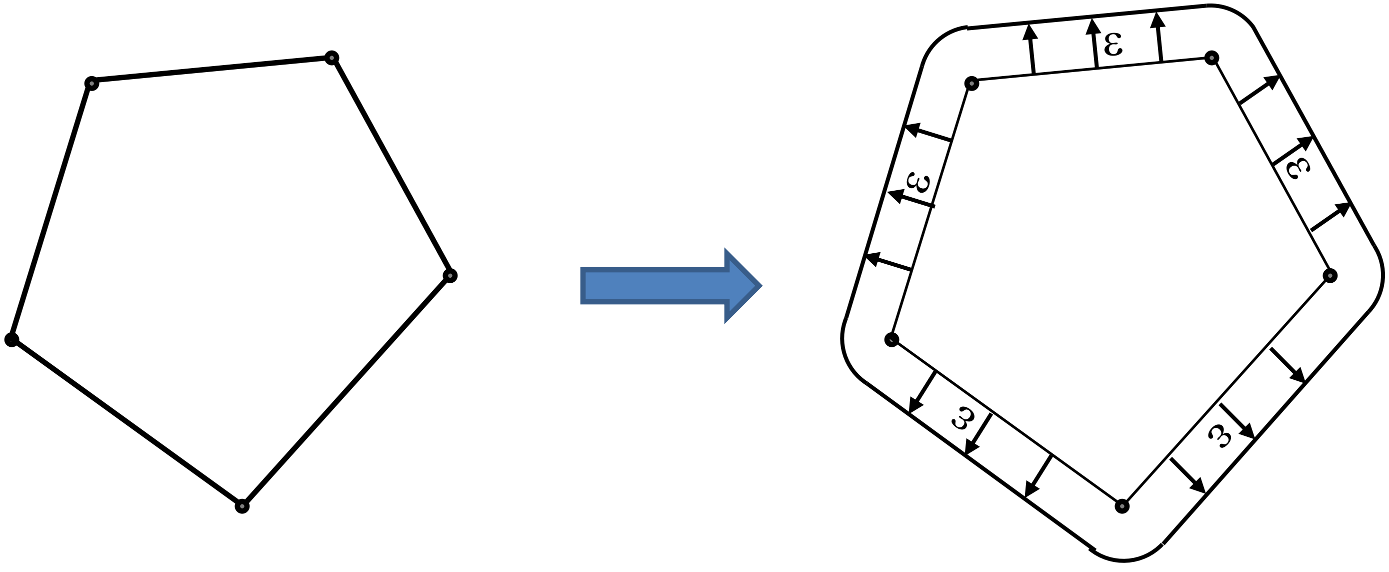


Discrete Curvature (Curves)

Given a closed curve, consider the curve obtained by offsetting by ε in the normal direction.

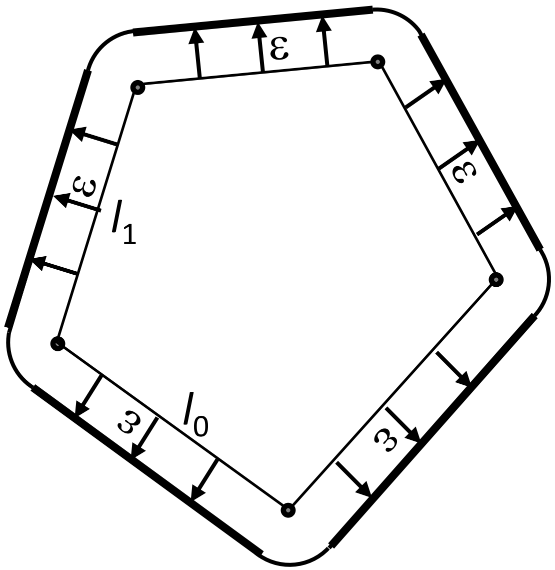


Discrete Curvature (Curves)

Given a closed curve, consider the curve obtained by offsetting by ε in the normal direction.

The length of the offset curve is the length of the old curve…

$$l = \sum_{i=0}^{N-1} l_i$$



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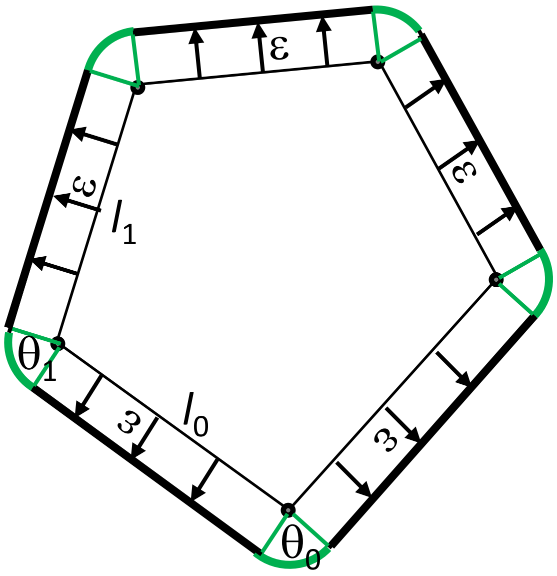
Given a closed curve, consider the curve obtained by offsetting by ε in the normal direction.

The length of the offset curve is the length of the old curve…

$$l = \sum_{i=0}^{N-1} l_i + \varepsilon\theta_i$$

plus the lengths of the arcs.



