Differential Geometry:

Surfaces and Parameterizations

Differential Geometry of Curves and Surfaces, Do Carmo (Chapter 2)

Math Review

Derivatives:

Given a function $F:\mathbf{R}^n \rightarrow \mathbf{R}^m$, the derivative of F at a point $p \in \mathbf{R}^n$ is the matrix dF_p which describes the "small change" in the position at $F(p)$ that would correspond to a "small change" in the position at p.

Math Review

Derivatives:

If F:$\boldsymbol{R}^n \to \boldsymbol{R}^m$ is expressed in terms of its coordinate functions $F(p)=(\, f_1(p),\ldots,f_m(p)\,)$ then the derivative is the nxm matrix:

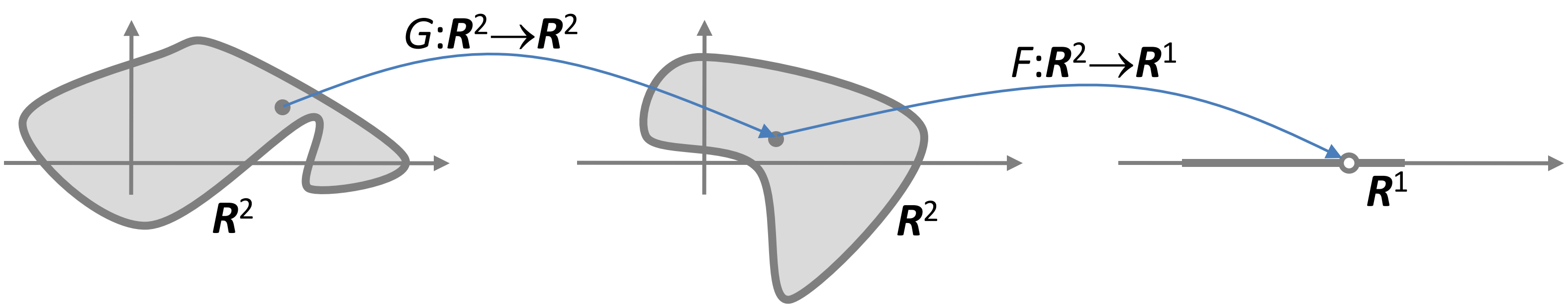
$$dF = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

Math Review

<u>Chain Rule</u>:

Given a function $F:\mathbf{R}^n \rightarrow \mathbf{R}^m$ and given a function $G:\mathbf{R}^n \rightarrow \mathbf{R}^n$, the derivative of the function $F \circ G$ is:

$$d\left(F \circ G\right)_p = dF_{G(p)} dG_p$$

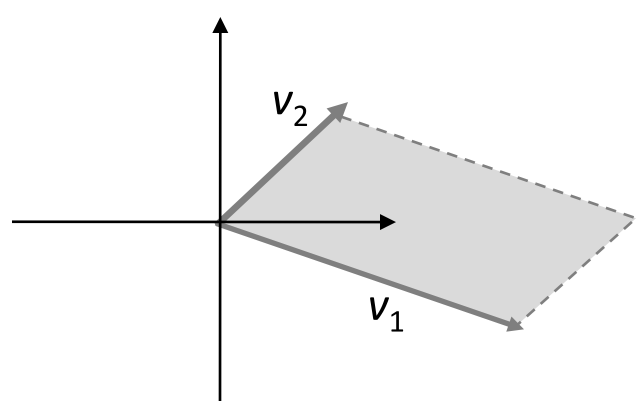


Math Review

Determinant:

Given vectors $v_1,\ldots,v_n$ in $\boldsymbol{R}^n$, the area of the parallelepiped defined by the vectors is equal to the determinant of the matrix:

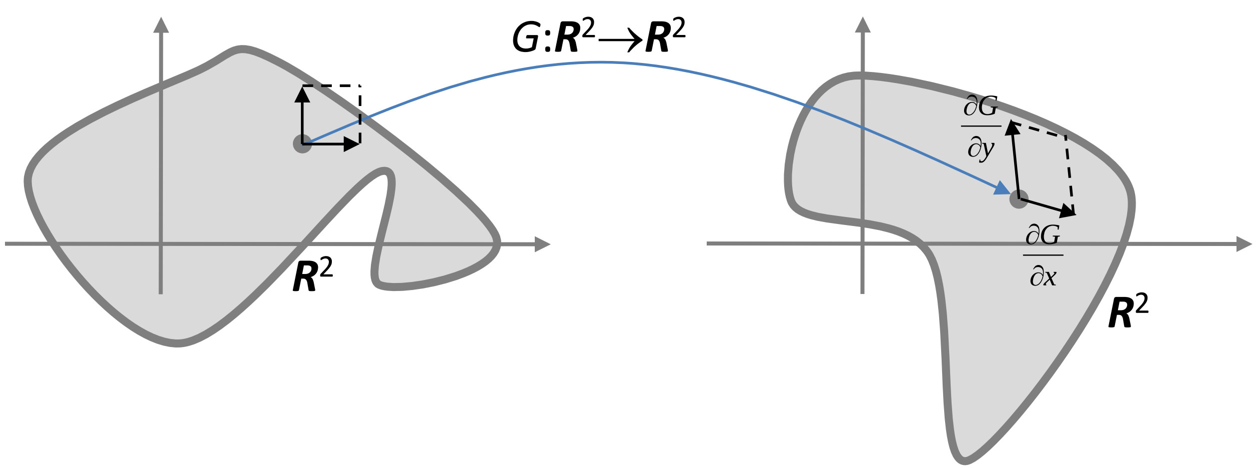
$$\left(v_1 \mid v_2 \mid \ldots \mid v_{n-1} \mid v_n\right)$$



Math Review

Determinant:

Given a function $G:\mathbf{R}^n \rightarrow \mathbf{R}^n$, the determinant of the derivative of G at a point p gives the area of the parallelepiped that is the image of a "small square" at p.

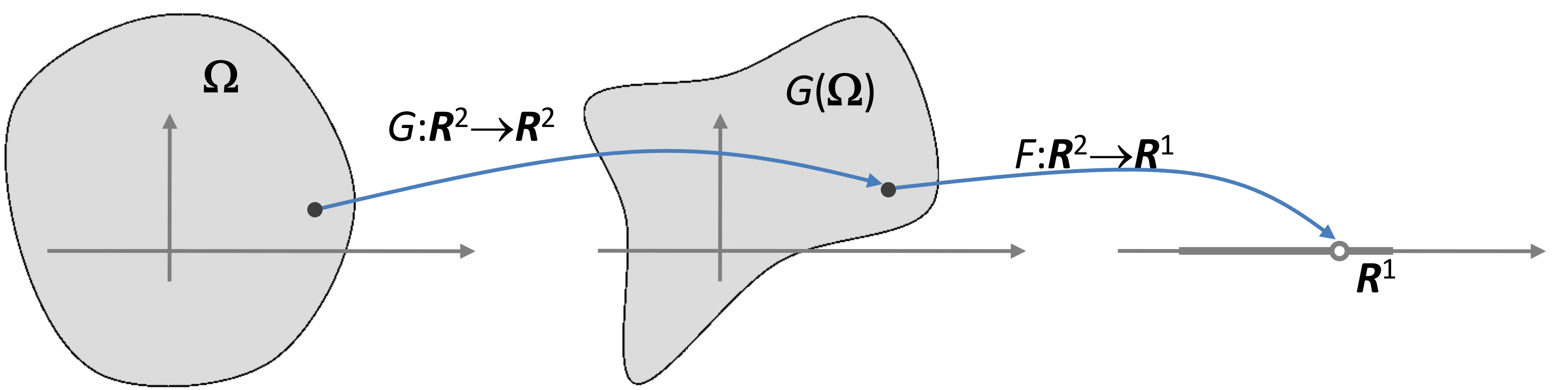


Math Review

Integration:

Given a function $F:\mathbf{R}^n \rightarrow \mathbf{R}^m$ and given a invertible function $G:\mathbf{R}^n \rightarrow \mathbf{R}^n$, the integral of the function $F \circ G$ over a domain $\Omega \subset \mathbf{R}^n$ is:

$$\int_{\Omega} (F \circ G)(p)\,dp = \int_{G(\Omega)} F(q) \left| \det d\left(G^{-1}\right)_q \right| dq$$

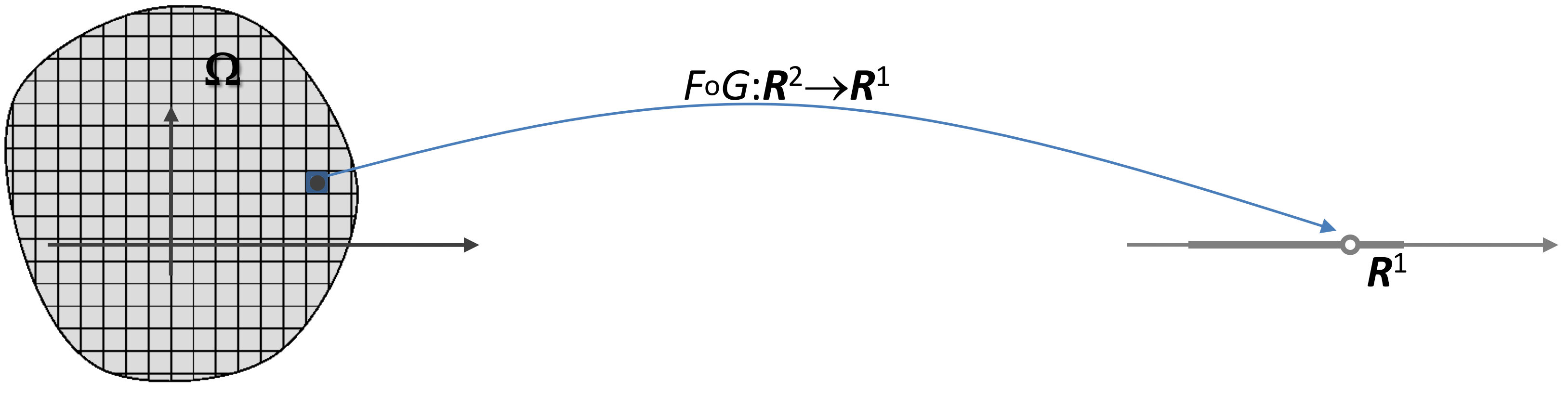


Math Review

<u>Integration</u>:

