Differential Geometry: Surfaces and Parameterizations
Derivatives:
Given a function $F: \mathbb{R}^n \rightarrow \mathbb{R}^m$, the derivative of $F$ at a point $p \in \mathbb{R}^n$ is the matrix $dF_p$ which describes the “small change” in the position at $F(p)$ that would correspond to a “small change” in the position at $p$. 
Math Review

Derivatives:
If \( F: \mathbb{R}^n \rightarrow \mathbb{R}^m \) is expressed in terms of its coordinate functions \( F(p) = (f_1(p), ..., f_m(p)) \) then the derivative is the \( n \times m \) matrix:

\[
dF = \begin{pmatrix}
\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\frac{\partial f_2}{\partial x_1} & \cdots & \frac{\partial f_2}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n}
\end{pmatrix}
\]
Chain Rule:

Given a function $F: \mathbb{R}^n \rightarrow \mathbb{R}^m$ and given a function $G: \mathbb{R}^n \rightarrow \mathbb{R}^n$, the derivative of the function $F \circ G$ is:

$$d(F \circ G)_p = dF_{G(p)}dG_p$$
Math Review

Determinant:
Given vectors $v_1, ..., v_n$ in $\mathbb{R}^n$, the area of the parallelepiped defined by the vectors is equal to the determinant of the matrix:

$$
\begin{vmatrix}
  v_1 & v_2 & \cdots & v_{n-1} & v_n \\
  \end{vmatrix}
$$
Determinant:
Given a function $G: \mathbb{R}^n \rightarrow \mathbb{R}^n$, the determinant of the derivative of $G$ at a point $p$ gives the area of the parallelepiped that is the image of a “small square” at $p$. 
Math Review

Integration:
Given a function $F: \mathbb{R}^n \to \mathbb{R}^m$ and given a invertible function $G: \mathbb{R}^n \to \mathbb{R}^n$, the integral of the function $F \circ G$ over a domain $\Omega \subset \mathbb{R}^n$ is:

$$\int_{\Omega} (F \circ G)(p) dp = \int_{G(\Omega)} F(q) \left| \det d(G^{-1})_q \right| dq$$
Math Review

Integration:
The integral of $F \circ G$ over $\Omega$ can be obtained by tessellating the $\Omega$ and then taking the sum of the values of $F \circ G$ weighted by the area of the squares.

$$\int_{\Omega} (F \circ G)(p) dp \approx \sum_{i} (F \circ G)(p_i^\Omega) \cdot \text{Area}(S_i^\Omega)$$
Math Review

Integration:
Alternatively, we can tessellate $G(\Omega)$ and weight the contribution by the area of the pre-image of the squares on $\Omega$:

\[
\int_{\Omega} (F \circ G)(p) dp \approx \sum_i F\left(q_i^{G(\Omega)}\right) \cdot \text{Area}\left(G^{-1}\left(S_i^{G(\Omega)}\right)\right)
\]
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\]

\[
\approx \sum_i F(q_i^{G(\Omega)}) \cdot \text{Area}(S_i^{G(\Omega)}) \cdot \left| \det d(G^{-1})_{q_i^{G(\Omega)}} \right|
\]
Math Review

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$$\approx \sum_{i} F\left(q_{i}^{G(\Omega)}\right) \cdot \text{Area}\left(S_{i}^{G(\Omega)}\right) \cdot \left|\det d\left(G^{-1}\right)_{q_{i}^{G(\Omega)}}\right|$$

$$\approx \int_{G(\Omega)} F(q) \left|\det d\left(G^{-1}\right)_{q}\right| dq$$
Math Review

Taylor Series:
Given a function $F: \mathbb{R}^2 \rightarrow \mathbb{R}$, we can approximate the function near the point $(0,0)$ by its *Taylor Series*:

$$F(x, y) \approx F(0,0) + \left. \frac{\partial F}{\partial x} \right|_{(0,0)} x + \left. \frac{\partial F}{\partial y} \right|_{(0,0)} y + \frac{1}{2} \left( \left. \frac{\partial^2 F}{\partial x^2} \right|_{(0,0)} x^2 + \left. \frac{\partial^2 F}{\partial y^2} \right|_{(0,0)} y^2 + 2 \left. \frac{\partial^2 F}{\partial x \partial y} \right|_{(0,0)} xy \right)$$