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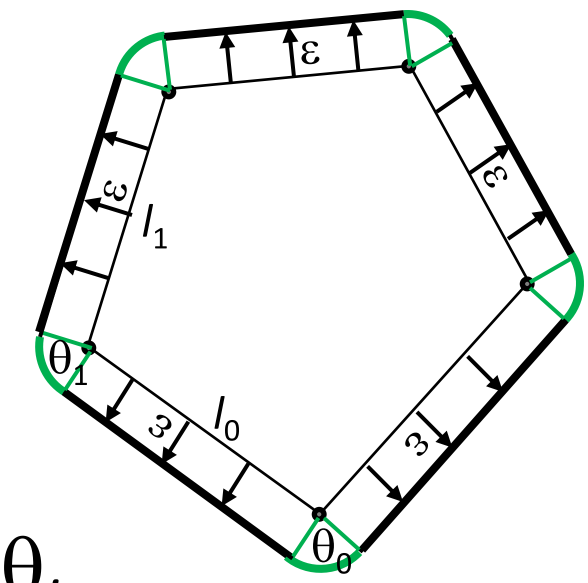
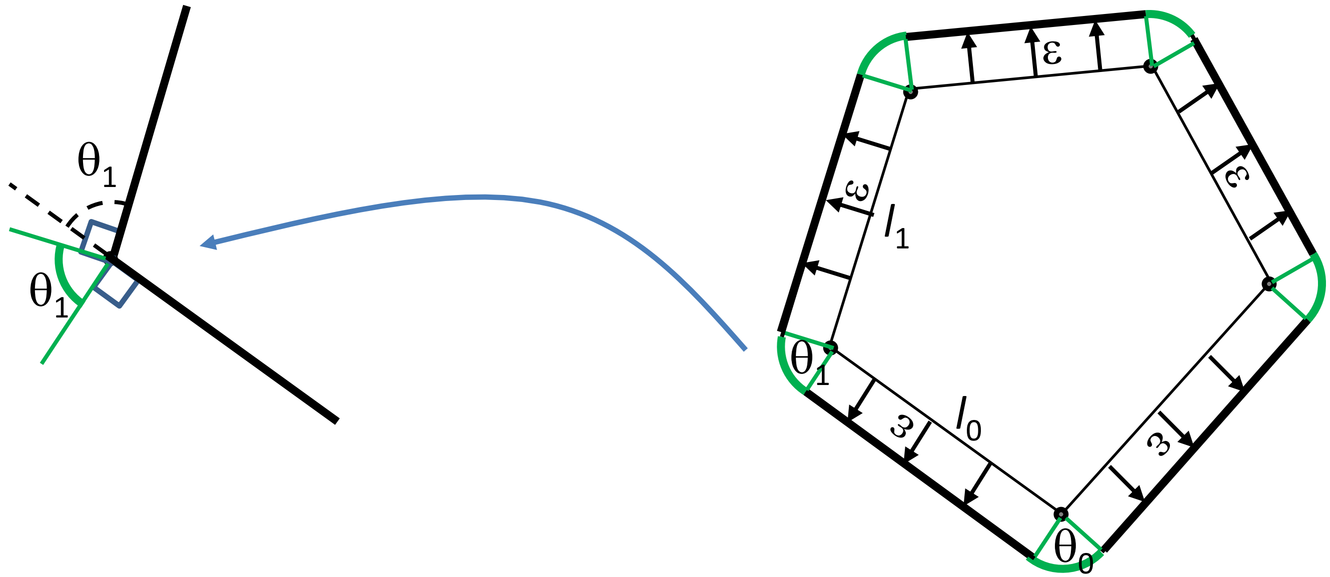
plus the lengths of the arcs.

Thus, the rate of change in length through the vertex *i* is θ_i.



Discrete Curvature (Curves)

But the angle of the arc is exactly the deficit angle, so we get the same definition as before.



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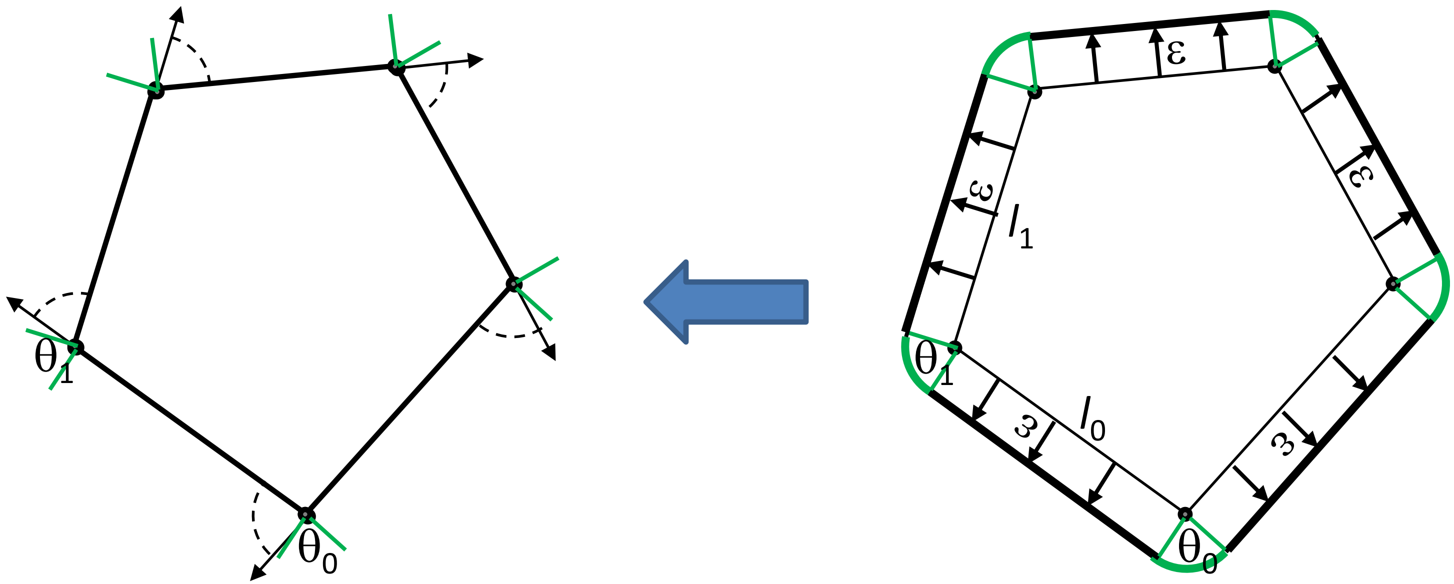

Smooth Curvature (Surfaces)

In a similar fashion, we can consider what happens to the area of a surface as we offset it in the normal direction by a distance of ε.

Smooth Curvature (Surfaces)

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In this case we consider both the rate of change and acceleration in area, and we get:

$$A_\varepsilon(p) \approx A_\varepsilon(p) + \varepsilon H(p) + \varepsilon^2 K(p)$$

where *H* is the mean curvature and *K* is the Gaussian curvature.

