Differential Geometry:

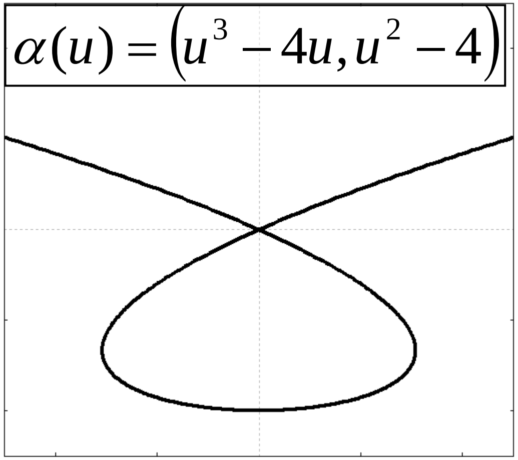
Surfaces and Parameterizations

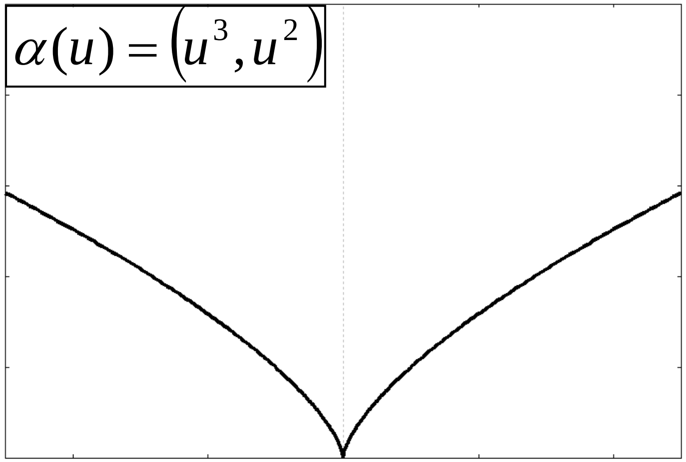
Differential Geometry of Curves and Surfaces, Do Carmo (Chapter 2)

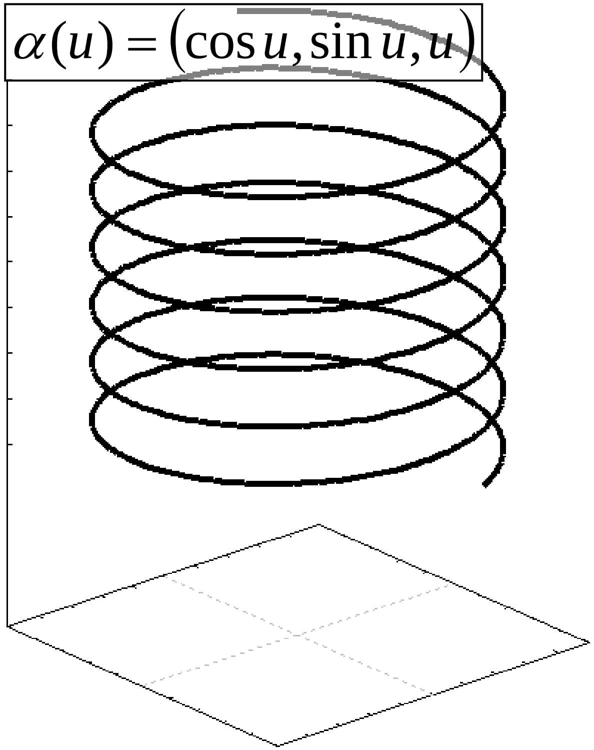
Differentiable Curves

<u>Recall</u>:

A *parameterized differentiable curve* is a differentiable map $\alpha:I\to\boldsymbol{R}^n$ of an open interval $I=(a,b)$ of the real line $\boldsymbol{R}$ into $\boldsymbol{R}^n$.



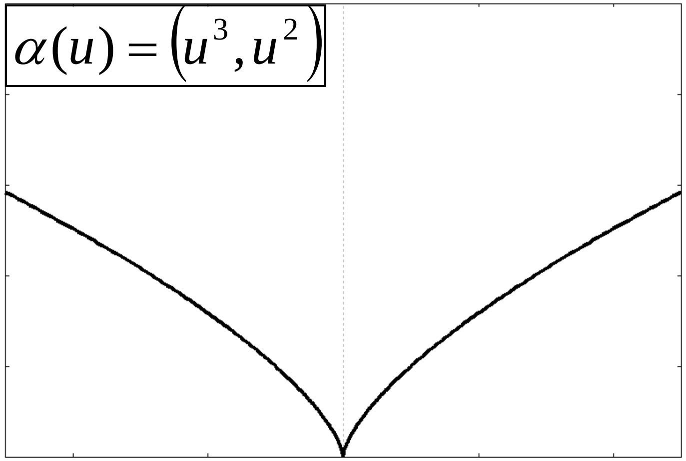


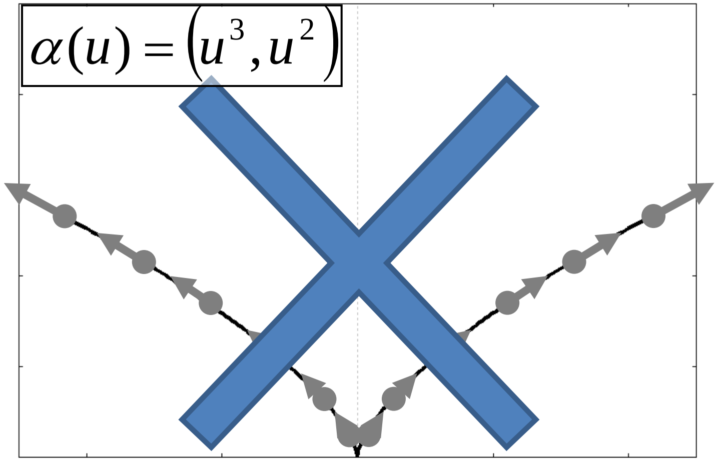


Regular Curves

<u>Recall</u>:

A parameterized curve, $\alpha : I \to \boldsymbol{R}^n$, is *regular* if $\alpha'(u) \neq 0$ for all $u \in I$. That is, if it has a well-defined tangent line at each point.





Regular Curves

Recall:

Although we started by looking at parameterizations of the curve, we ended up focusing on properties of the curve that were independent of the parameterization.

Regular Surfaces

For surfaces, it can be hard to find an open interval in the plane, $I \subset \boldsymbol{R}^2$, over which to parameterize.

So, instead of looking at global mappings from open sets in the plane onto the surface, we will consider local mappings.

Regular Surfaces

Definition:

A subset $S \subset \boldsymbol{R}^3$, is a *regular surface* if for every $p \in S$ there exists a neighborhood $V \subset \boldsymbol{R}^3$, and a map $\Phi{:}U \to V \cap S$ of an open set $U \subset \boldsymbol{R}^2$ onto $V \cap S$ such that:

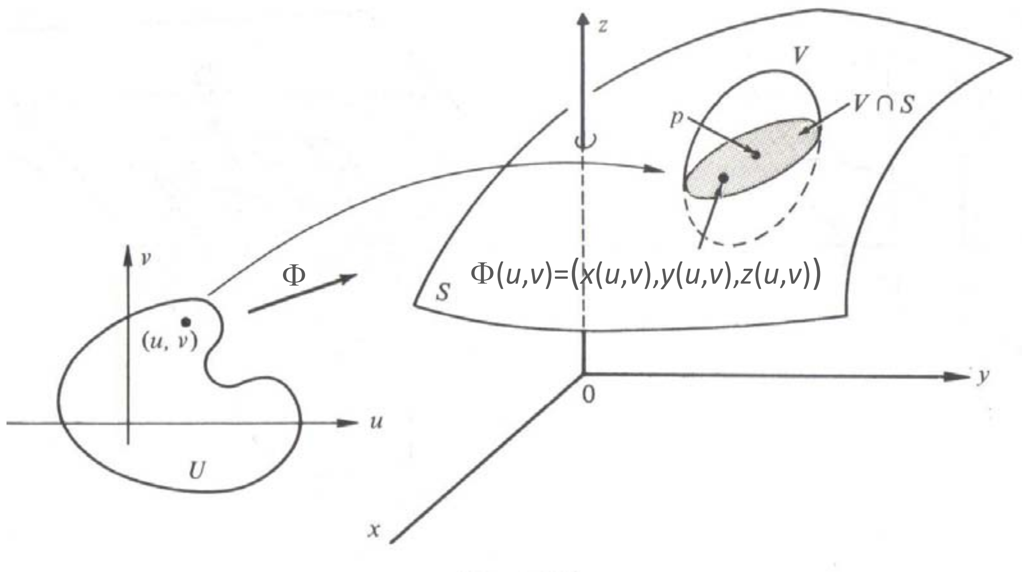


Figure 2-1

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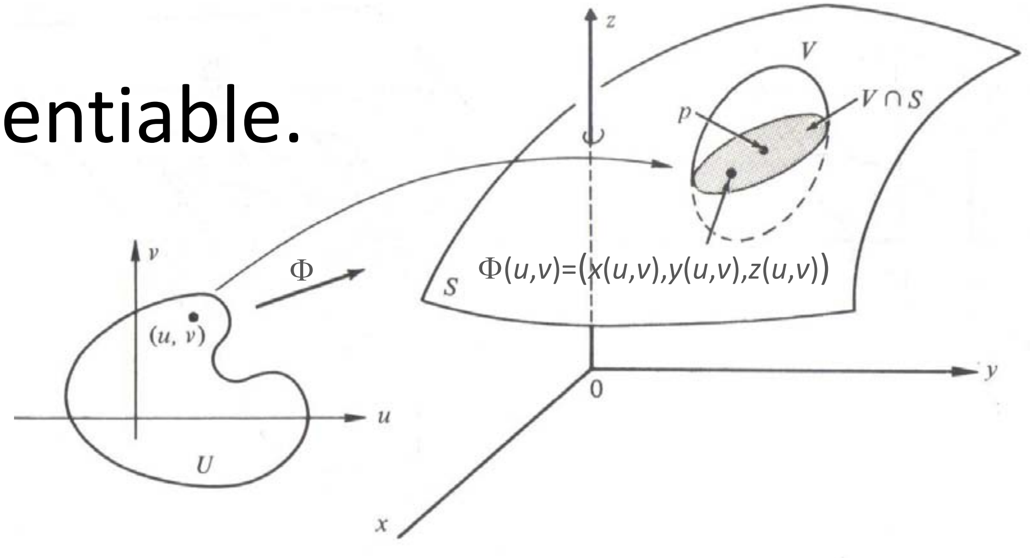
1. The map Φ is differentiable.



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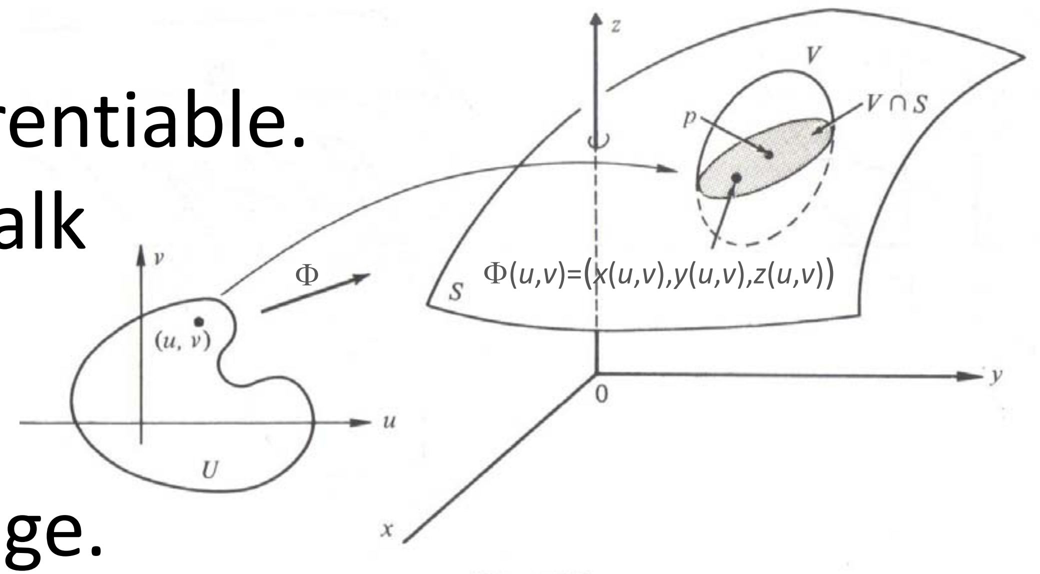
   It makes sense to talk about planes that are tangent to S and how they change.



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2. At each point p, the differential $d\Phi_p: \boldsymbol{R}^2 \rightarrow \boldsymbol{R}^3$ is one-to-one.

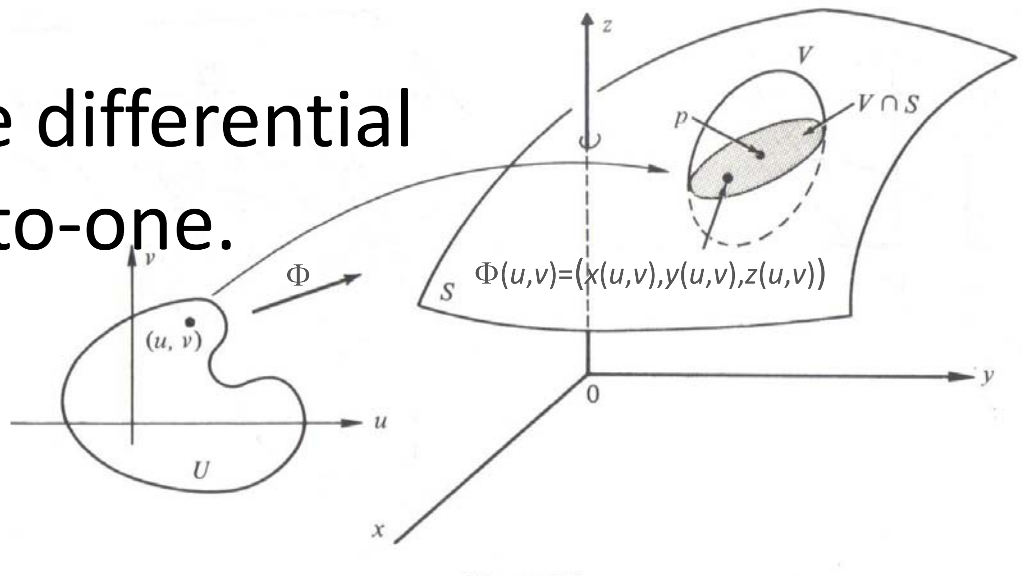


Figure 2-1

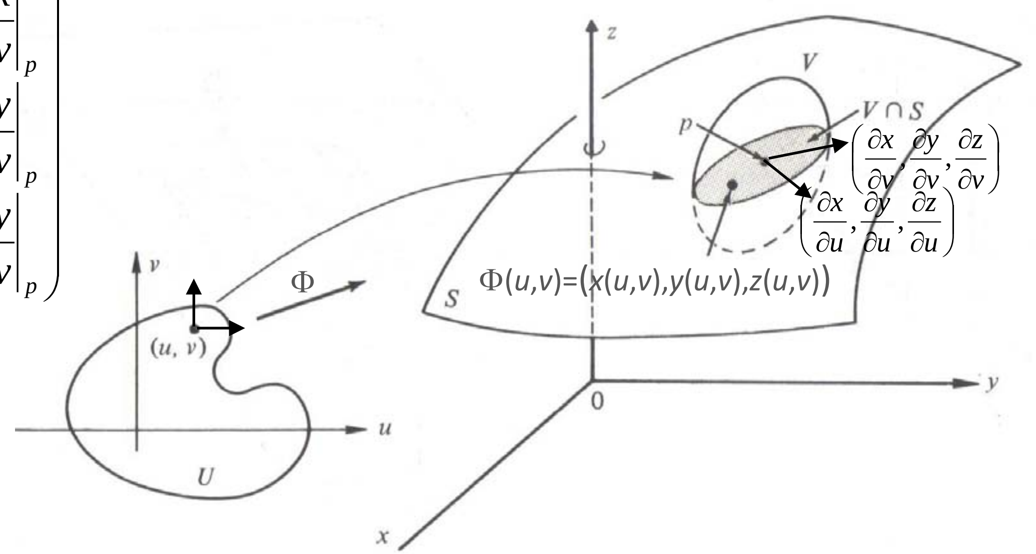
Regular Surfaces

Definition:

2. At each point p, the differential $d\Phi_p$:$\boldsymbol{R}^2 \rightarrow \boldsymbol{R}^3$ is one-to-one.

The differential of the map is defined as:

$$d\Phi_p = \begin{pmatrix} \left.\frac{\partial x}{\partial u}\right|_p & \left.\frac{\partial x}{\partial v}\right|_p \\ \left.\frac{\partial y}{\partial u}\right|_p & \left.\frac{\partial y}{\partial v}\right|_p \\ \left.\frac{\partial z}{\partial u}\right|_p & \left.\frac{\partial y}{\partial v}\right|_p \end{pmatrix}$$



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Describing how change in (u,v) coords. corresponds to change in (x,y,z) coords.