- Constant
- Linear
- Quadratic
Math Review

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- Constant
- Linear
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If $F(0,0)=0$ and $(\partial F/\partial x, \partial F/\partial y)(0,0)=0$ the Taylor Series simplifies to:

$$F(x, y) \approx \frac{1}{2} \left( \left. \frac{\partial^2 F}{\partial x^2} \right|_{(0,0)} x^2 + \left. \frac{\partial^2 F}{\partial y^2} \right|_{(0,0)} y^2 + 2 \left. \frac{\partial^2 F}{\partial x \partial y} \right|_{(0,0)} xy \right)$$
Math Review

**Quadratic Forms:**

Given a quadratic form \( F: \mathbb{R}^2 \rightarrow \mathbb{R} \):

\[
F(x, y) = ax^2 + 2bxy + cy^2
\]

We can re-write \( F \) as:

\[
F(x, y) = \begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
\]
Symmetric Matrices:

Any symmetric matrix $M$ can be expressed as:

$$M = R^t \Delta R$$

where $R$ is a rotation and $\Delta$ is a diagonal matrix.
Math Review

Symmetric Matrices:

Any symmetric matrix $M$ can be expressed as:

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where $R$ is a rotation and $\Delta$ is a diagonal matrix.

In particular, this implies that if we perform a change of coordinates $(u,v) = R(x,y)$, we get:

$$F(u,v) = \begin{pmatrix} u \\ v \end{pmatrix}^T \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$F(x,y) = \begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$
Math Review

Quadratic Forms:

Up to rotation, all quadratic form look like:

\[ F(x, y) = \lambda_1 x^2 + \lambda_2 y^2 \]

\[
F(x, y) \approx \frac{1}{2} \left( \frac{\partial^2 F}{\partial x^2} \right)_{(0,0)} x^2 + \frac{\partial^2 F}{\partial y^2} \right)_{(0,0)} y^2 + 2 \frac{\partial^2 F}{\partial x \partial y} \right)_{(0,0)} xy
\]
Math Review

Quadratic Forms:

So that, up to rotation, the quadratic form will look like:

$$F(x, y) = \lambda_1 x^2 + \lambda_2 y^2$$

with $\lambda_1 \geq \lambda_2$:

- If $\lambda_1, \lambda_2 > 0$: Upward Parabola
Math Review

Quadratic Forms:

So that, up to rotation, the quadratic form will look like:

\[
F(x, y) \approx \frac{1}{2} \left( \frac{\partial^{2} F}{\partial x^{2}} \right)_{(0,0)} \left( x^{2} + \frac{\partial^{2} F}{\partial y^{2}} \right)_{(0,0)} y^{2} + 2 \frac{\partial^{2} F}{\partial x \partial y} \left( xy \right)_{(0,0)}
\]

\[
F(x, y) = \lambda_{1} x^{2} + \lambda_{2} y^{2}
\]

with \( \lambda_{1} \geq \lambda_{2} \):

• If \( \lambda_{1}, \lambda_{2} < 0 \): Downward Parabola
Math Review

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Math Review

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\]

with \( \lambda_1 \geq \lambda_2 \):

• If \( \lambda_1, \lambda_2 = 0 \): Plane
Math Review

Quadratic Forms:

So that, up to rotation, the quadratic form will look like:

\[
F(x, y) = \frac{1}{2} \begin{vmatrix}
\frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} & \frac{\partial^2 F}{\partial y^2} \\
(0,0) & (0,0) & (0,0)
\end{vmatrix} x^2 + \frac{\partial^2 F}{\partial x \partial y} x y + \frac{\partial^2 F}{\partial y^2} y^2
\]

with \( \lambda_1 \geq \lambda_2 \):