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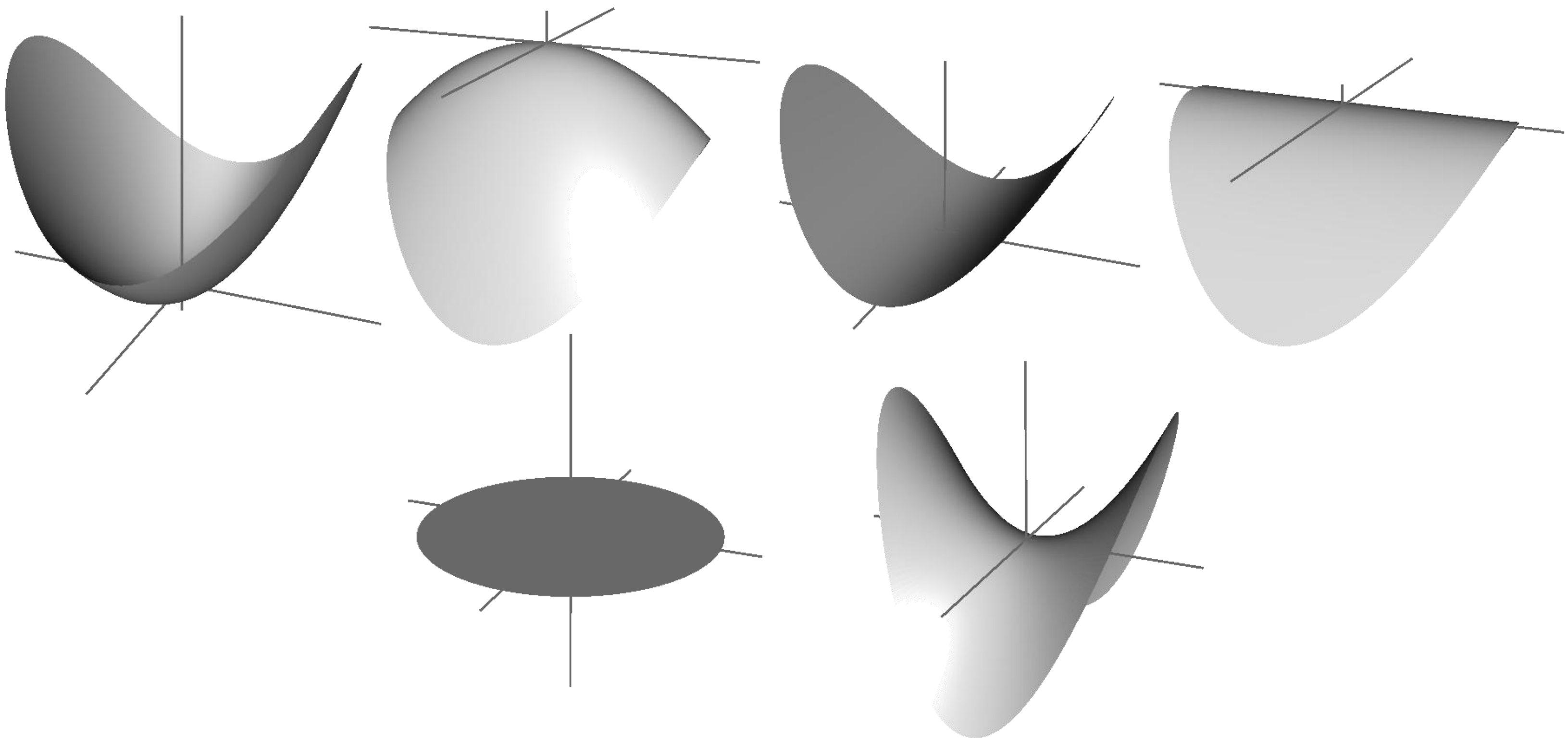
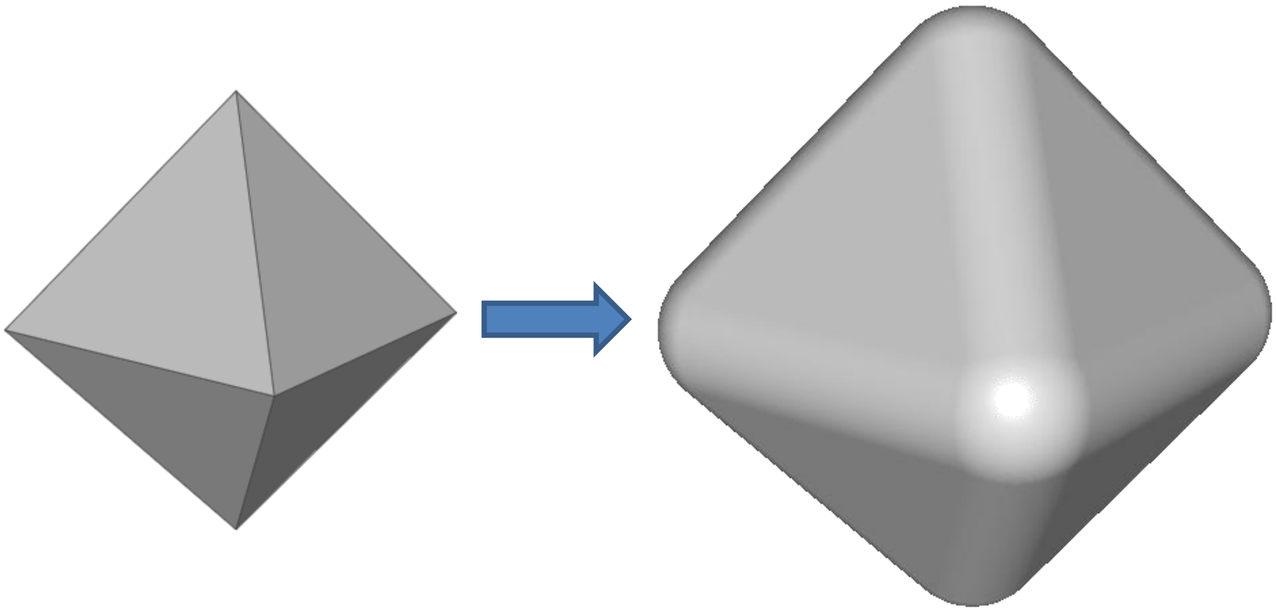


Discrete Curvature (Surfaces)

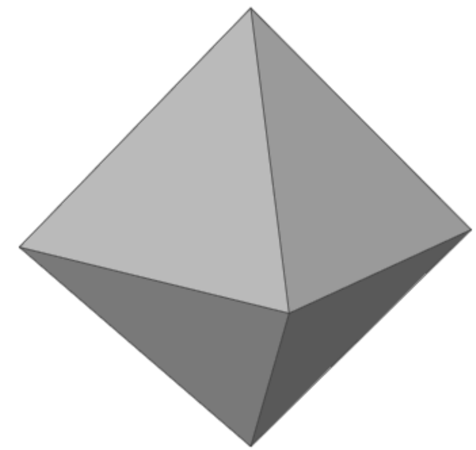
What happens when we offset points on a discrete surface?

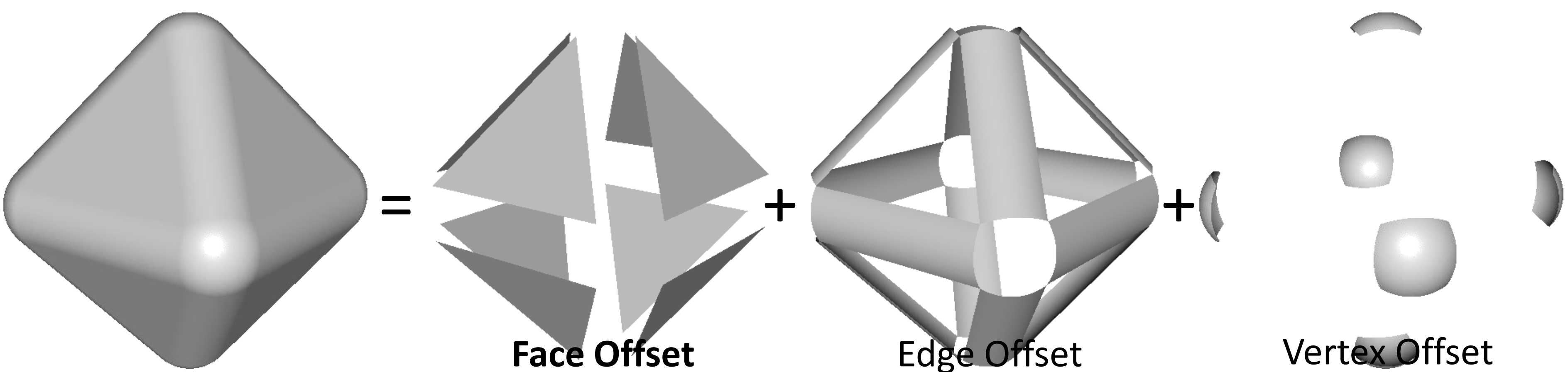


Discrete Curvature (Surfaces)

What happens when we offset points on a discrete surface?