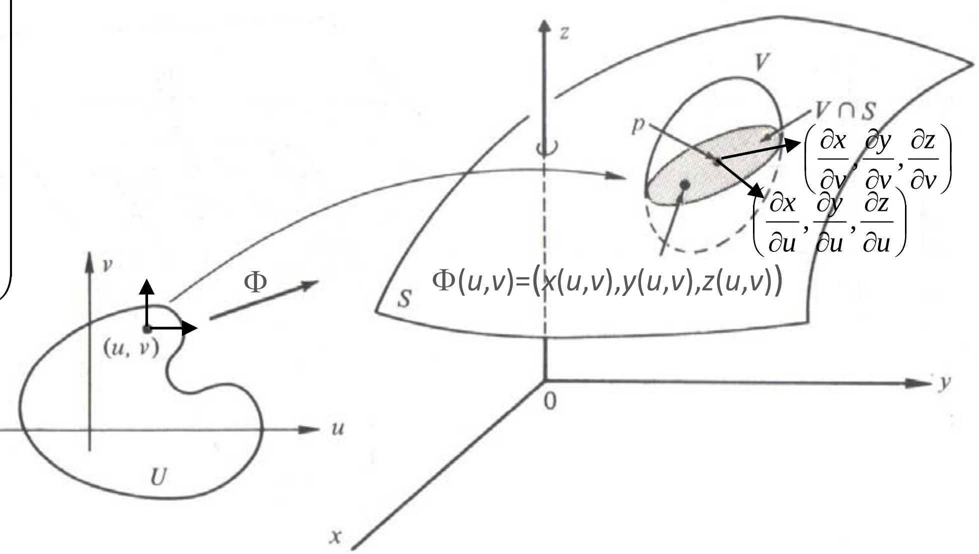


Figure 2-1

Regular Surfaces

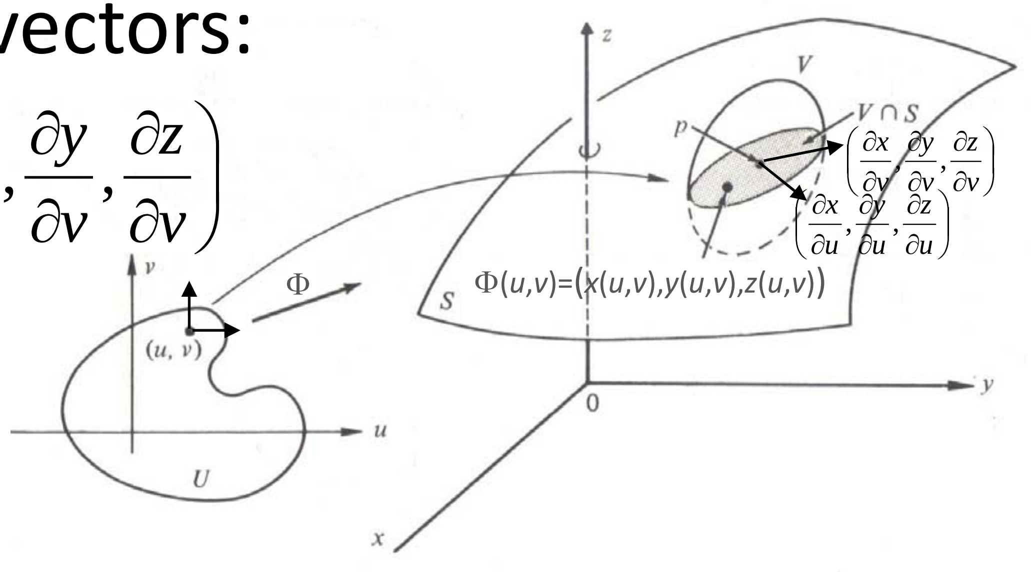
Definition:

2. At each point p, the differential $d\Phi_p$:$\boldsymbol{R}^2 \rightarrow \boldsymbol{R}^3$ is one-to-one.

If the differential is not one-to-one at some point p, then the two vectors:

$$\left(\frac{\partial x}{\partial u},\frac{\partial y}{\partial u},\frac{\partial z}{\partial u}\right) \quad \left(\frac{\partial x}{\partial v},\frac{\partial y}{\partial v},\frac{\partial z}{\partial v}\right)$$

lie on the same line.



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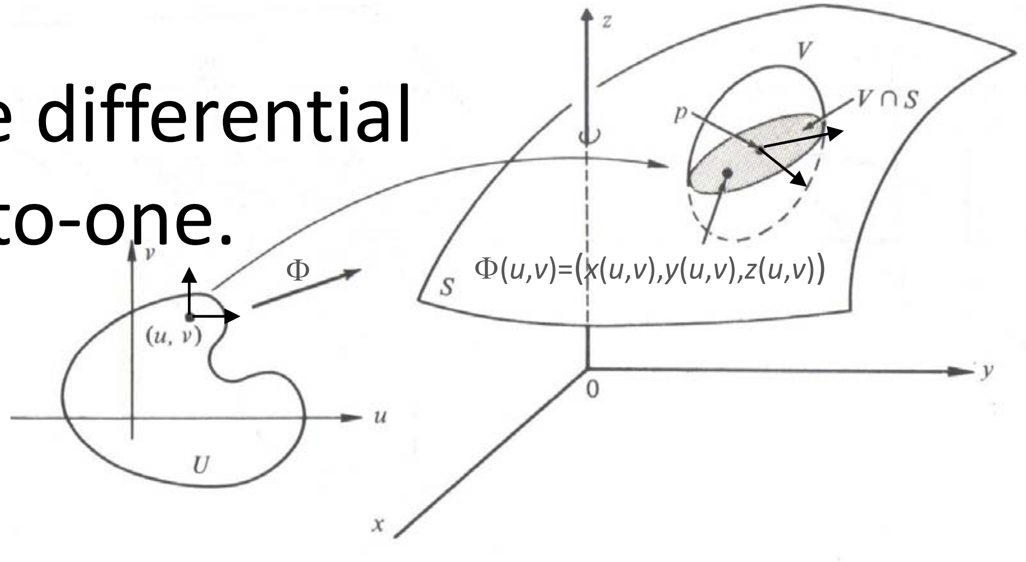
Regular Surfaces

<u>Definition</u>:

A subset $S \subset \boldsymbol{R}^3$, is a *regular surface* if for every $p \in S$ there exists a neighborhood $V \subset \boldsymbol{R}^3$, and a map $\Phi: U \rightarrow V \cap S$ of an open set $U \subset \boldsymbol{R}^2$ onto $V \cap S$ such that:

2. At each point p, the differential $d\Phi_p: \boldsymbol{R}^2 \rightarrow \boldsymbol{R}^3$ is one-to-one. Ensures that the surface is, in fact, two-dimensional.



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3. The map Φ has a continuous inverse $\Phi^{-1}: V \cap S \to U$.