- If \( \lambda_1 > 0, \lambda_2 < 0 \): Hyperbola
Tangent Planes

Give a parameterization $\Phi: U \rightarrow S$ of a regular surface, we define the tangent plane, $T_p(S)$, to be the 2D subspace of $\mathbb{R}^3$ that is the span of $d\Phi_p$.

$$d\Phi_p = \begin{pmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\
\frac{\partial z}{\partial u} & \frac{\partial z}{\partial v}
\end{pmatrix}$$
Tangent Planes

Note that if $Φ:U→S$ and $Ψ:W→S$ are two parameterizations of $S$ then they will define the same tangent plane, even if the don’t define the same partials:

\[
dΦ_p = d\left(Φ \circ Θ \circ Θ^{-1}\right)_p = d(Φ \circ Θ)_{Θ^{-1}(p)} d(Θ^{-1})_p = dΨ_{Θ^{-1}(p)} d(Θ^{-1})_p
\]
Tangent Planes

At each point $p$, the tangent plane is perpendicular to a normal line.
Tangent Planes

We say that a surface is orientable if we can define a differentiable function, \( N:S \to S^2 \), that assigns a unit-normal vector to each point \( p \in S \).
Tangent Planes

We say that a surface is *orientable* if we can define a differentiable function, \( N:S \rightarrow S^2 \), that assigns a unit-normal vector to each point \( p \in S \). Note that if the normal function \( N:S \rightarrow S^2 \) is continuous, then so is the function \(-N\).
We say that a surface is *orientable* if we can define a differentiable function, \( N:S \to S^2 \), that assigns a unit-normal vector to each point \( p \in S \).

Note that if the normal function \( N:S \to S^2 \) is continuous, then so is the function \(-N\).

Note that even if the surface is not orientable, we can use the parameterization to assign a normal locally:

\[
N(p) = \frac{\partial \Phi}{\partial u} \bigg|_p \times \frac{\partial \Phi}{\partial v} \bigg|_p
\]
Recall:

For a regular curve $\alpha: [a, b] \rightarrow \mathbb{R}^2$, the curvature is defined as:

$$\kappa(u_0) = \frac{|t'(u_0)|}{|\alpha'(u_0)|}$$

where $t(u)$ is the tangent vector at $u$. 

![Diagram showing a curve and a tangent vector](image-url)
Recall:
For a regular curve $\alpha : [a, b] \rightarrow \mathbb{R}^2$, the curvature is defined as:

$$\kappa(u_0) = \frac{|t'(u_0)|}{|\alpha'(u_0)|}$$

where $t(u)$ is the tangent vector at $u$.

Note that the curvature does not change if we rotate or translate the curve.
Curvature and Graphs

If we rotate the curve so that $p$ goes to $(0,0)$ and the tangent line at $p$ aligns with $x$-axis, we can describe the curve (locally) as the graph:

$$\beta(u) = (u, f(u))$$
Curvature and Graphs

If we rotate the curve so that $p$ goes to $(0,0)$ and the tangent line at $p$ aligns with $x$-axis, we can describe the curve (locally) as the graph:

$$\beta(u) = (u, f(u))$$

Since we align the tangent line with the $x$-axis, we have:

$$\beta'(u) = (1, f'(u))$$

so that $f'(0) = 0$. 