We can decompose the offset surface into three parts.

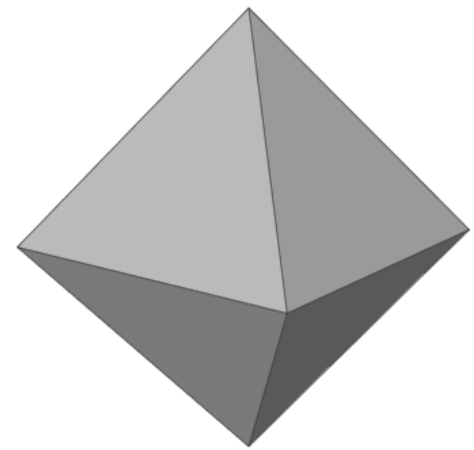


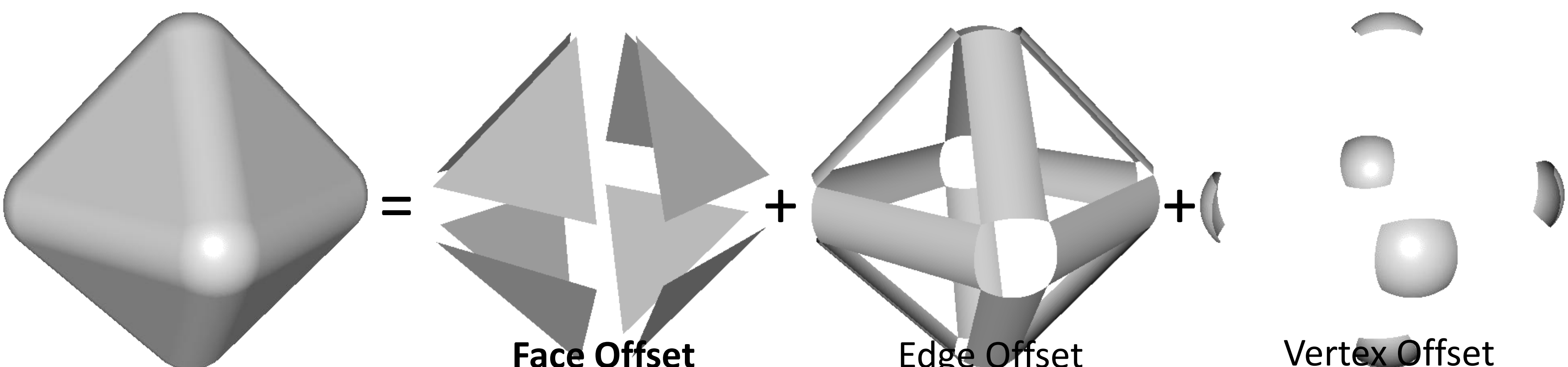


Discrete Curvature (Surfaces)

The area of the offset surface is the sum of:

- The area of the original surface

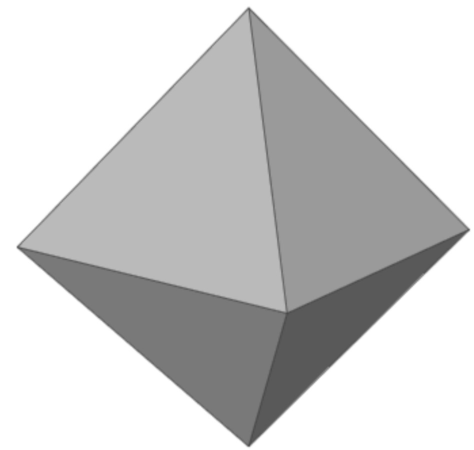


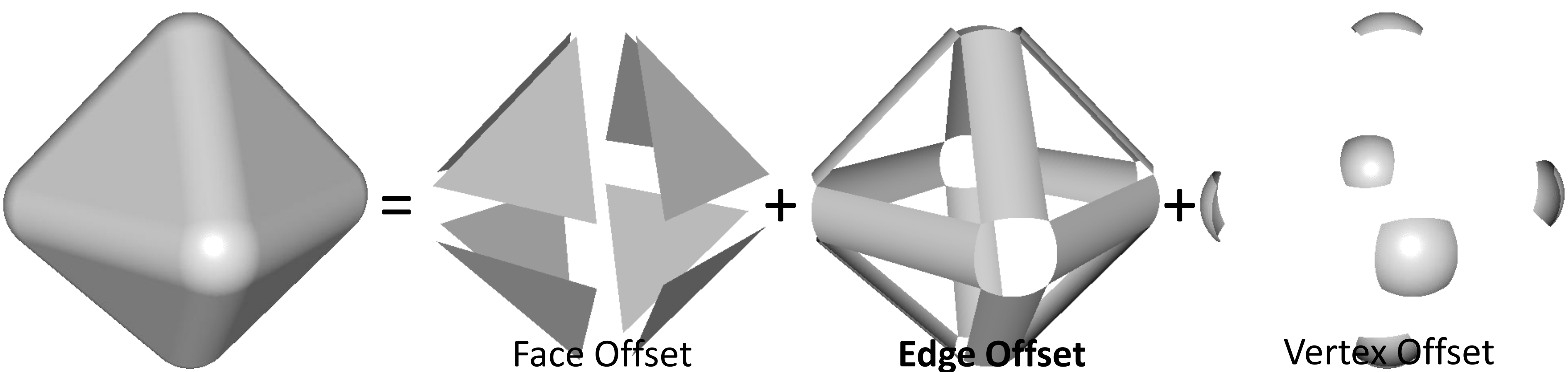


Discrete Curvature (Surfaces)

The area of the offset surface is the sum of:

- The area of the original surface
- The area of the cylindrical arcs defined by the edges