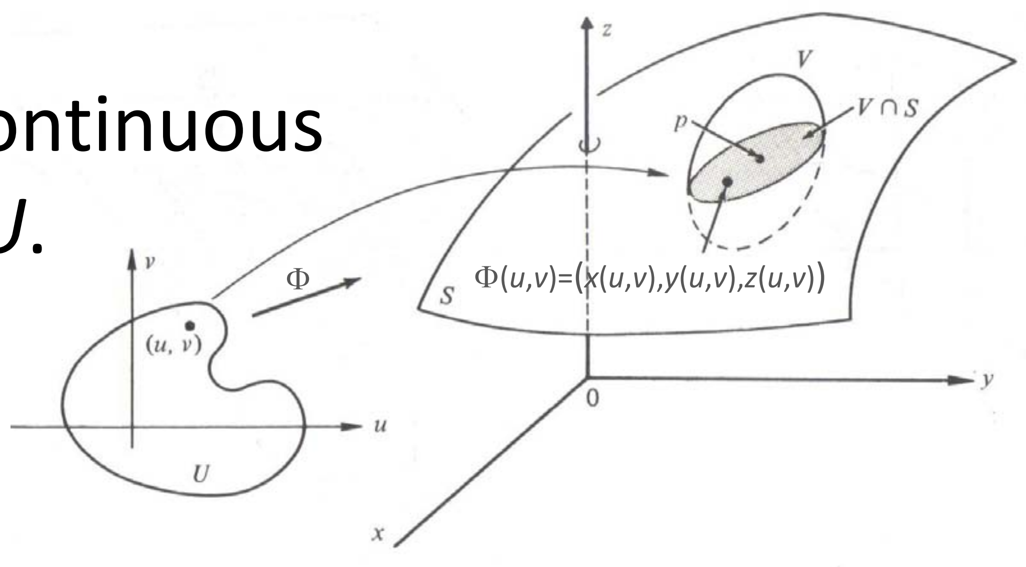


Figure 2-1

Regular Surfaces

Definition:

3. The map Φ has a continuous inverse $\Phi^{-1}: V \cap S \rightarrow U$.

This ensures that if $\Phi: U \rightarrow V \cap S$ and $\Psi: W \rightarrow V \cap S$ are two different mappings into V, then the mapping:

$$\Psi^{-1} \circ \Phi : U \rightarrow W$$

is differentiable.

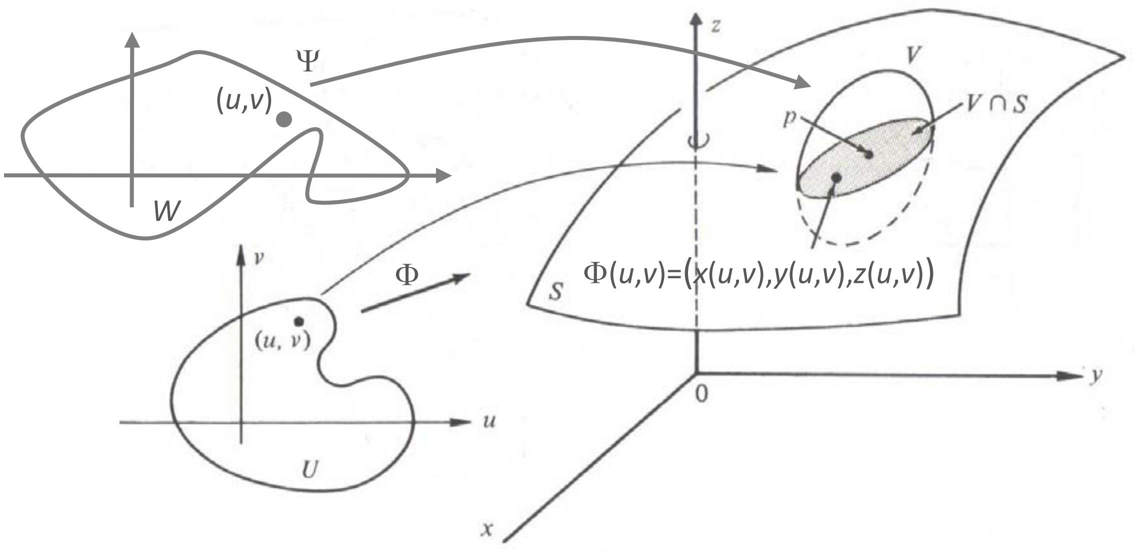
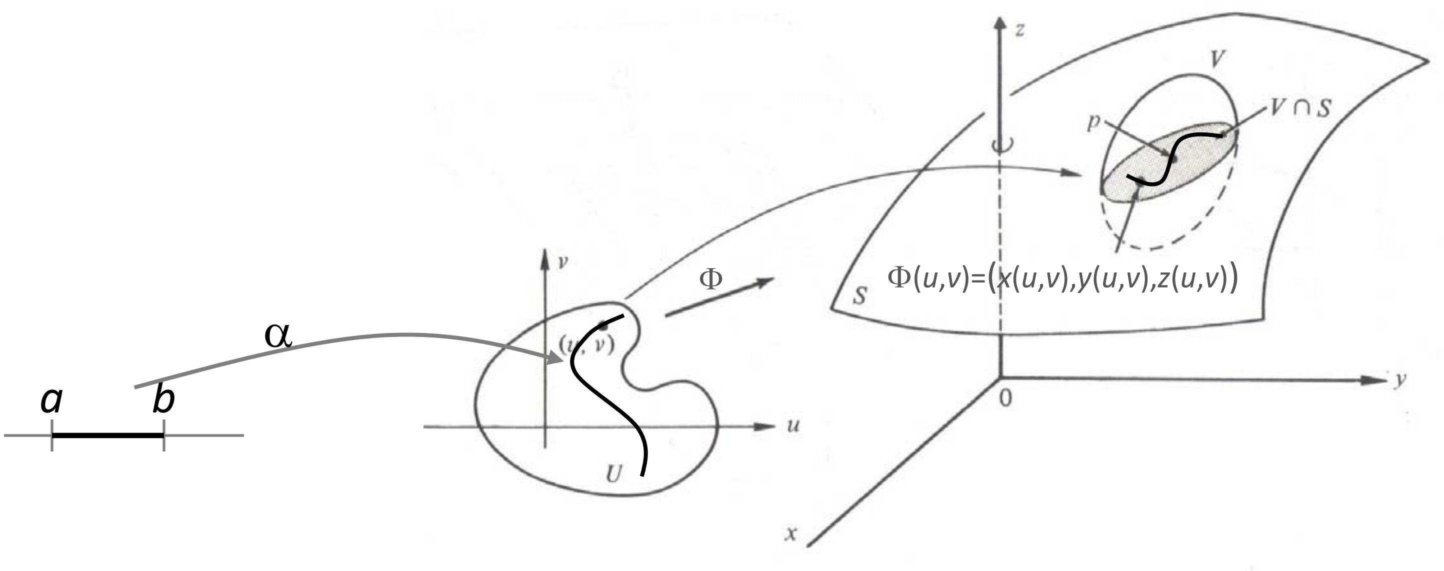


Figure 2-1

Regular Curves on Surfaces

Given a regular surface, with a mapping $\Phi:U\to V\cap S$, if $\alpha:[a,b]\to U\subset \boldsymbol{R}^2$ is a regular curve, then so is $(\Phi\circ\alpha):[a,b]\to V\subset \boldsymbol{R}^3$.