The integral of *F*∘*G* over Ω can be obtained by tessellating the Ω and then taking the sum of the values of *F*∘*G* weighted by the area of the squares.

$$\int_{\Omega} (F \circ G)(p)\,dp \approx \sum_i (F \circ G)\left(p_i^{\Omega}\right) \cdot \mathrm{Area}\left(S_i^{\Omega}\right)$$

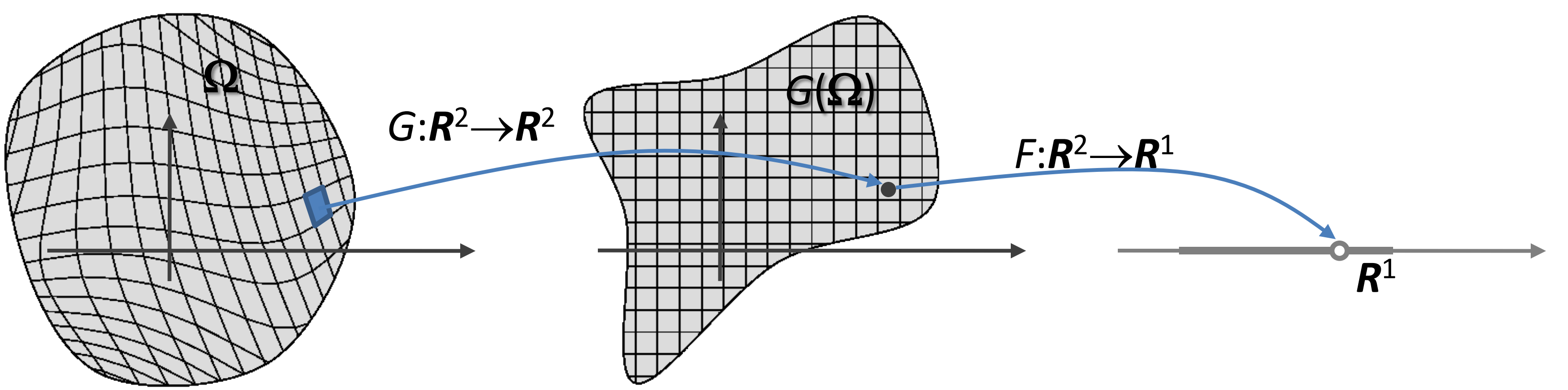


Math Review

Integration:

Alternatively, we can tessellate $G(\Omega)$ and weight the contribution by the area of the pre-image of the squares on Ω:

$$\int_{\Omega} (F \circ G)(p)\,dp \approx \sum_i F\left(q_i^{G(\Omega)}\right) \cdot \mathrm{Area}\left(G^{-1}\left(S_i^{G(\Omega)}\right)\right)$$



Math Review

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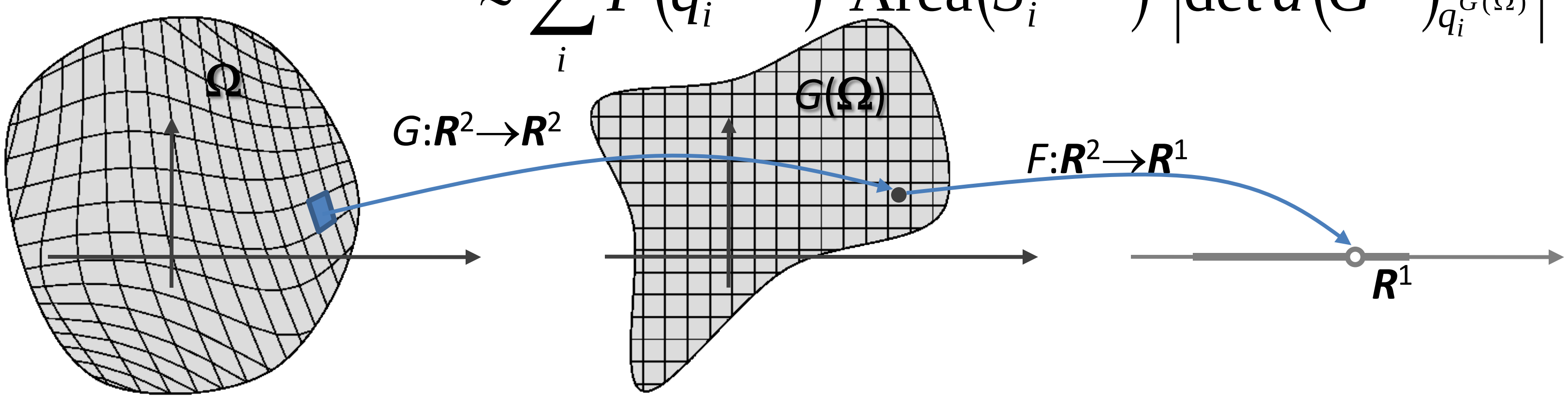
$$\approx \sum_i F\left(q_i^{G(\Omega)}\right) \cdot \text{Area}\left(S_i^{G(\Omega)}\right) \cdot \left|\det d\left(G^{-1}\right)_{q_i^{G(\Omega)}}\right|$$



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$$\approx \int_{G(\Omega)} F(q)\left|\det d\left(G^{-1}\right)_q\right| dq$$

Math Review

Taylor Series:

Given a function F:$\boldsymbol{R}^2 \rightarrow \boldsymbol{R}$, we can approximate the function near the point (0,0) by its *Taylor Series*:

$$F(x,y) \approx \underbrace{F(0,0)}_{\text{Constant}} + \underbrace{\left.\frac{\partial F}{\partial x}\right|_{(0,0)} x + \left.\frac{\partial F}{\partial y}\right|_{(0,0)} y}_{\text{Linear}} + \underbrace{\frac{1}{2}\left(\left.\frac{\partial^2 F}{\partial x^2}\right|_{(0,0)} x^2 + \left.\frac{\partial^2 F}{\partial y^2}\right|_{(0,0)} y^2 + 2\left.\frac{\partial^2 F}{\partial x \partial y}\right|_{(0,0)} xy \right)}_{\text{Quadratic}}$$

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If $F(0,0)=0$ and $(\partial F/\partial x, \partial F/\partial y)(0,0)=0$ the Taylor Series simplifies to:

$$F(x,y) \approx \frac{1}{2}\left(\left.\frac{\partial^2 F}{\partial x^2}\right|_{(0,0)} x^2 + \left.\frac{\partial^2 F}{\partial y^2}\right|_{(0,0)} y^2 + 2\left.\frac{\partial^2 F}{\partial x \partial y}\right|_{(0,0)} xy \right)$$

Math Review

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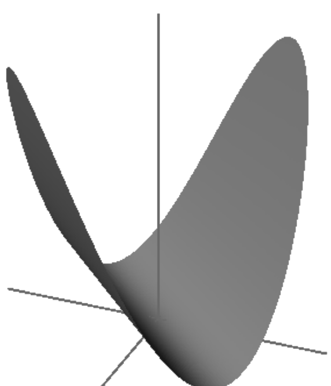
<u>Quadratic Forms</u>:

Given a *quadratic form* F:$\mathbf{R}^2 \rightarrow \mathbf{R}$:

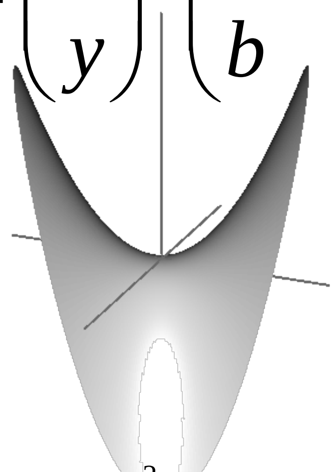
$$F(x,y) = ax^2 + 2bxy + cy^2$$

We can re-write F as:

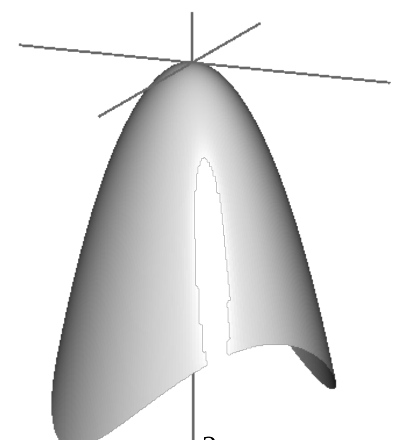
$$F(x,y) = \begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$



$F(x,y) = x^2 + 2xy + y^2$



$F(x,y) = x^2 + 2xy - y^2$



$F(x,y) = -2x^2 - xy - 2y^2$

Math Review

$$F(x,y) = \begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

<u>Symmetric Matrices</u>:

Any symmetric matrix *M* can be expressed as:

$$M = R^t \Delta R$$

where *R* is a rotation and Δ is a diagonal matrix.

Math Review

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Symmetric Matrices:

Any symmetric matrix *M* can be expressed as:

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where *R* is a rotation and Δ is a diagonal matrix.