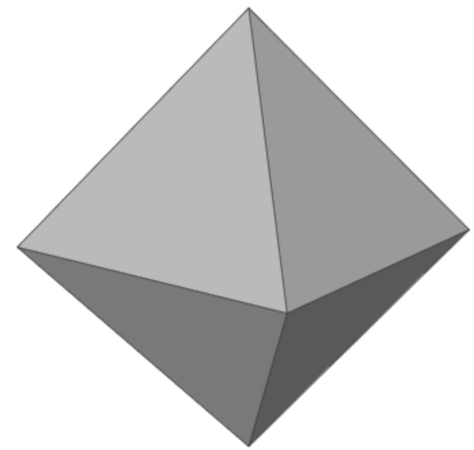


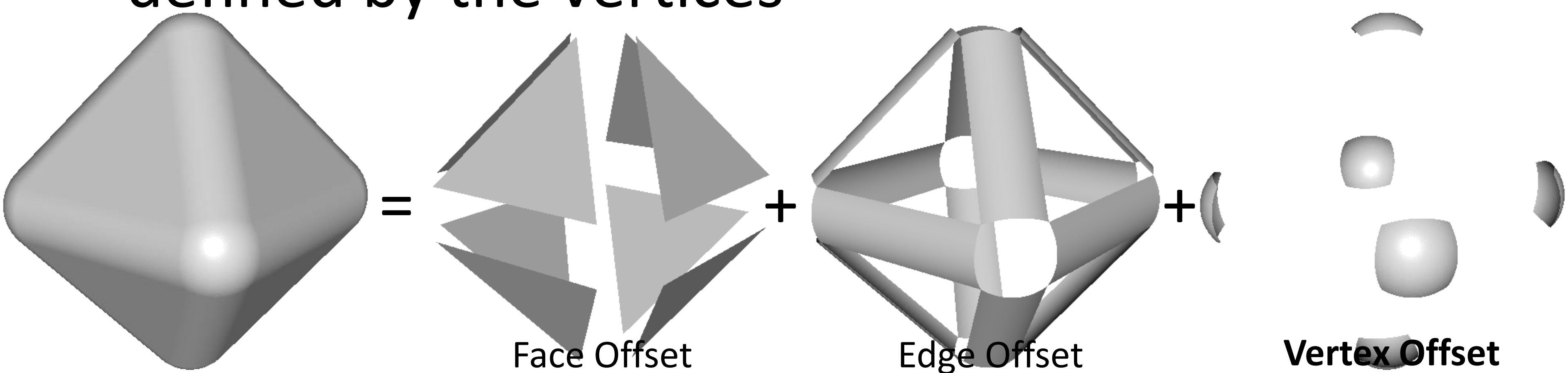


Discrete Curvature (Surfaces)

The area of the offset surface is the sum of:

- The area of the original surface
- The area of the cylindrical arcs defined by the edges
- The area of the spherical caps defined by the vertices





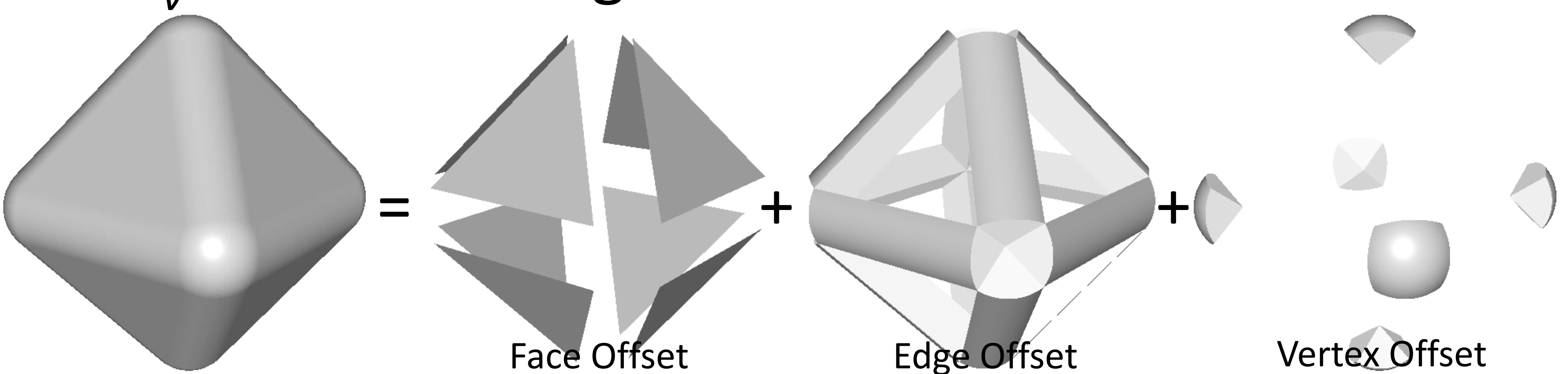
Discrete Curvature (Surfaces)

The area of the offset surface is the sum of:

$$A_\varepsilon = \sum_{t \in Tris.} A(t) + \varepsilon \sum_{e \in Edges.} |e| \theta_e + \varepsilon^2 \sum_{v \in Verts.} \theta_v$$

where:

- θ_e is the angle at edge *e*
- θ_v is the solid angle at vertex *v*



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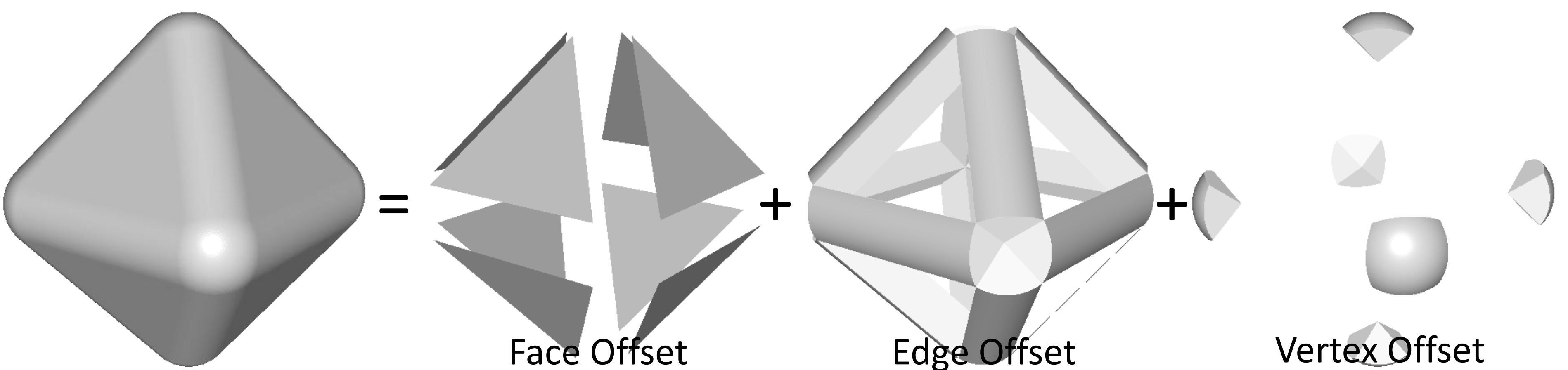
So the offset surface has $|e|\theta_e$ as the 1st-order term of the area, and θ_v as the 2nd-order term.