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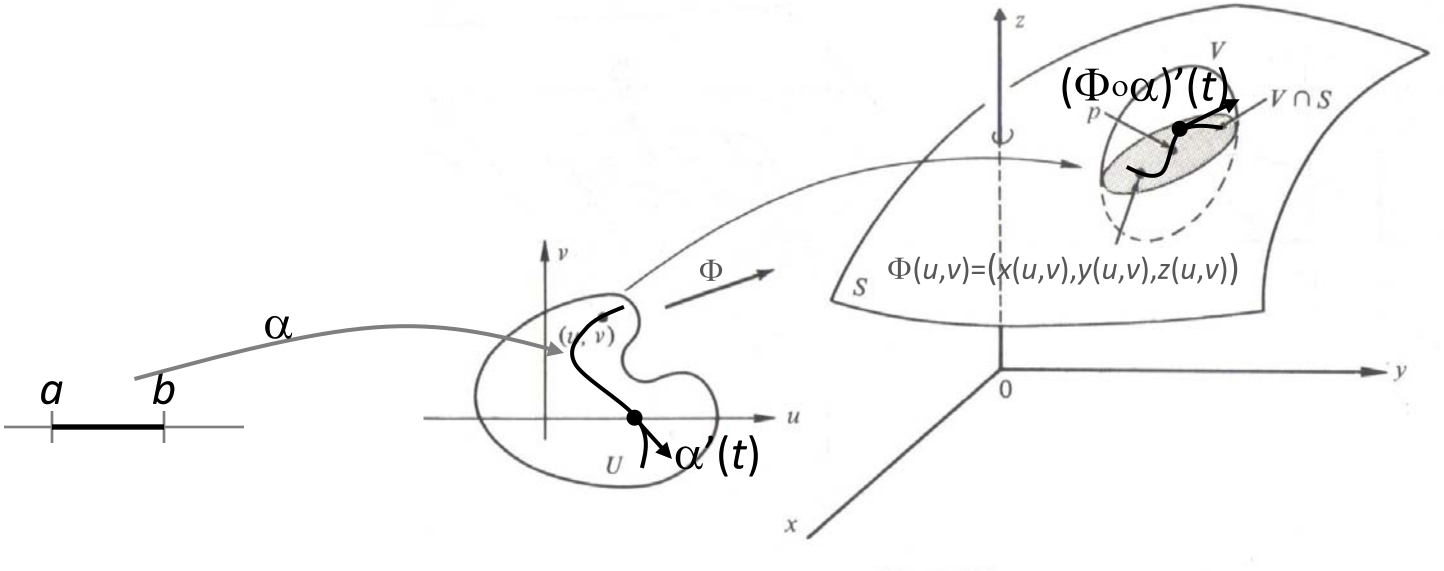
$$(\Phi\circ\alpha)'(t) = d\Phi_{\alpha(t)}(\alpha'(t))$$



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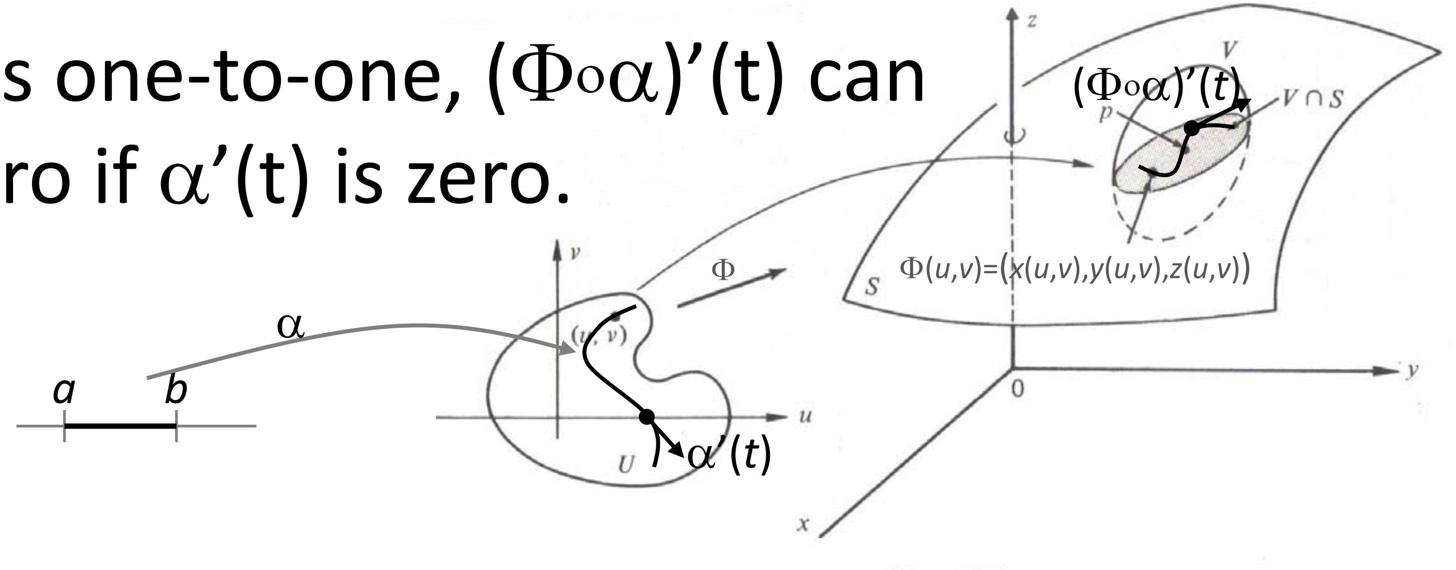
$$\left(\Phi \circ \alpha\right)'(t) = d\Phi_{\alpha(t)}\left(\alpha'(t)\right)$$

Since $d\Phi$ is one-to-one, $(\Phi \circ \alpha)'(t)$ can only be zero if $\alpha'(t)$ is zero.



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Since α is regular, so is $\Phi\circ\alpha$.

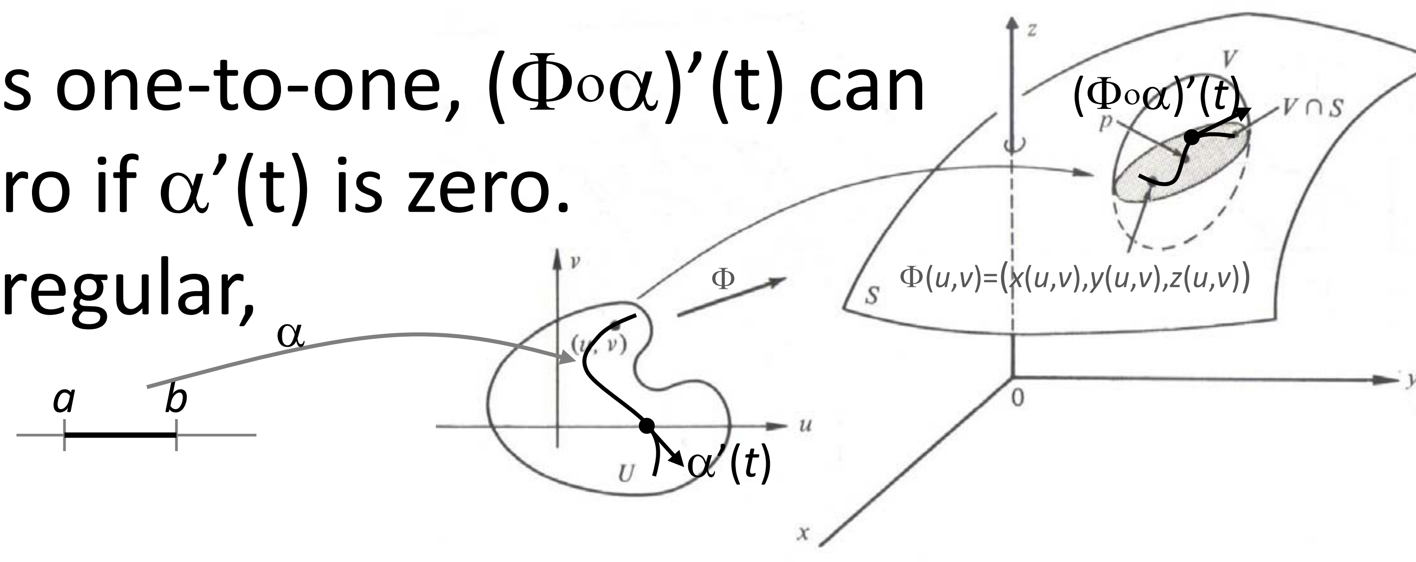


Figure 2-1

Regular Curves on Surfaces

If Φ:$U \to V \cap S$ and Ψ:$W \to V \cap S$ are two mappings into V, then α:[a,b]$\to U$ and β:[c,d]$\to W$ define the same regular curve on S, if there exists a function ϕ:[c,d]$\to$[a,b] with $\phi'(t) \neq 0$ such that:

$$(\Psi \circ \beta)(t) = (\Phi \circ \alpha \circ \phi)(t)$$

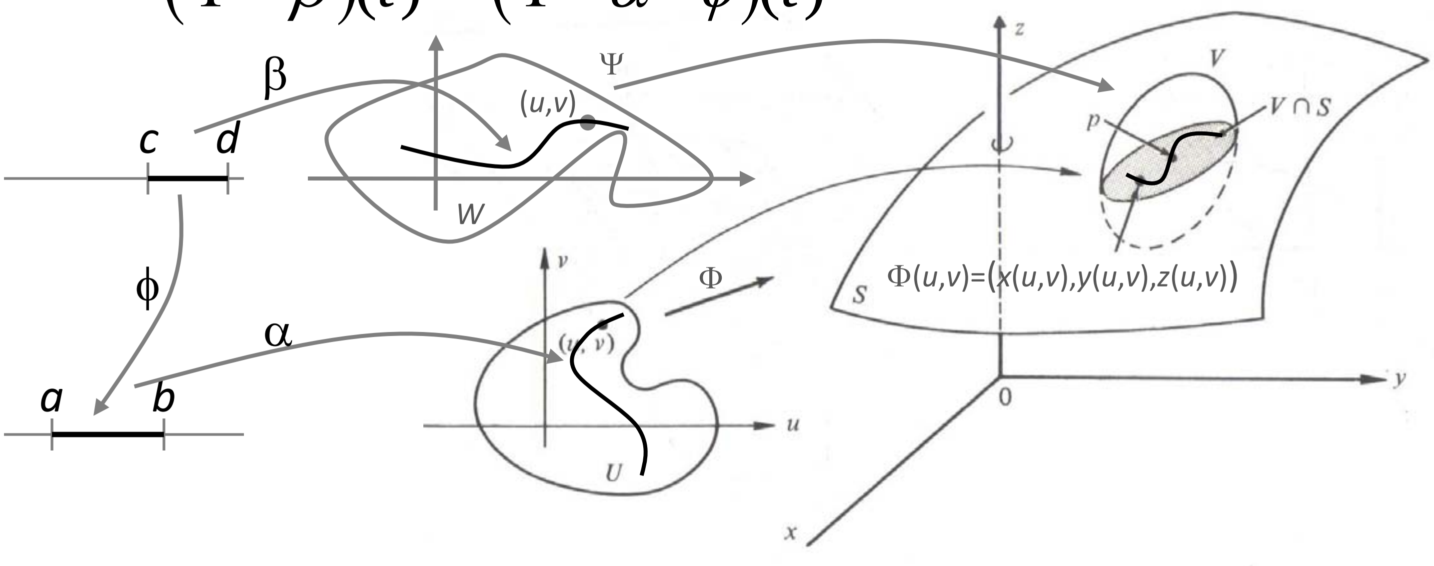


Figure 2-1

Lengths of Curves on Surfaces

Given a regular surface, with a mapping $\Phi : U \to V \cap S$, if we have a regular curve, α, defined on *U* then

$$l(\alpha) = \int_a^b \left| (\Phi \circ \alpha)'(t) \right| dt$$

$$= \int_a^b \left| d\Phi_{\alpha(t)}(\alpha'(t)) \right| dt$$

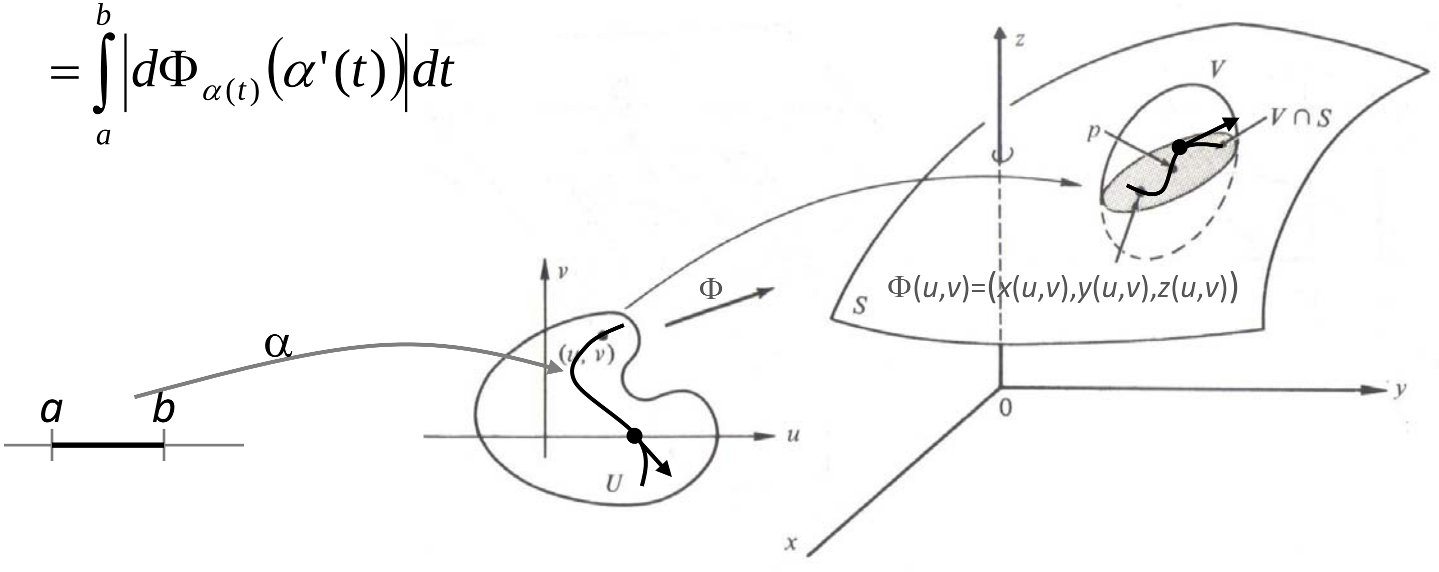
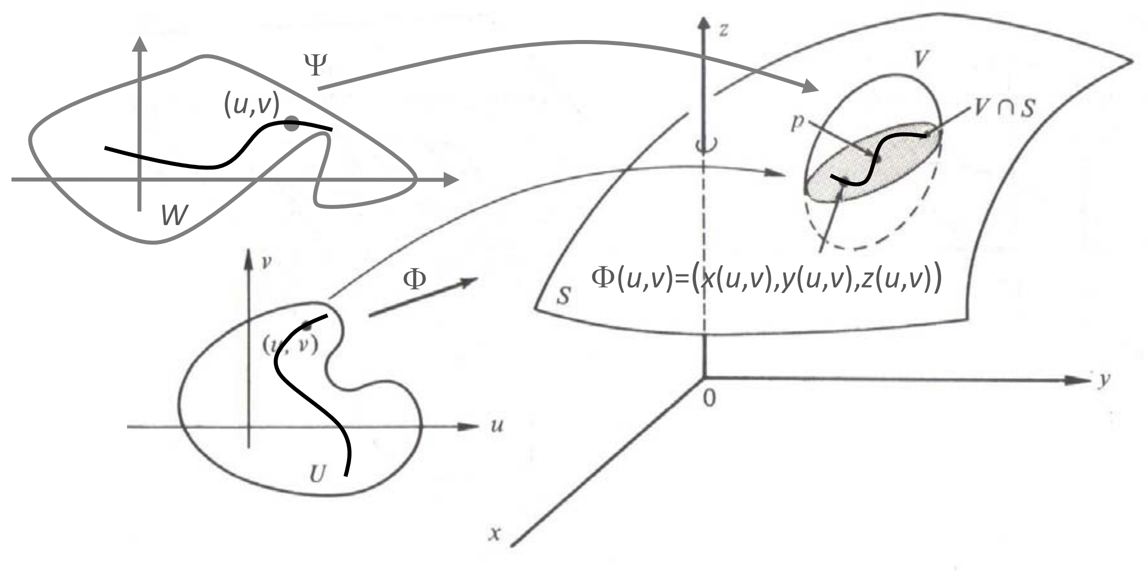


Figure 2-1

Lengths of Curves on Surfaces

If $\Phi:U \to V \cap S$ and $\Psi:W \to V \cap S$ are two mappings into V, and $\alpha:[a,b] \to U$ and $\beta:[c,d] \to W$ define the same regular curve on S, then the length of the curve on S is independent of the mapping.