In particular, this implies that if we perform a change of coordinates (*u*,*v*)=*R*(*x*,*y*), we get:

$$F(u,v) = \begin{pmatrix} u \\ v \end{pmatrix}^T \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

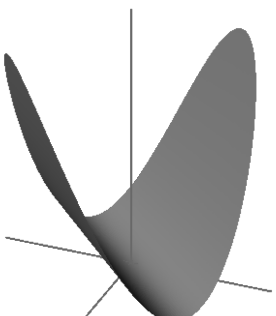
Math Review

<u>Quadratic Forms</u>:

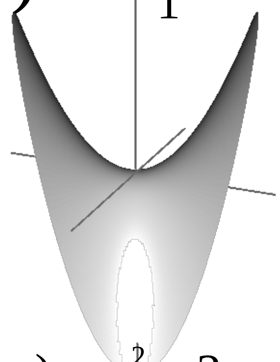
$$F(x,y) \approx \frac{1}{2}\left(\left.\frac{\partial^2 F}{\partial x^2}\right|_{(0,0)} x^2 + \left.\frac{\partial^2 F}{\partial y^2}\right|_{(0,0)} y^2 + 2\left.\frac{\partial^2 F}{\partial x \partial y}\right|_{(0,0)} xy \right)$$

Up to rotation, all quadratic form look like:

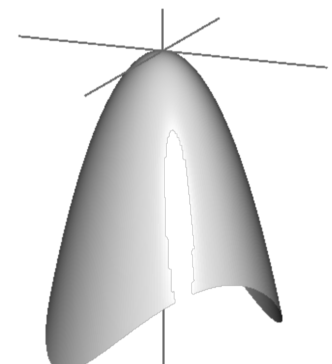
$$F(x,y) = \lambda_1 x^2 + \lambda_2 y^2$$



$F(x,y) = x^2 + 2xy + y^2$



$F(x,y) = x^2 + 2xy - y^2$



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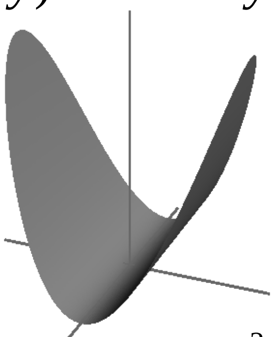
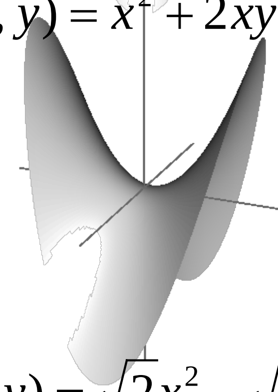
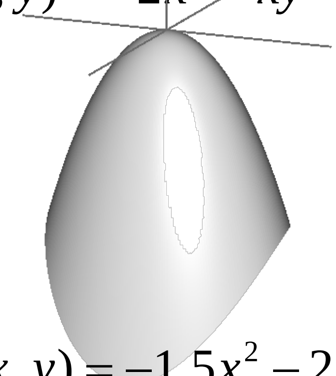


$F(x,y) = x^2$



$F(x,y) = \sqrt{2}x^2 - \sqrt{2}y^2$



$F(x,y) = -1.5x^2 - 2.5y^2$

Math Review

$$F(x,y) \approx \frac{1}{2}\left(\left.\frac{\partial^2 F}{\partial x^2}\right|_{(0,0)} x^2 + \left.\frac{\partial^2 F}{\partial y^2}\right|_{(0,0)} y^2 + 2\left.\frac{\partial^2 F}{\partial x \partial y}\right|_{(0,0)} xy \right)$$

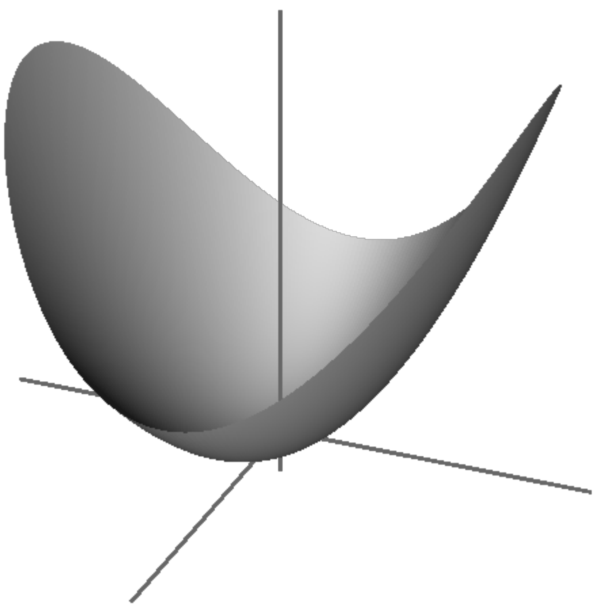
<u>Quadratic Forms</u>:

So that, up to rotation, the quadratic form will look like:

$$F(x,y) = \lambda_1 x^2 + \lambda_2 y^2$$

with $\lambda_1 \geq \lambda_2$:

- If $\lambda_1, \lambda_2 > 0$: Upward Parabola



Math Review

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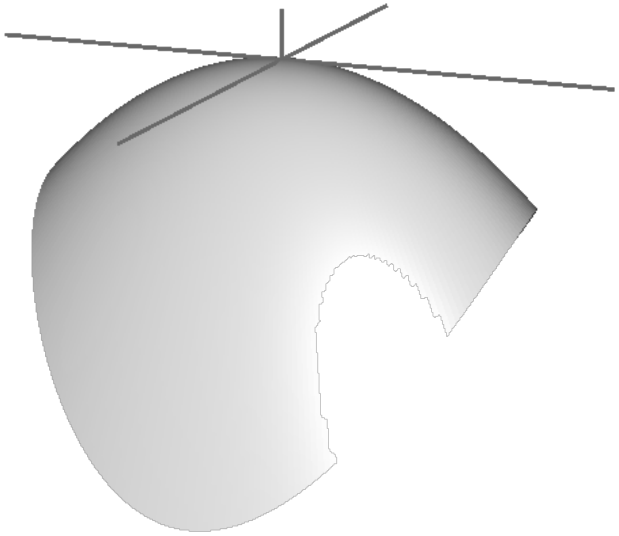
with $\lambda_1 \geq \lambda_2$:

- If $\lambda_1, \lambda_2 < 0$: Downward Parabola



Math Review

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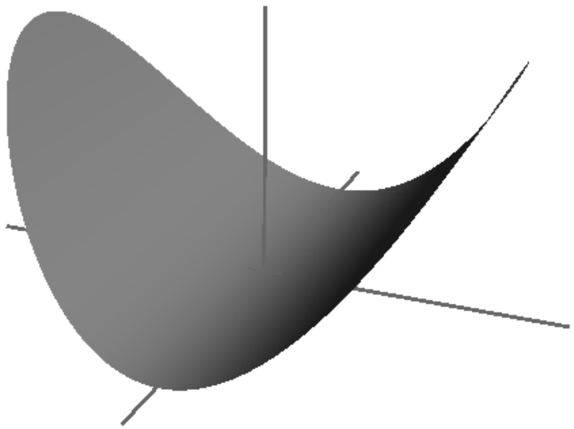
So that, up to rotation, the quadratic form will look like:

$$F(x,y) = \lambda_1 x^2 + \lambda_2 y^2$$

with $\lambda_1 \geq \lambda_2$:

- If $\lambda_1 > 0, \lambda_2 = 0$: Upward Cylinder



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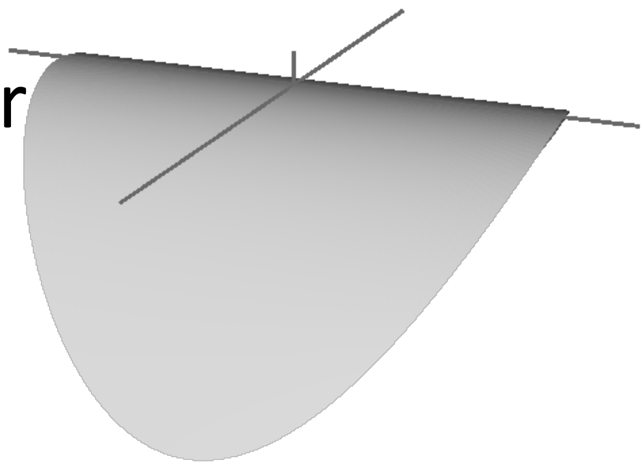
So that, up to rotation, the quadratic form will look like:

$$F(x,y) = \lambda_1 x^2 + \lambda_2 y^2$$

with $\lambda_1 \geq \lambda_2$:

- If $\lambda_1=0, \lambda_2<0$: Downward Cylinder



Math Review

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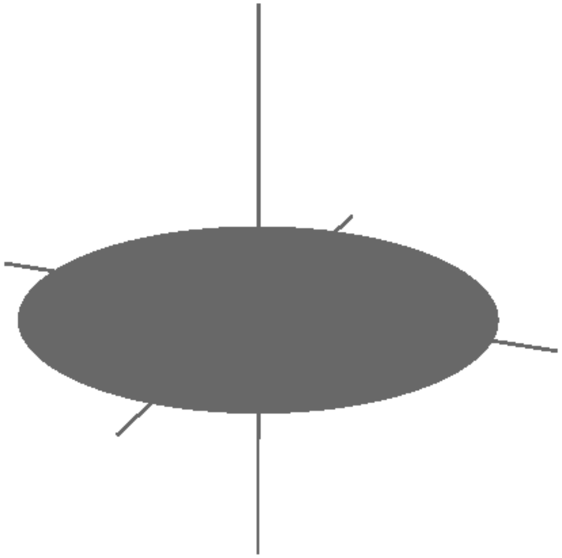
Quadratic Forms:

So that, up to rotation, the quadratic form will look like:

$$F(x,y) = \lambda_1 x^2 + \lambda_2 y^2$$

with $\lambda_1 \geq \lambda_2$:

- If $\lambda_1, \lambda_2 = 0$: Plane



Math Review

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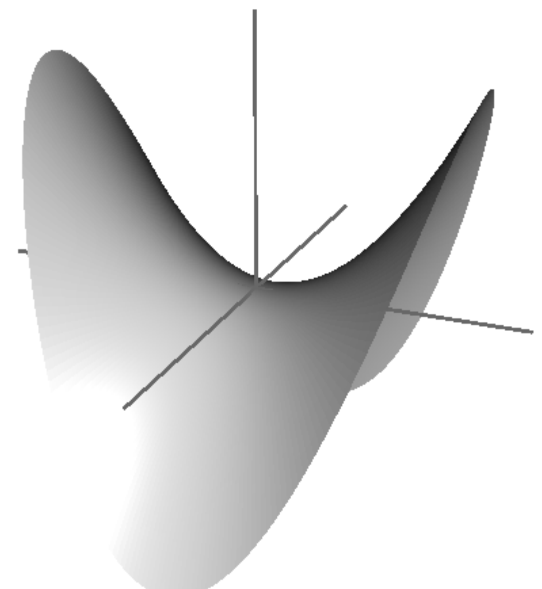
<u>Quadratic Forms</u>:

So that, up to rotation, the quadratic form will look like:

$$F(x,y) = \lambda_1 x^2 + \lambda_2 y^2$$

with $\lambda_1 \geq \lambda_2$:

- If $\lambda_1 > 0, \lambda_2 < 0$: Hyperbola



Tangent Planes

Give a parameterization Φ:$U\rightarrow S$ of a regular surface, we define the *tangent plane, $T_p(S)$*, to be the 2D subspace of $\boldsymbol{R}^3$ that is the span of $d\Phi_p$.

$$d\Phi_p = \begin{pmatrix} \left.\frac{\partial x}{\partial u}\right|_p & \left.\frac{\partial x}{\partial v}\right|_p \\ \left.\frac{\partial y}{\partial u}\right|_p & \left.\frac{\partial y}{\partial v}\right|_p \\ \left.\frac{\partial z}{\partial u}\right|_p & \left.\frac{\partial y}{\partial v}\right|_p \end{pmatrix}$$

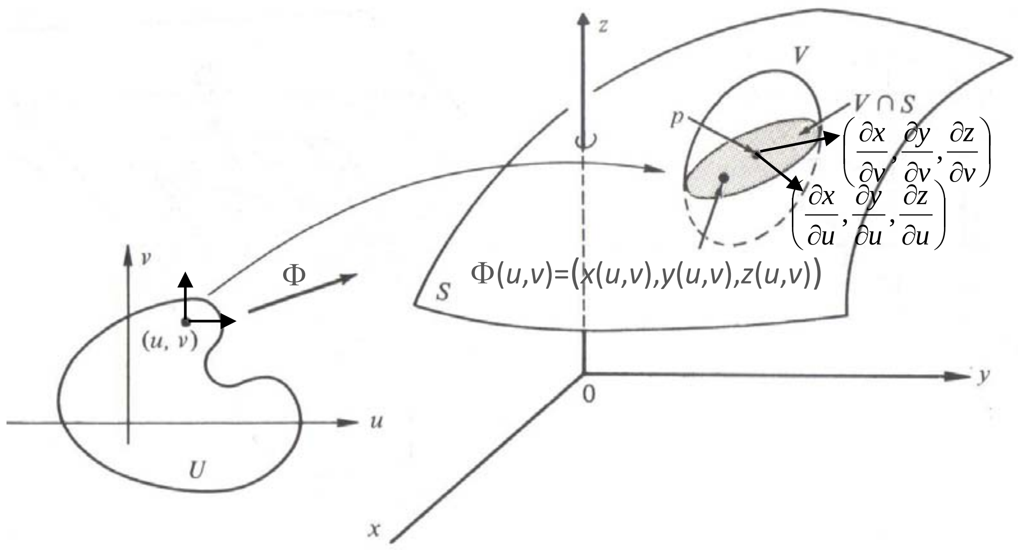


Figure 2-1

Tangent Planes

Note that if Φ:*U*→*S* and Ψ:*W*→*S* are two parameterizations of *S* then they will define the same tangent plane, even if the don’t define the same partials:

$$
\begin{aligned}
d\Phi_p &= d\left(\Phi \circ \Theta \circ \Theta^{-1}\right)_p \\
&= d\left(\Phi \circ \Theta\right)_{\Theta^{-1}(p)} d\left(\Theta^{-1}\right)_p \\
&= d\Psi_{\Theta^{-1}(p)} d\left(\Theta^{-1}\right)_p
\end{aligned}
$$

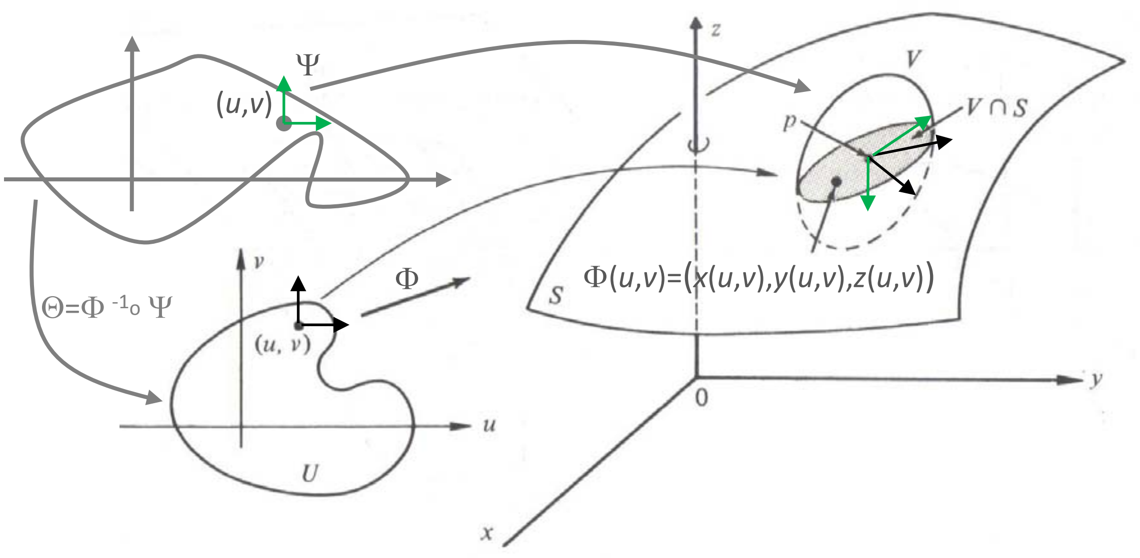


Figure 2-1

Tangent Planes

At each point *p*, the tangent plane is perpendicular to a normal line.

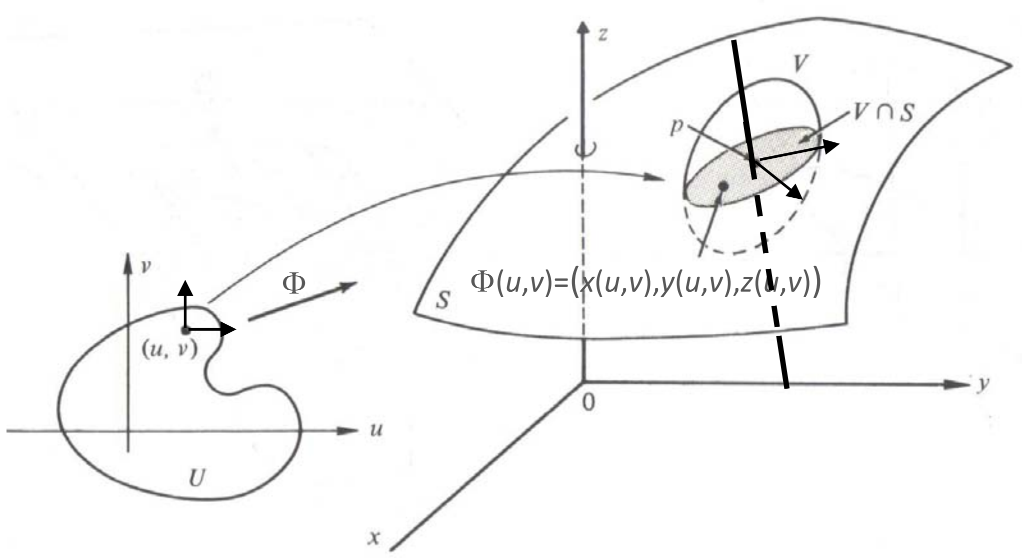


Figure 2-1

Tangent Planes

We say that a surface is *orientable* if we can define a differentiable function, $N\!:\!S \rightarrow S^2$, that assigns a unit-normal vector to each point $p \in S$.