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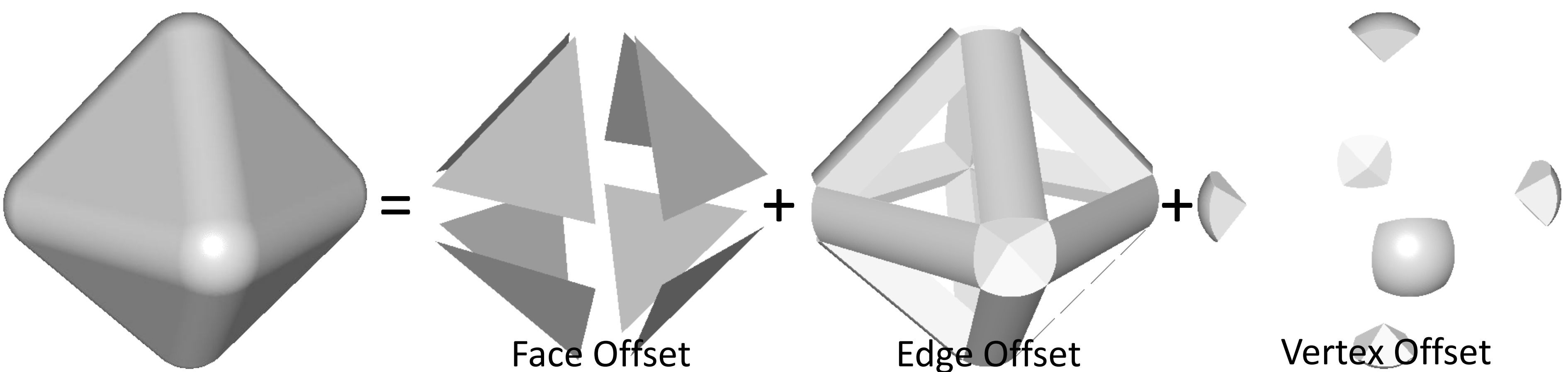
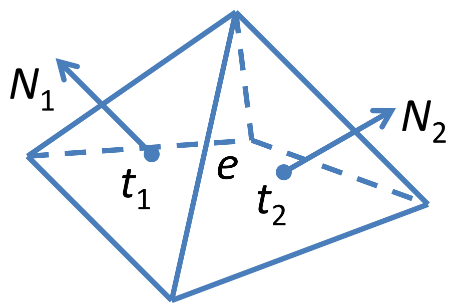
We associate the *discrete mean curvature* $|e|\theta_e$ with the edges of the polygon and *discrete Gaussian curvature* θ_v with the vertices.

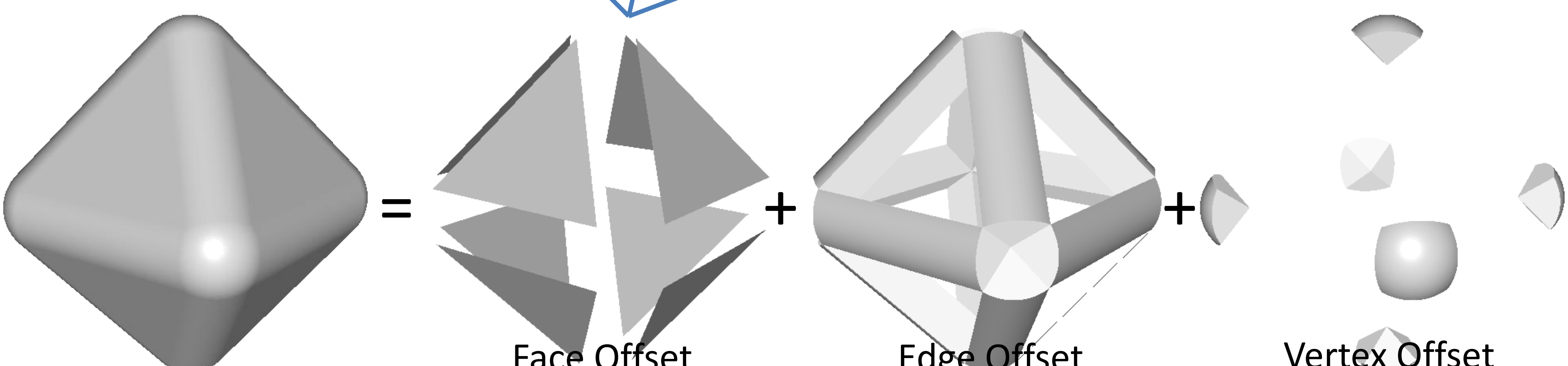


Discrete Curvature (Surfaces)

If triangles t_1 and t_2 meet at edge e, the angle θ_e is defined as:

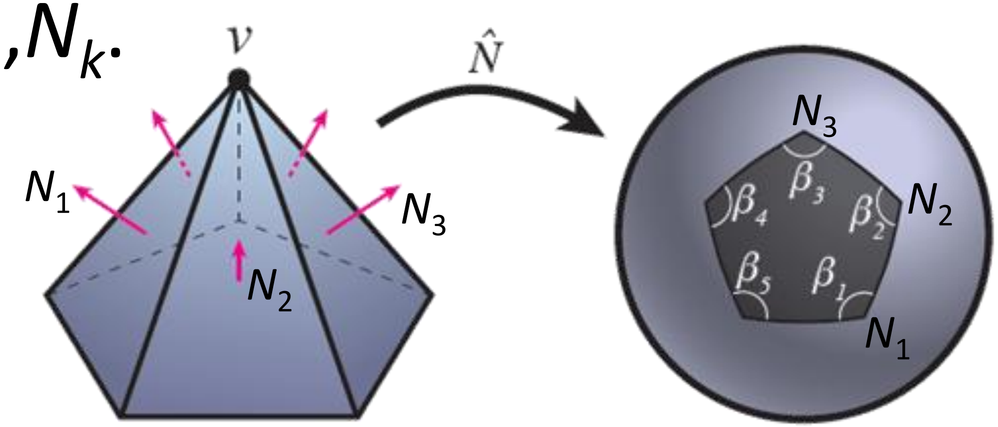
$$\cos(\theta_e) = \langle N_1, N_2 \rangle$$

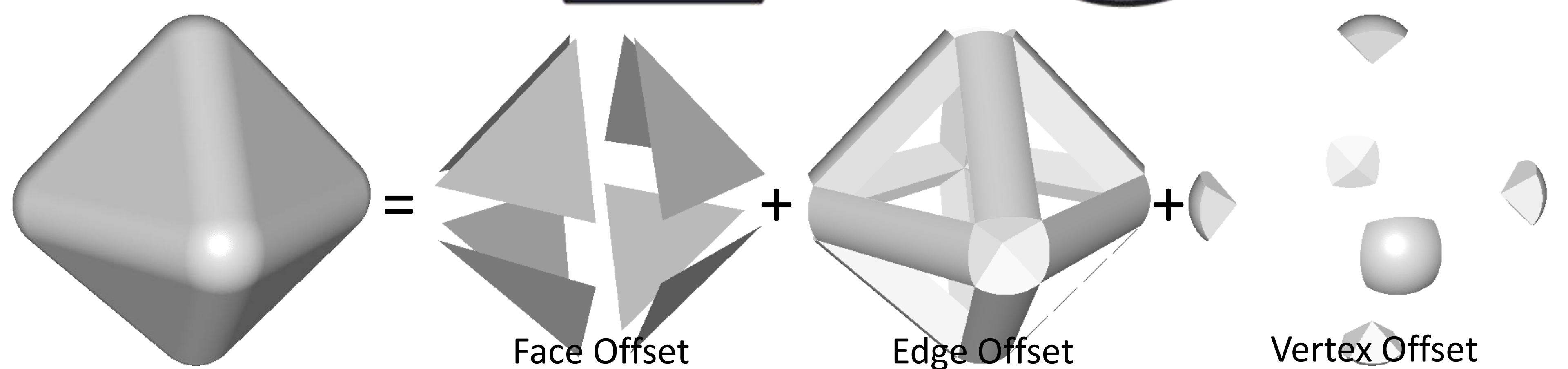




Discrete Curvature (Surfaces)

If triangles $t_1,\ldots,t_k$ meet at vertex v, the solid angle θ_v is the area of the spherical wedge going through $N_1,\ldots,N_k$.