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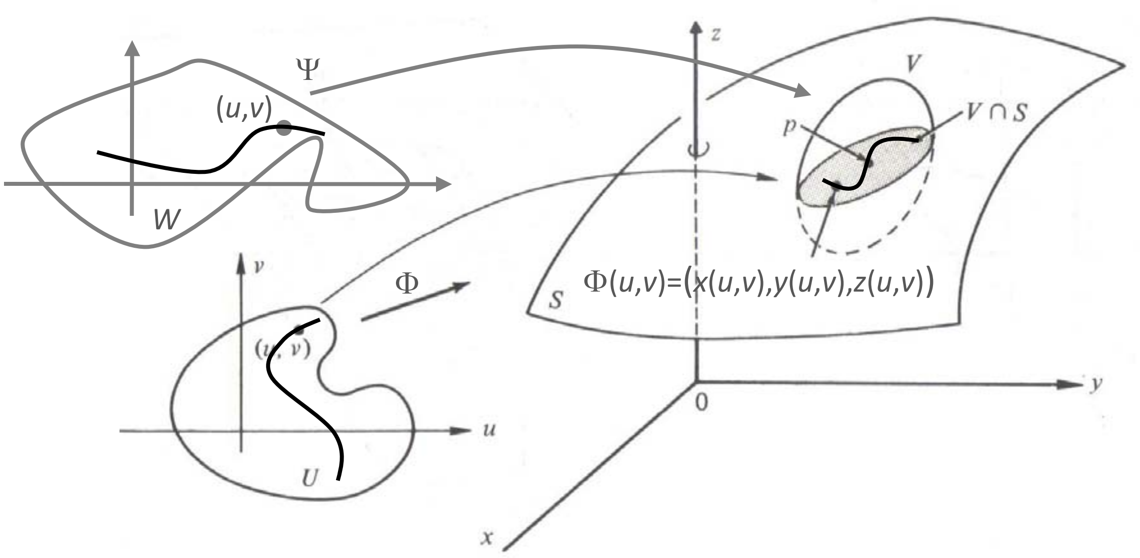
Lengths of Curves on Surfaces

If Φ:*U*→*V*∩*S* and Ψ:*W*→*V*∩*S* are two mappings into *V*, and α:[*a*,*b*]→*U* and β:[*c*,*d*]→*W* define the same regular curve on *S*, then the length of the curve on *S* is independent of the mapping.

$$l(\beta) = \int_c^d \left| (\Psi \circ \beta)'(t) \right| dt$$



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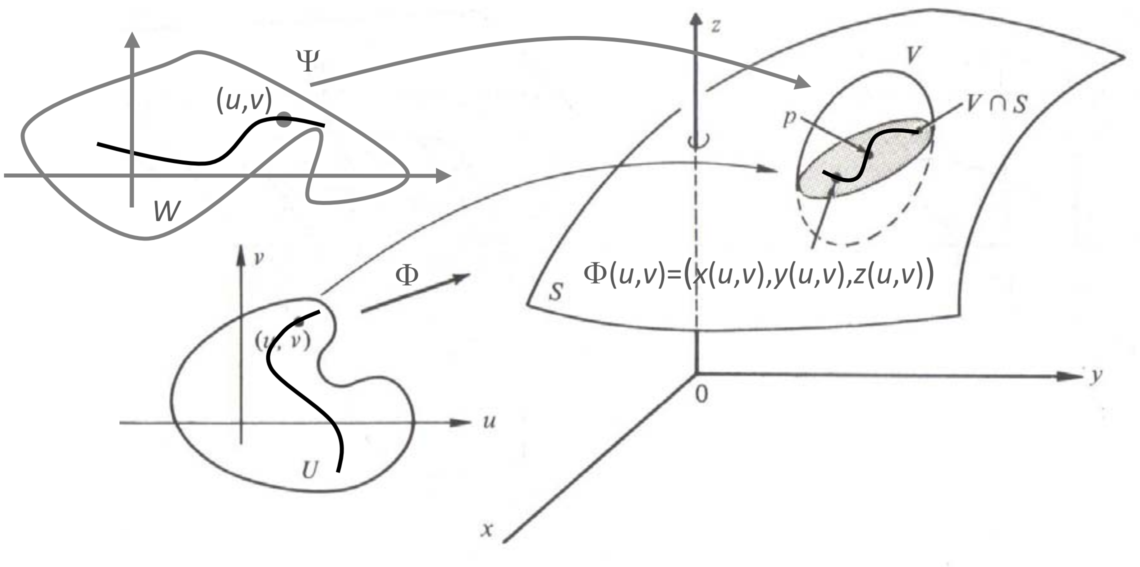
Lengths of Curves on Surfaces

If Φ:U→V∩S and Ψ:W→V∩S are two mappings into V, and α:[a,b]→U and β:[c,d]→W define the same regular curve on S, then the length of the curve on S is independent of the mapping.

$$l(\beta) = \int_c^d \left|(\Psi \circ \beta)'(t)\right| dt$$

$$= \int_c^d \left|(\Phi \circ \alpha \circ \phi)'(t)\right| dt$$



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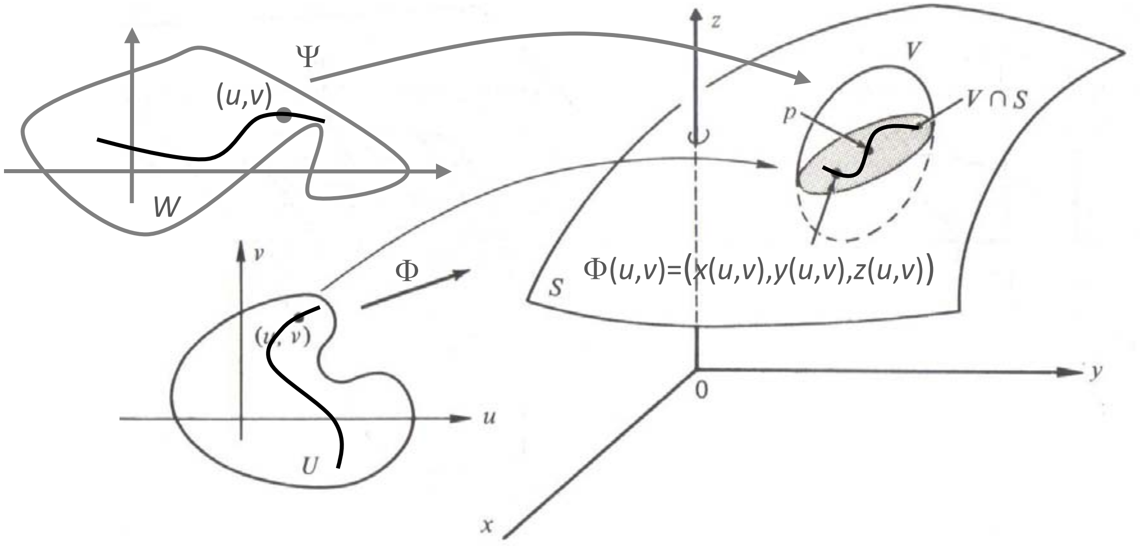
If $\Phi{:}U\to V\cap S$ and $\Psi{:}W\to V\cap S$ are two mappings into V, and $\alpha{:}[a,b]\to U$ and $\beta{:}[c,d]\to W$ define the same regular curve on S, then the length of the curve on S is independent of the mapping.

$$l(\beta)=\int_c^d\left|(\Psi\circ\beta)'(t)\right|dt$$

$$=\int_c^d\left|(\Phi\circ\alpha\circ\phi)'(t)\right|dt$$

$$=\int_c^d\left|(\Phi\circ\alpha)'_{\phi(t)}\right|\cdot\left|\phi'(t)\right|dt$$