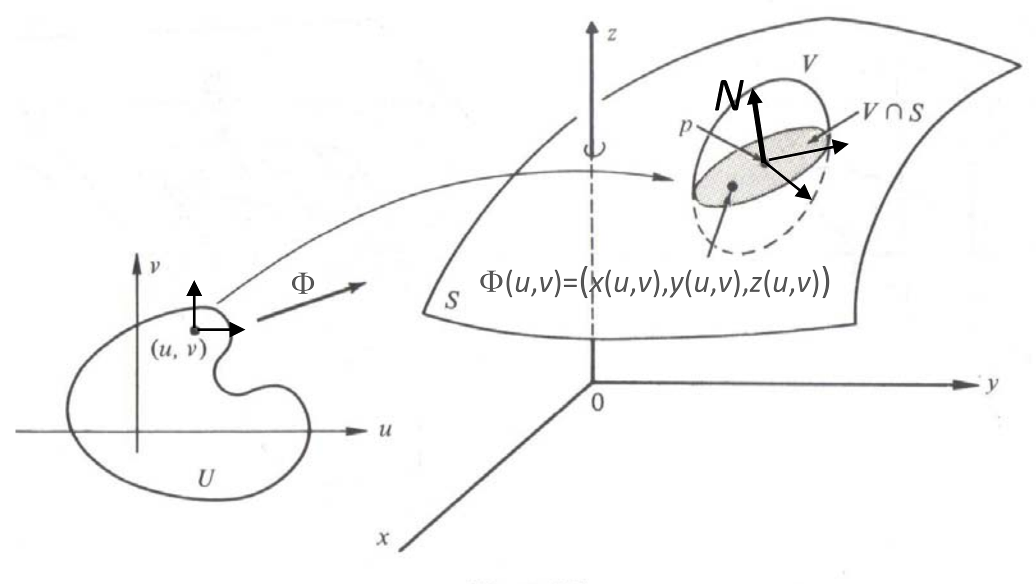
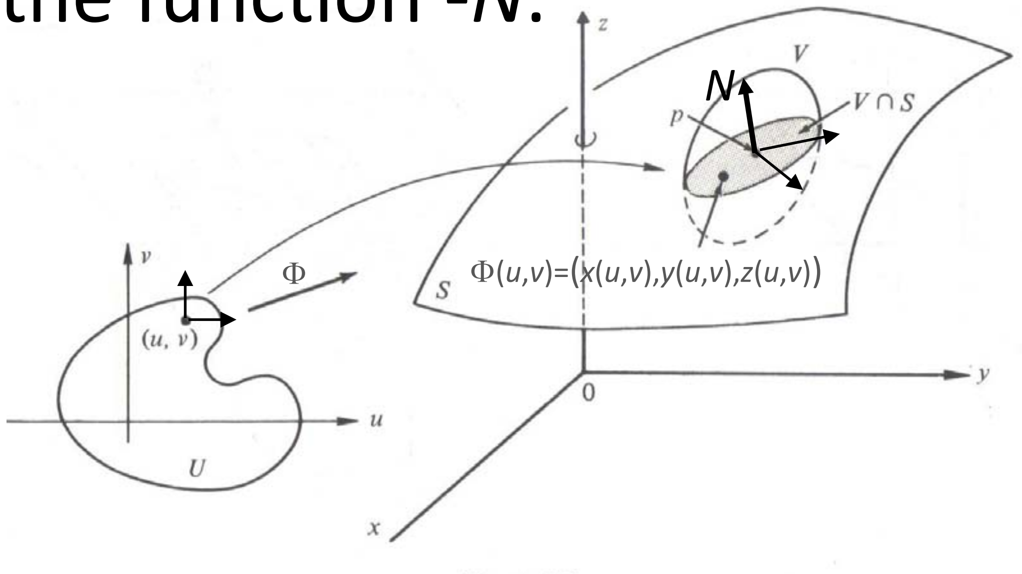


Figure 2-1

Tangent Planes

We say that a surface is *orientable* if we can define a differentiable function, $N{:}S\rightarrow S^2$, that assigns a unit-normal vector to each point $p\in S$.

Note that if the normal function $N{:}S\rightarrow S^2$ is continuous, then so is the function $-N$.



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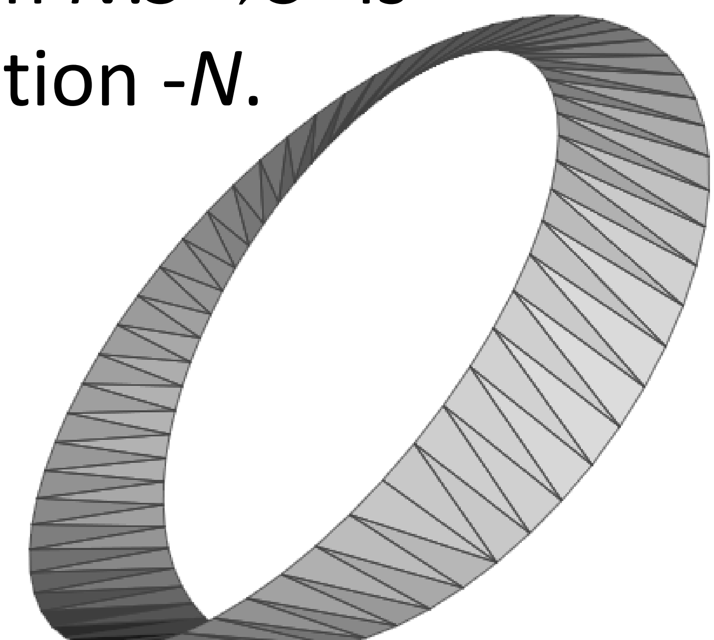
Note that even if the surface is not orientable, we can use the parameterization to assign a normal locally:

$$N(p)=\left.\frac{\partial\Phi}{\partial u}\right|_p\times\left.\frac{\partial\Phi}{\partial v}\right|_p$$



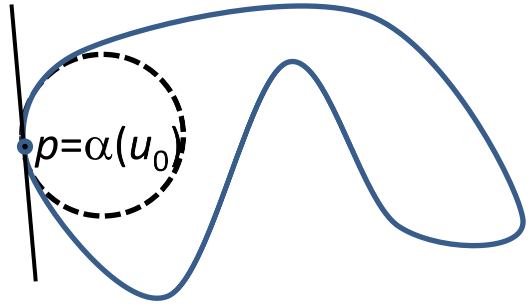
Curvature and Graphs

<u>Recall</u>:

For a regular curve $\alpha:[a,b]\rightarrow \boldsymbol{R}^2$, the curvature is defined as:

$$\kappa(u_0) = \frac{|t'(u_0)|}{|\alpha'(u_0)|}$$

where $t(u)$ is the tangent vector at u.



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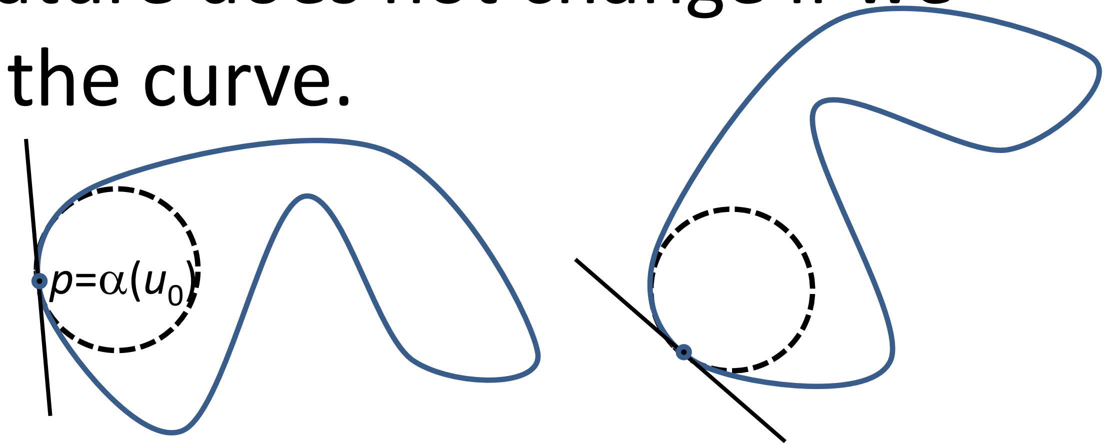
where $t(u)$ is the tangent vector at u.

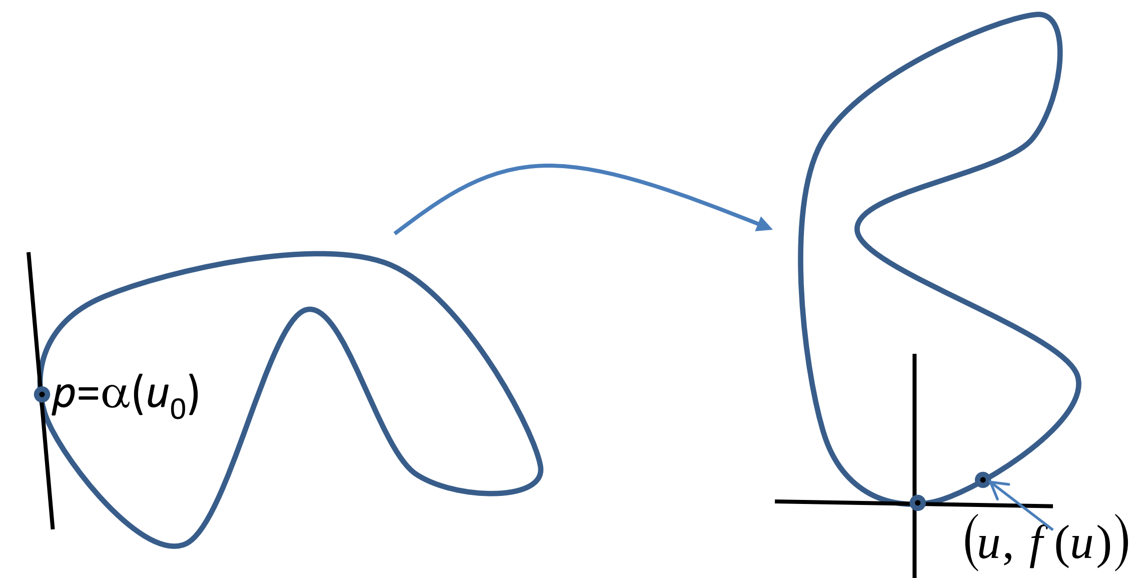
Note that the curvature does not change if we rotate or translate the curve.