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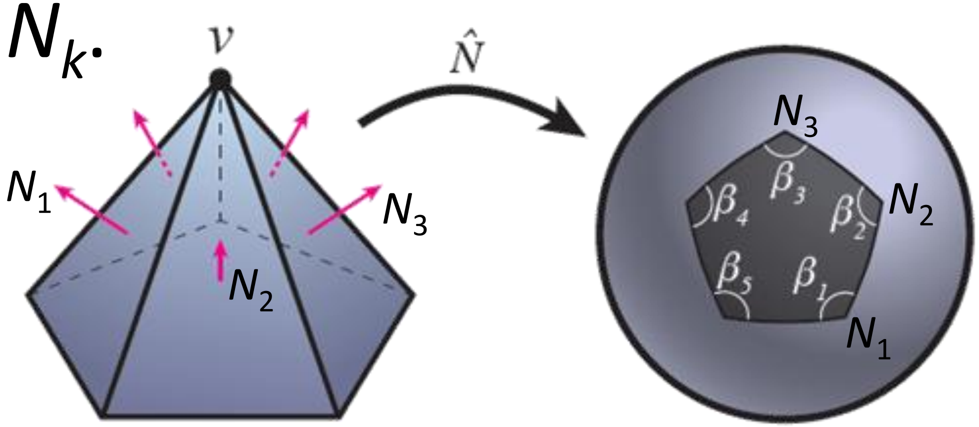


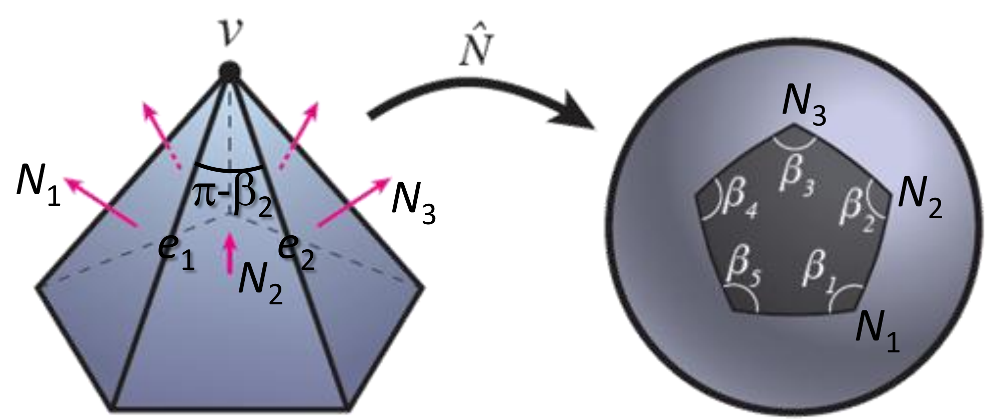
On a sphere, the area of a polygon with angles $\beta_1,\ldots,\beta_k$ is:

$$A = (2-k)\pi + \sum_{i=1}^{k} \beta_i$$

Discrete Curvature (Surfaces)

<u>Claim</u>:

The angle β_i at the intersection of arcs $N_{i-1}N_i$ and N_iN_{i+1} is π minus the angle between e_{i-1} and e_i.



Discrete Curvature (Surfaces)

<u>Claim</u>:

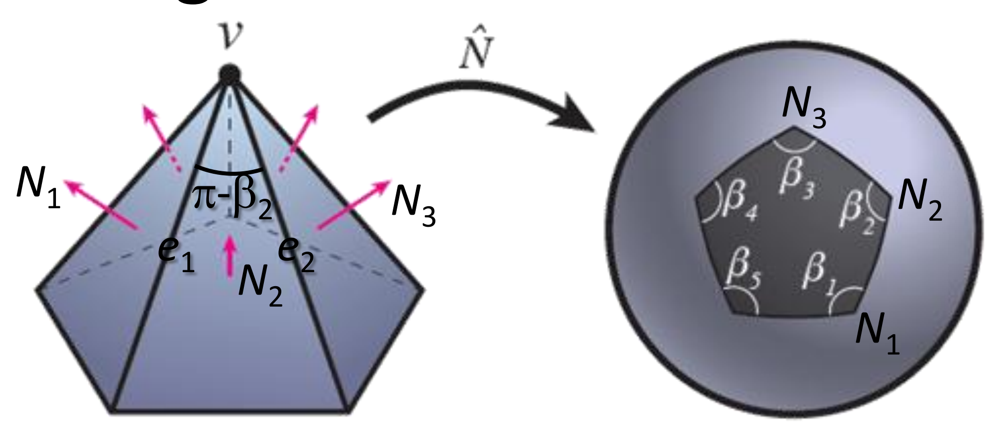
The angle β_i at the intersection of arcs $N_{i\text{-}1}N_i$ and N_iN_{i+1} is π minus the angle between $e_{i\text{-}1}$ and e_i.

<u>Implications</u>:

If α_i is the angle (at v) between $e_{i\text{-}1}$ and e_i, the Gaussian curvature is the angle of deficit at v:

$$A = (2-k)\pi + \sum_{i=1}^{k} \beta_i$$

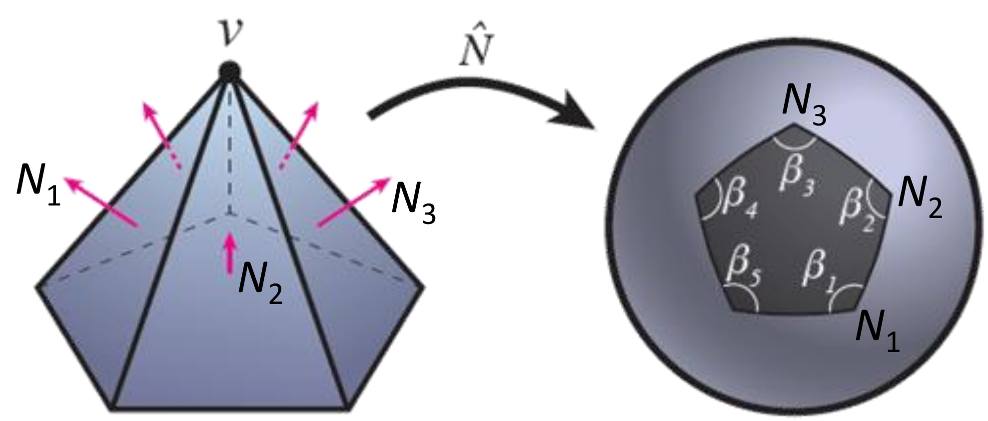
$$= 2\pi - \sum_{i=1}^{k} \alpha_i$$



Discrete Curvature (Surfaces)