$p = \alpha(u_0)$

$(u, f(u))$
Curvature and Graphs

Thus, at the point \( p \), the curvature can be expressed as:

\[
\kappa(p) = \frac{|t'(0)|}{|\beta'(0)|}
\]
Curvature and Graphs

The tangent at the point $u$ is given by:

$$t(u) = \frac{\beta'(u)}{|\beta'(u)|}$$

\[
\begin{align*}
\beta(u) &= (u, f(u)) \\
\kappa(p) &= \frac{|t'(0)|}{|\beta'(0)|}
\end{align*}
\]
Curvature and Graphs

The tangent at the point $u$ is given by:

$$t(u) = \frac{\beta'(u)}{|\beta'(u)|}$$

$$= \frac{(1, f'(u))}{(1 + f'(u)f'(u))^{1/2}}$$

$\beta(u) = (u, f(u))$

$\kappa(p) = \frac{|t'(0)|}{|\beta'(0)|}$

$f'(0) = 0$
Curvature and Graphs

So the derivative of the tangent at the point (0,0) is:

\[
t'(0) = \frac{(0, f''(0))}{(1 + f'(0)f''(0))^{1/2}} - \frac{(1, f'(0))}{(1 + f'(0)f''(0))^{3/2}} f'(0)f''(0)
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Curvature and Graphs

So the derivative of the tangent at the point (0,0) is:

\[
\begin{align*}
t'(0) &= \frac{(0, f''(0))}{(1 + f'(0)f'(0))^{1/2}} - \frac{(1, f'(0))}{(1 + f'(0)f'(0))^{3/2}} f'(0)f''(0) \\
&= (0, f''(0))
\end{align*}
\]
Thus, the curvature at the point (0,0) is:

\[ \kappa(p) = \frac{|t'(0)|}{|\beta'(0)|} = f'''(0) \]
For a regular surface, we can define the curvatures in an analogous fashion – describing the surface locally as the graph of a function and considering the second-order terms.
Curvature and Graphs

For a regular surface, we can define the curvatures in an analogous fashion – describing the surface locally as the graph of a function and considering the second-order terms.

We translate the surface so that $p$ goes to the origin, and rotate so that the tangent plane aligns with the $x$-$y$ plane.
Curvature and Graphs

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We translate the surface so that $p$ goes to the origin, and rotate so that the tangent plane aligns with the $x$-$y$ plane.

About the origin, we can describe the surface as the graph $(x, y, f(x,y))$. 
Curvature and Graphs

Note:

1. Since \( p \) is sent to the origin, \( f(x, y) = 0 \).
2. Since the tangent plane at \( p \) is sent to the \( x-y \) plane, the partials satisfy:

\[
\left. \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \right|_{(0,0)} = 0
\]
Thus, up to a rotation in the x-y plane, we have:

\[ f(x, y) \approx \frac{\lambda_1}{2} x^2 + \frac{\lambda_2}{2} y^2 \]
Curvature and Graphs

Thus, up to a rotation in the $x$-$y$ plane, we have:

$$f(x, y) \approx \frac{\lambda_1}{2} x^2 + \frac{\lambda_2}{2} y^2$$

Fixing $y=0$, we get a curve with curvature $\lambda_1$. 
Curvature and Graphs

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Fixing $y=0$, we get a curve with curvature $\lambda_1$.
Fixing $x=0$, we get a curve with curvature $\lambda_2$. 
Curvature and Graphs

Thus, up to a rotation in the x-y plane, we have:

$$f(x, y) \approx \frac{\lambda_1}{2} x^2 + \frac{\lambda_2}{2} y^2$$

Fixing $y=0$, we get a curve with curvature $\lambda_1$. Fixing $x=0$, we get a curve with curvature $\lambda_2$. Any other line in the tangent plane will generate a curve with curvature between $\lambda_1$ and $\lambda_2$. 

![Graphs showing curvature and tangent planes](image)
Curvature and Graphs

Thus, up to a rotation in the $x$-$y$ plane, we have:

$$f(x, y) \approx \frac{\lambda_1}{2} x^2 + \frac{\lambda_2}{2} y^2$$

The values $\lambda_1$ and $\lambda_2$ are the principal curvatures at $p$ and the corresponding directions of the curves at the point $p$ are called the principal directions.
Curvature and Graphs

Definition:
The product of the principal curvatures, $\lambda_1 \cdot \lambda_2$, is the *Gaussian Curvature*.
The sum of the principal curvatures, $\lambda_1 + \lambda_2$, is the *Mean Curvature*.