Curvature and Graphs

If we rotate the curve so that *p* goes to (0,0) and the tangent line at *p* aligns with *x*-axis, we can describe the curve (locally) as the graph:

$$\beta(u) = \big(u, f(u)\big)$$



Curvature and Graphs

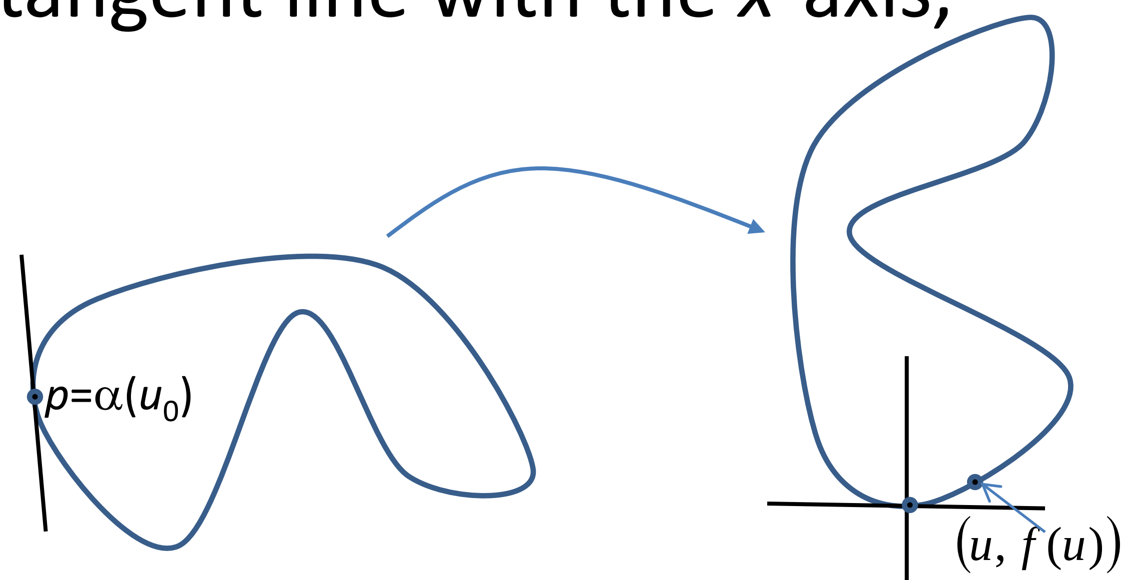
If we rotate the curve so that *p* goes to (0,0) and the tangent line at *p* aligns with *x*-axis, we can describe the curve (locally) as the graph:

$$\beta(u) = (u, f(u))$$

Since we align the tangent line with the *x*-axis, we have:

$$\beta'(u) = (1, f'(u))$$

so that $f'(0)=0$.



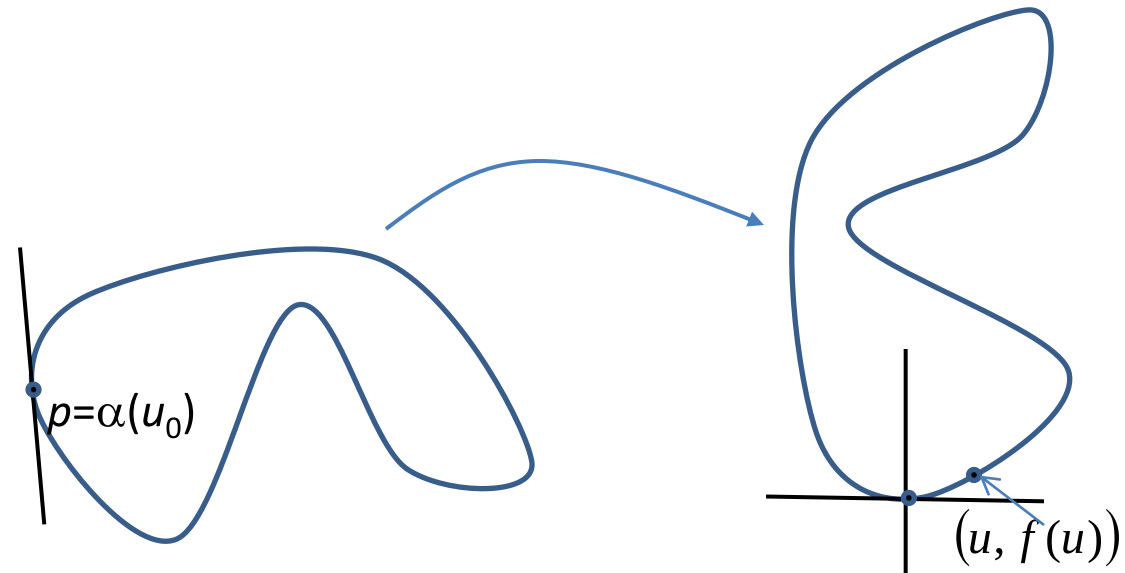
Curvature and Graphs

$$f'(0) = 0$$

$$\beta(u) = \left(u, f(u)\right)$$

Thus, at the point *p*, the curvature can be expressed as:

$$\kappa(p) = \frac{\left|t'(0)\right|}{\left|\beta'(0)\right|}$$



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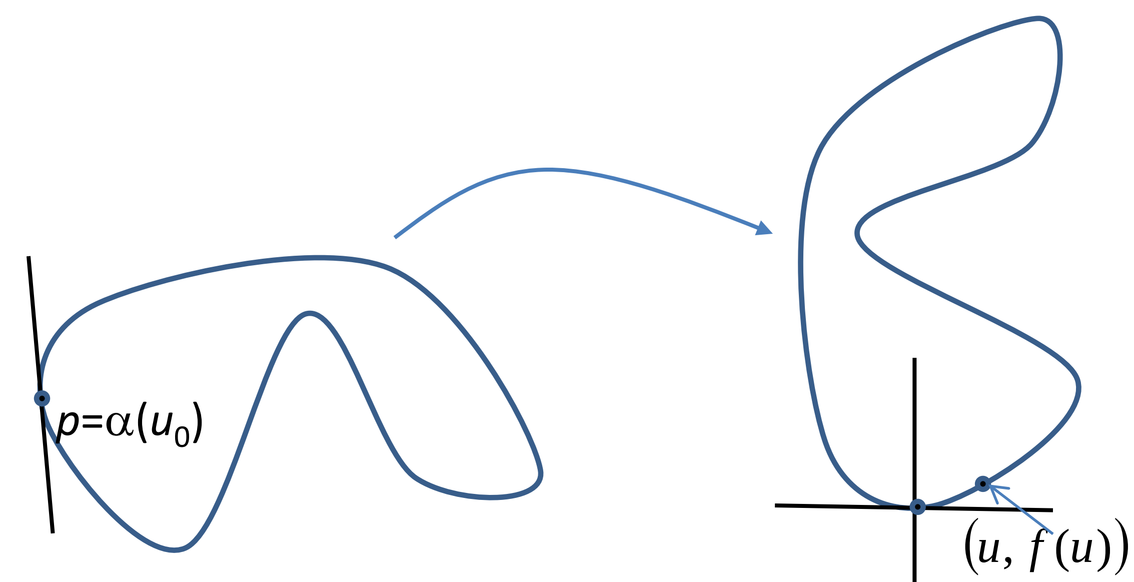

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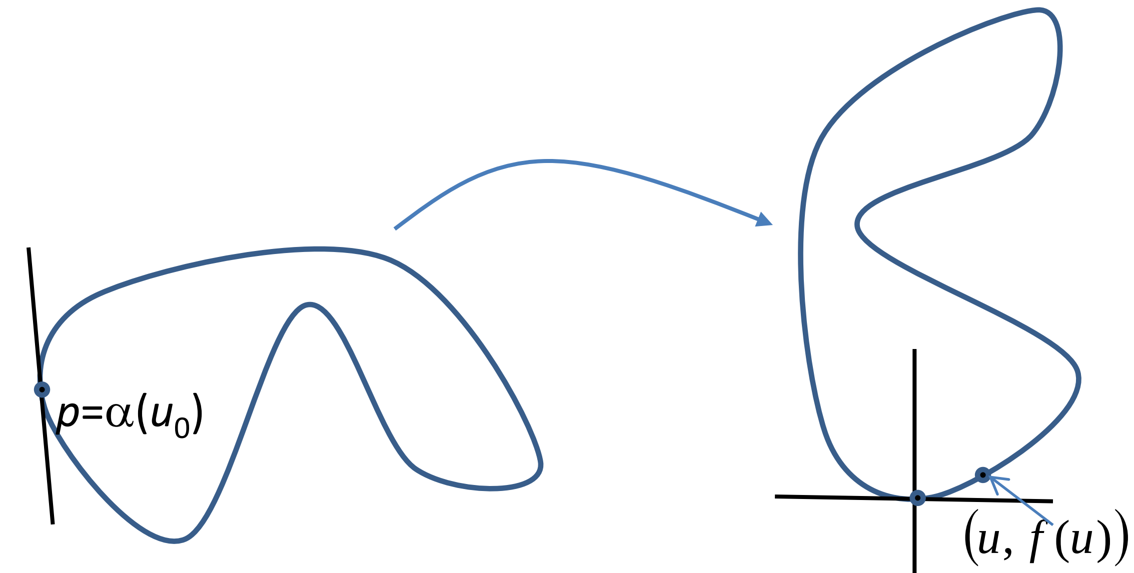

Curvature and Graphs

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So the derivative of the tangent at the point (0,0) is:

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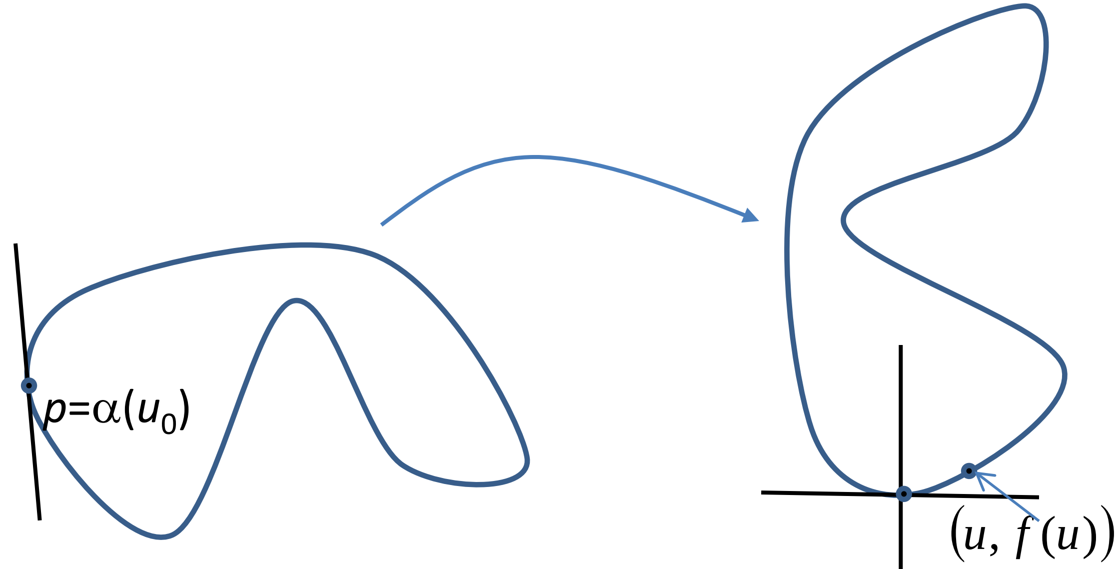