<u>What is the angles β_i?</u>

A (geodesic) arc between points *p* and *q* on the sphere is contained in the intersection of the sphere with the plane perpendicular to *p* and *q.*



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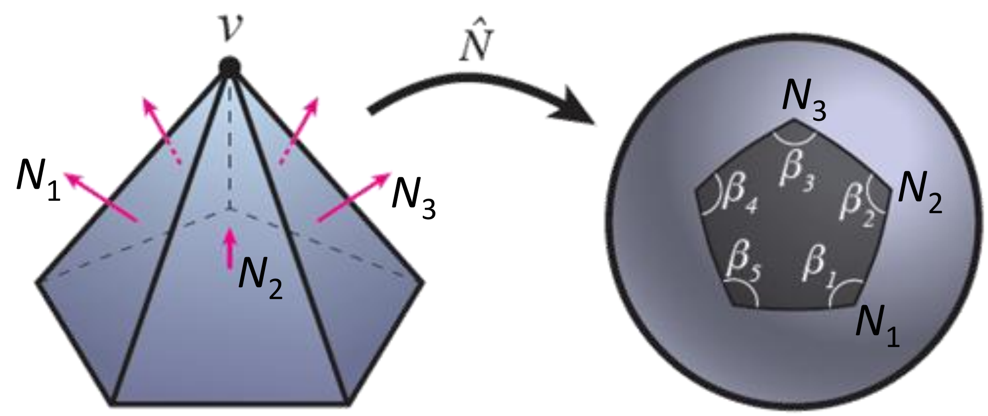
<u>What is the angles β_i?</u>

A (geodesic) arc between points *p* and *q* on the sphere is contained in the intersection of the sphere with the plane perpendicular to *p* and *q*.

The angle between two arcs is π minus the angle between the planes' normals.



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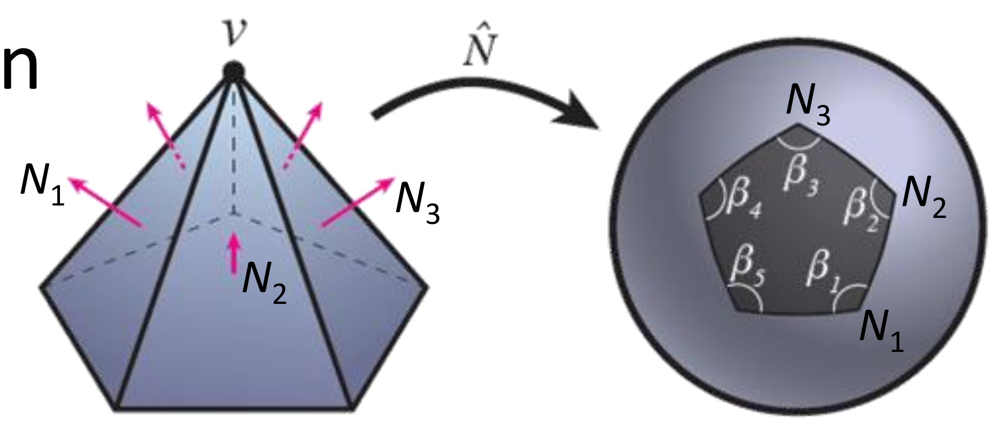
But the edge e_i between triangle $t_{i\text{-}1}$ and N_i is perpendicular to both the normals.



Gauss-Bonnet Theorem (Smooth)

Given a (closed surface) *S*, the integral of the Gaussian curvature over the surface is:

$$\int_S K(p)\,dp = 2\pi\chi_S$$

where χ_S is the *Euler Characteristic* of the surface *S* (an integer that is a topological invariant of the surface).

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What happens in the discrete case?

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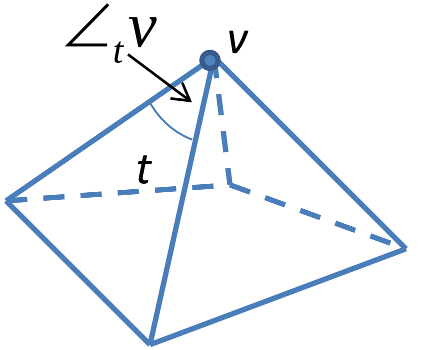
Summing the Gaussian curvatures we get:

$$\sum_{v \in V} K_v = \sum_{v \in V} 2\pi - \left(\sum_{t \in T | t \cap v \neq \emptyset} \angle_t v \right)$$

where *t* is a triangle containing *v* and $\angle_t v$ is the interior angle of *t* at *v*.



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$$= 2\pi |V| - \sum_{t \in T} \sum_{v \in t} \angle_t v$$

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$$= 2\pi|V| - \sum_{t\in T} \pi$$

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$$= 2\pi|V| - \sum_{t\in T} \pi$$

$$= 2\pi|V| - \pi|T|$$

Gauss-Bonnet Theorem (Smooth)

What happens in the discrete case?

Summing the Gaussian curvatures we get:

$$\begin{aligned}
\sum_{v\in V} K_v &= \sum_{v\in V} 2\pi - \left(\sum_{t\in T | t\cap v\neq\emptyset} \angle_t v \right) \\
&= 2\pi|V| - \sum_{t\in T}\sum_{v\in t} \angle_t v \\
&= 2\pi|V| - \sum_{t\in T} \pi \\
&= 2\pi|V| - \pi|T| \\
&= 2\pi|V| - \pi\left(2|E| - 2|T|\right)
\end{aligned}$$

Gauss-Bonnet Theorem (Smooth)

What happens in the discrete case?

Summing the Gaussian curvatures we get:

$$\begin{aligned}
\sum_{v\in V} K_v &= \sum_{v\in V} 2\pi - \left(\sum_{t\in T \mid t\cap v\neq\emptyset} \angle_t v \right) \\
&= 2\pi|V| - \sum_{t\in T}\sum_{v\in t} \angle_t v \\
&= 2\pi|V| - \sum_{t\in T} \pi \\
&= 2\pi|V| - \pi|T| \\
&= 2\pi|V| - \pi\left(2|E| - 2|T|\right) \\
&= 2\pi\left(|V| - |E| + |T|\right)
\end{aligned}$$

Gauss-Bonnet Theorem (Smooth)

What happens in the discrete case?

In the discrete case, the sum of the Gaussian curvature is equal to:

$$\sum_{v \in V} K_v = 2\pi\left(|V| - |E| + |T|\right)$$

Gauss-Bonnet Theorem (Smooth)

What happens in the discrete case?

In the discrete case, the sum of the Gaussian curvature is equal to:

$$\sum_{v \in V} K_v = 2\pi\left(|V| - |E| + |T|\right)$$

Note that for a closed polyhedron:

$$\chi = |V| - |E| + |T|$$

is the Euler Characteristic, and satisfies:

$$\chi = 2 - 2g$$

where *g* is the genus of the surface.