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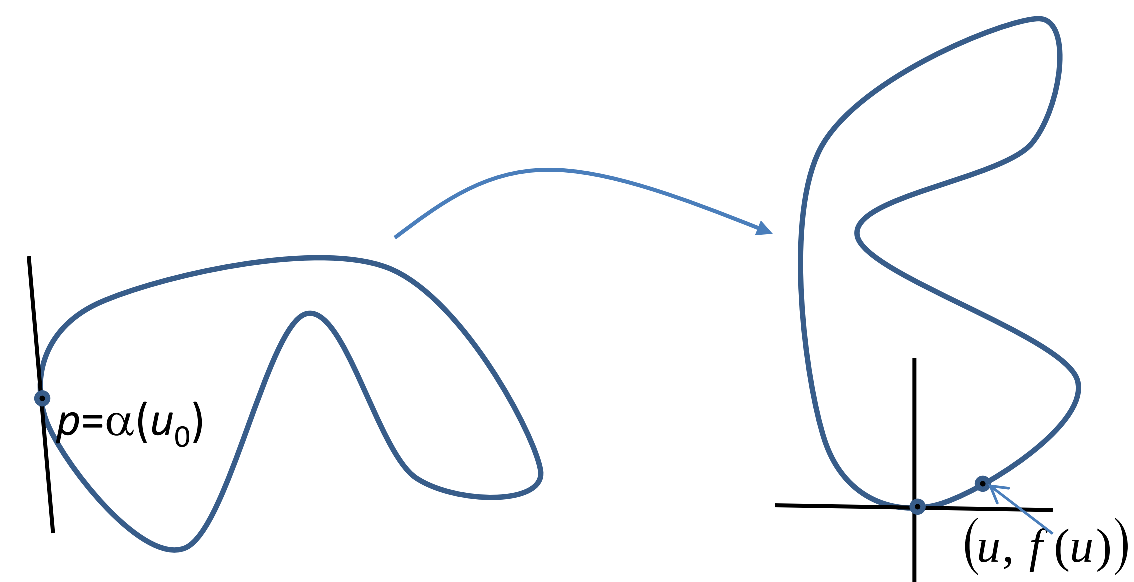

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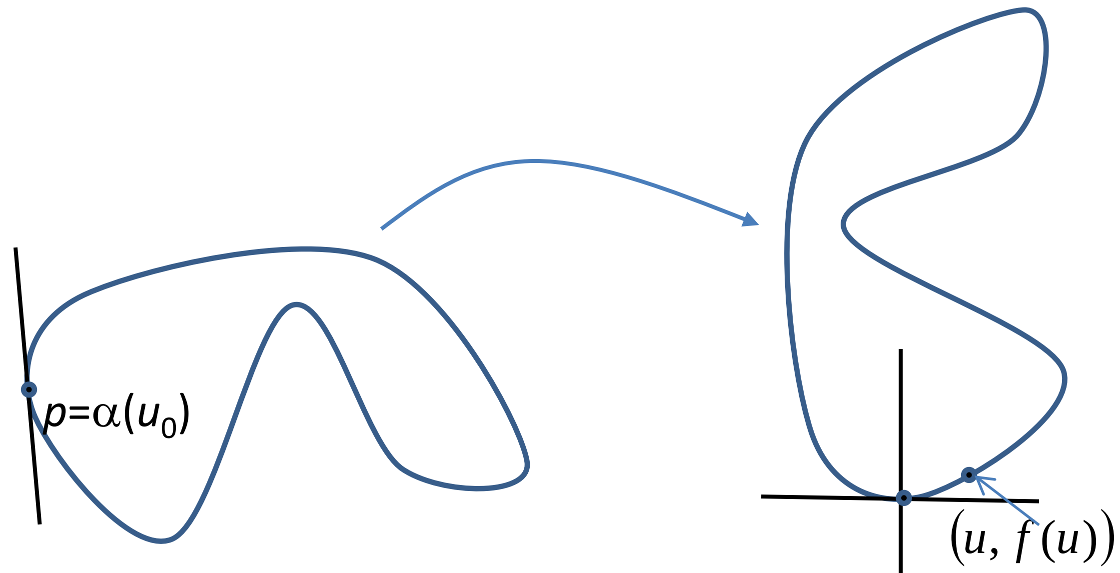
$$\kappa(p)=\frac{|t'(0)|}{|\beta'(0)|}=f''(0)$$



Curvature and Graphs

For a regular surface, we can define the curvatures in an analogous fashion – describing the surface locally as the graph of a function and considering the second-order terms.

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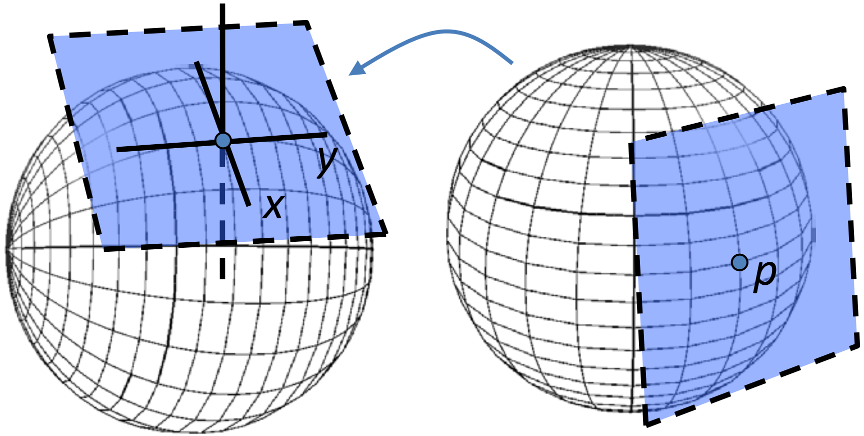
We translate the surface so that *p* goes to the origin, and rotate so that the tangent plane aligns with the *x*-*y* plane.



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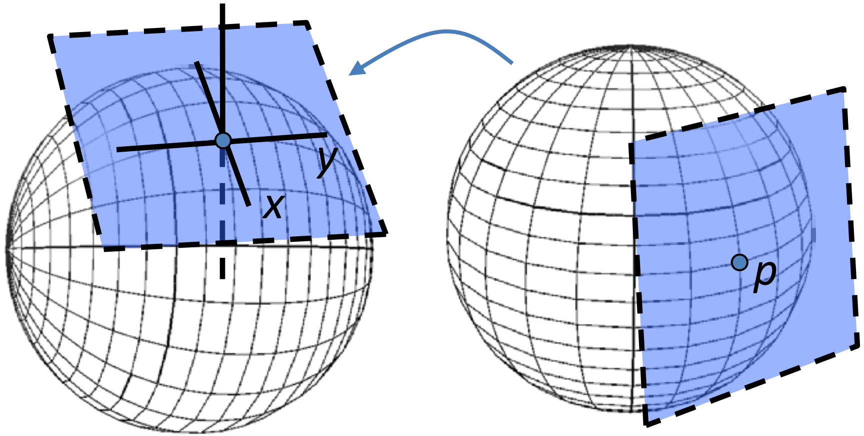
We translate the surface so that *p* goes to the origin, and rotate so that the tangent plane aligns with the *x*-*y* plane.

About the origin, we can describe the surface as the graph $(\,x\,,\,y\,,\,f(x,y)\,)$.

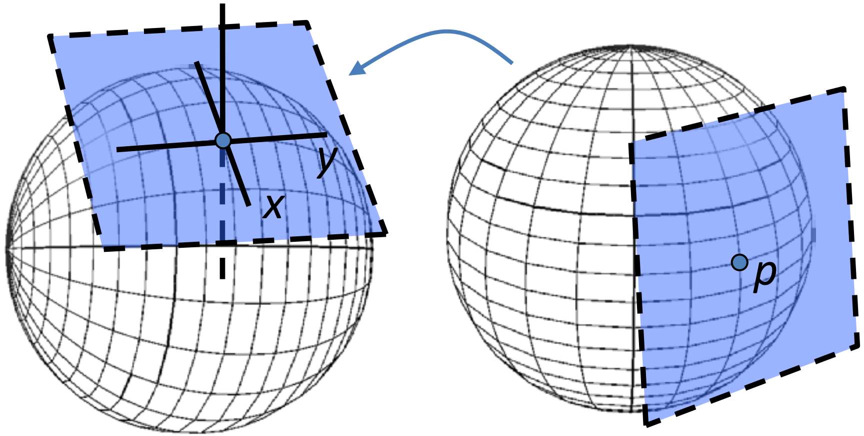


Curvature and Graphs

Note:

1. Since p is sent to the origin, $f(x,y)=0$.
2. Since the tangent plane at p is sent to the x-y plane, the partials satisfy:

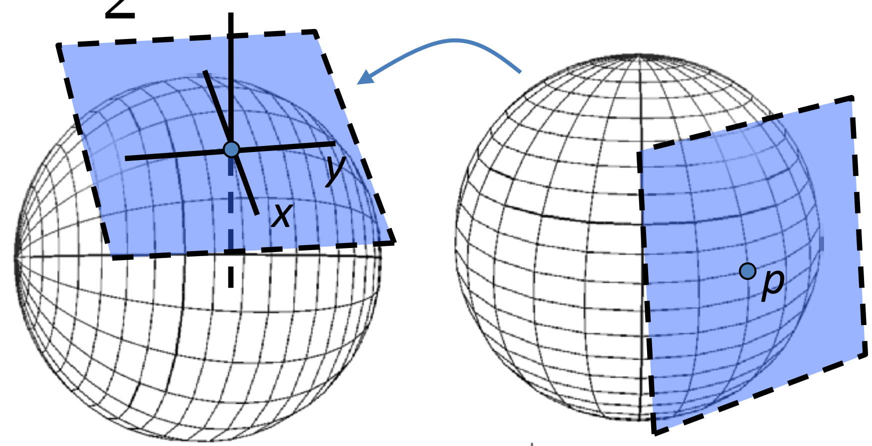
$$\left.\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)\right|_{(0,0)}=0$$

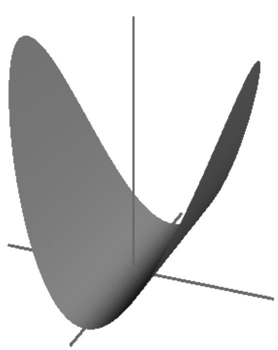


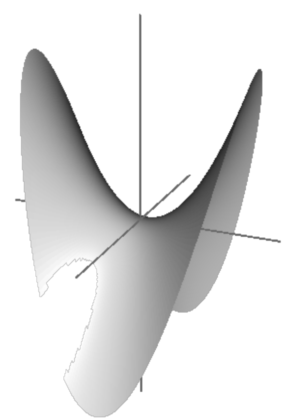
Curvature and Graphs

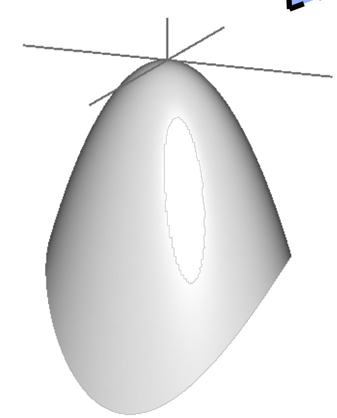
Thus, up to a rotation in the *x*-*y* plane, we have:

$$f(x,y) \approx \frac{\lambda_1}{2}x^2 + \frac{\lambda_2}{2}y^2$$









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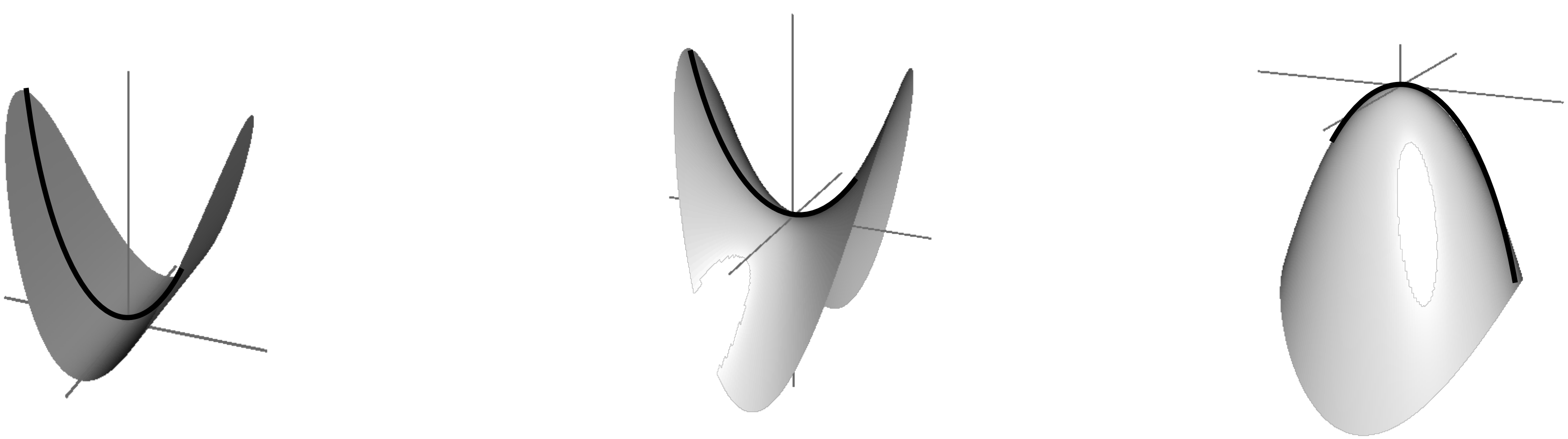
Fixing y=0, we get a curve with curvature λ_1.



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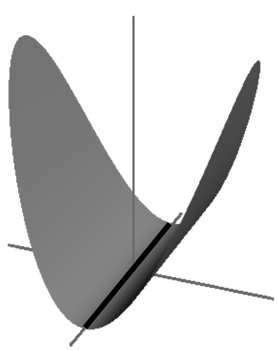
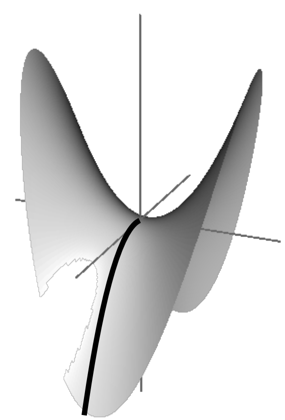
Thus, up to a rotation in the *x*-*y* plane, we have:

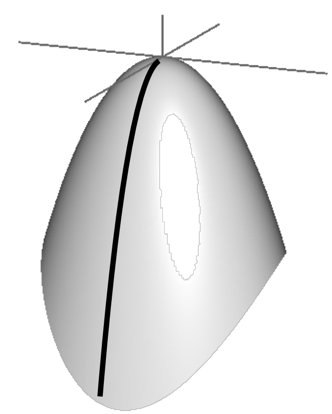
$$f(x,y) \approx \frac{\lambda_1}{2}x^2 + \frac{\lambda_2}{2}y^2$$

Fixing *y*=0, we get a curve with curvature λ_1.

Fixing *x*=0, we get a curve with curvature λ_2.







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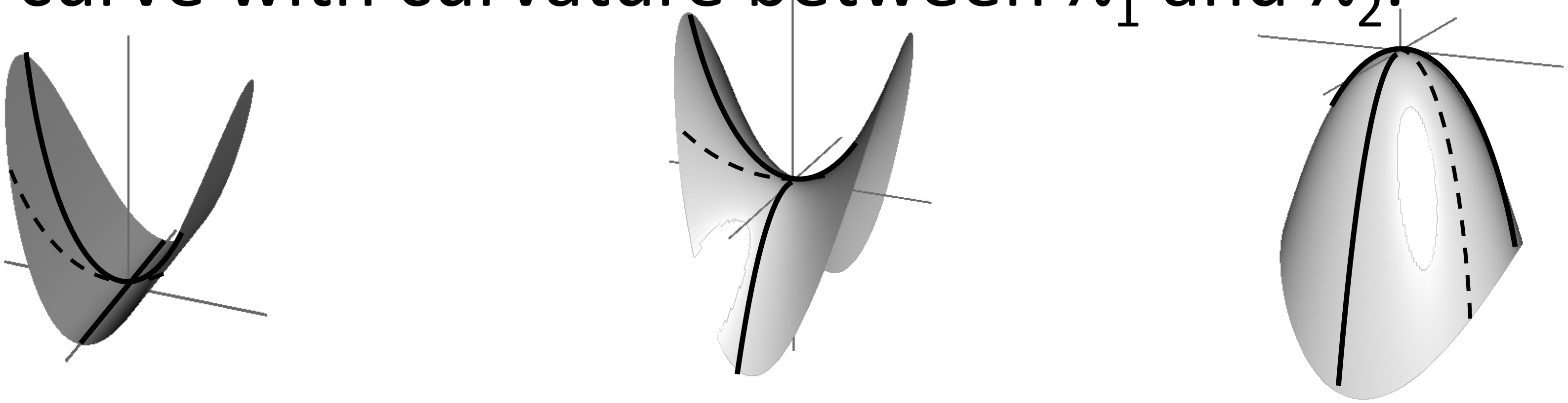
Any other line in the tangent plane will generate a curve with curvature between λ_1 and λ_2.

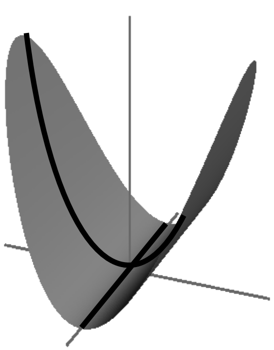


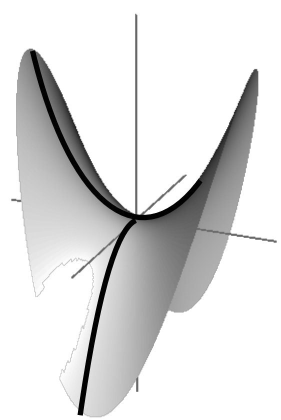
Curvature and Graphs

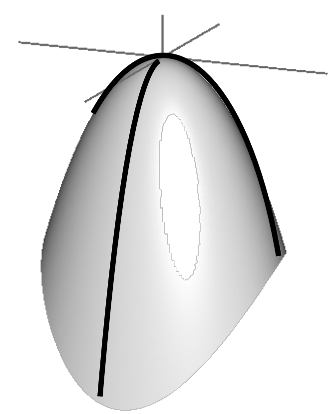
Thus, up to a rotation in the *x-y* plane, we have:

$$f(x,y) \approx \frac{\lambda_1}{2}x^2 + \frac{\lambda_2}{2}y^2$$

The values λ_1 and λ_2 are the *principal curvatures* at *p* and the corresponding directions of the curves at the point *p* are called the *principal directions*.







Curvature and Graphs

<u>Definition</u>:

The product of the principal curvatures, $\lambda_1 \cdot \lambda_2$, is the *Gaussian Curvature*.

The sum of the principal curvatures, $\lambda_1 + \lambda_2$, is the *Mean Curvature*.