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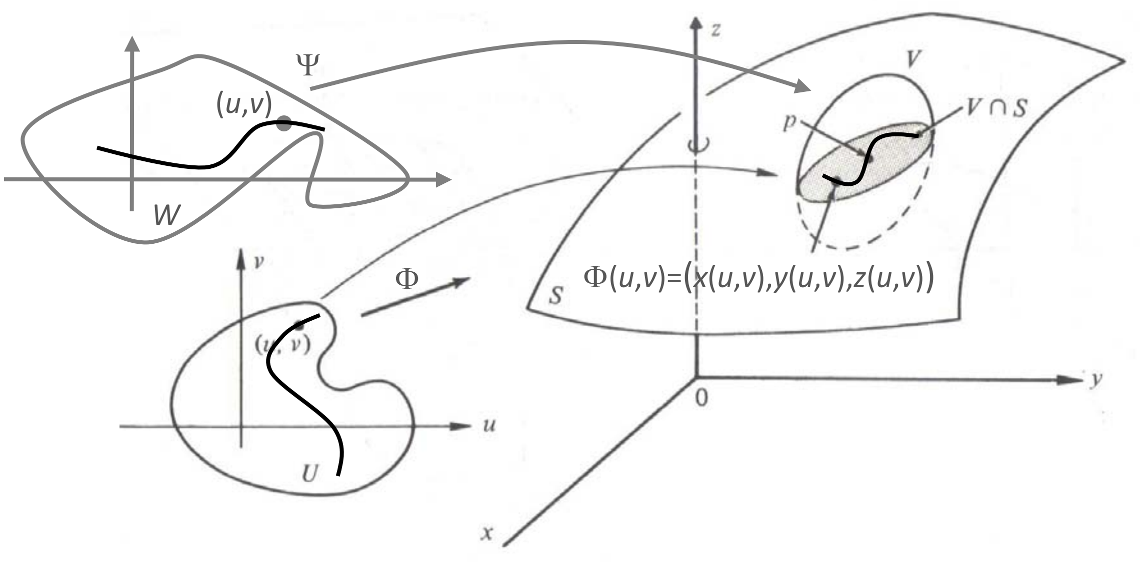
$$= \int_{\phi(c)}^{\phi(d)} \left|(\Phi \circ \alpha)'(t)\right| dt$$



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$$
\begin{aligned}
l(\beta) &= \int_c^d \left|(\Psi \circ \beta)'(t)\right| dt \\
&= \int_c^d \left|(\Phi \circ \alpha \circ \phi)'(t)\right| dt \\
&= \int_c^d \left|(\Phi \circ \alpha)'_{\phi(t)}\right| \cdot \left|\phi'(t)\right| dt \\
&= \int_{\phi(c)}^{\phi(d)} \left|(\Phi \circ \alpha)'(t)\right| dt = l(\alpha)
\end{aligned}
$$

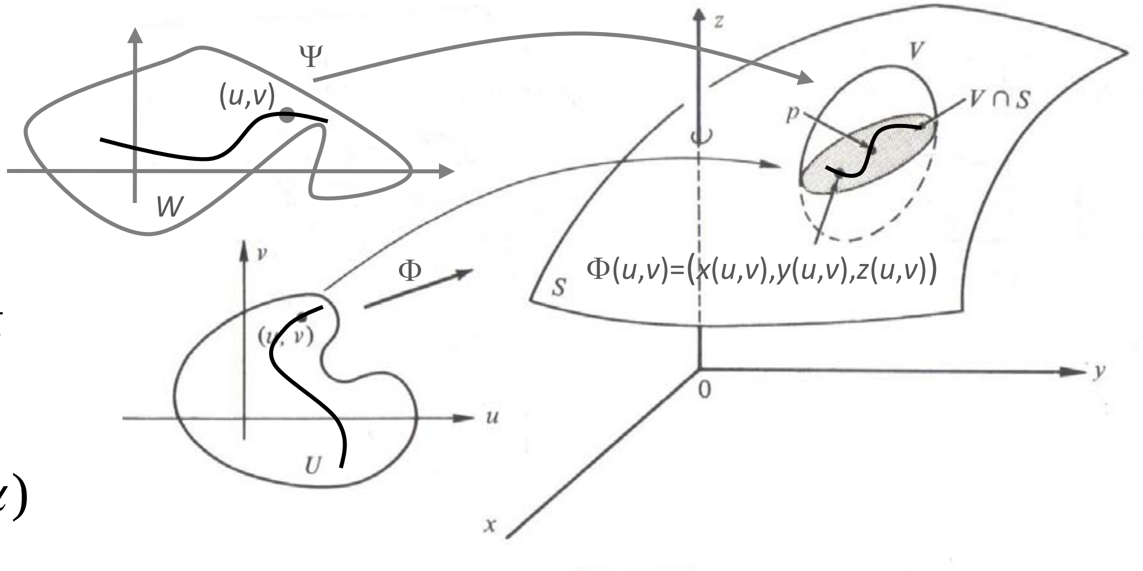
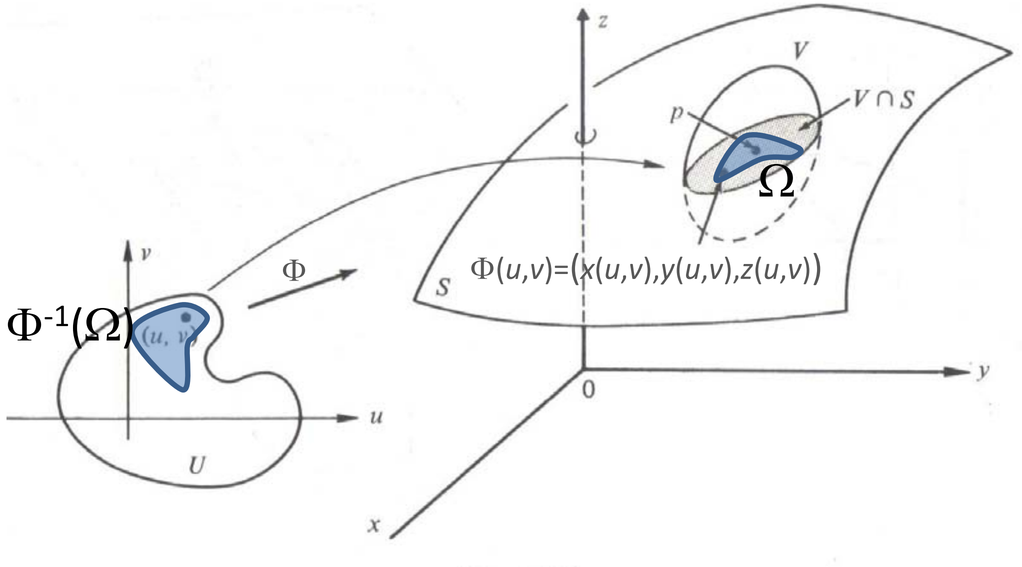


Figure 2-1

Areas of Patches on Surfaces

Given a regular surface, with a mapping $\Phi: U \to V \cap S$, if we have an open set $\Omega \subset V \cap S$, the area of Ω is:

$$A_\Phi(\Omega) = \int_{\Phi^{-1}(\Omega)} \left\| \left.\frac{\partial \Phi}{\partial u}\right|_p \times \left.\frac{\partial \Phi}{\partial v}\right|_p \right\| dp$$



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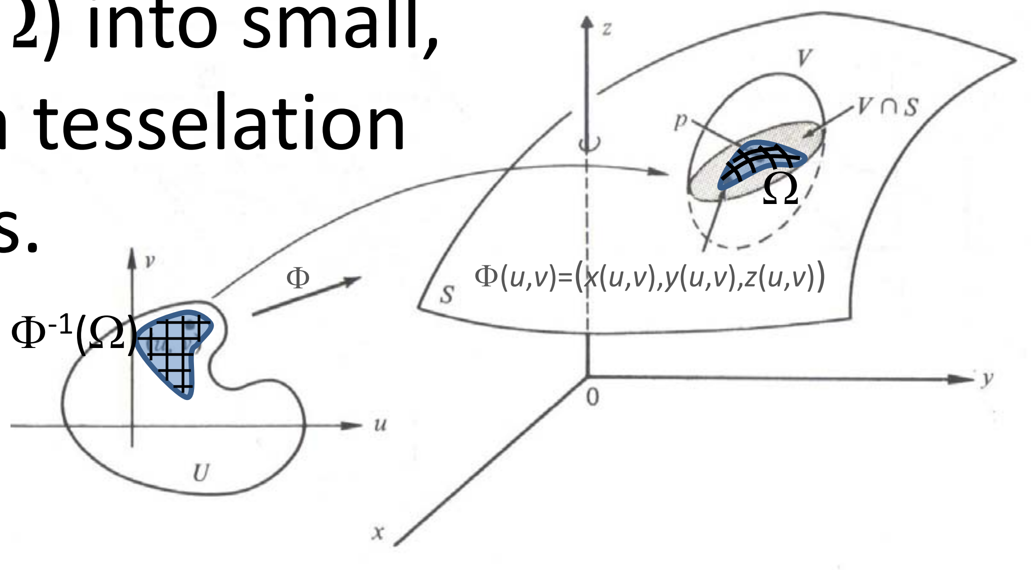
If we decompose $\Phi^{-1}(\Omega)$ into small, squares, this gives us a tesselation of Ω into quad patches.



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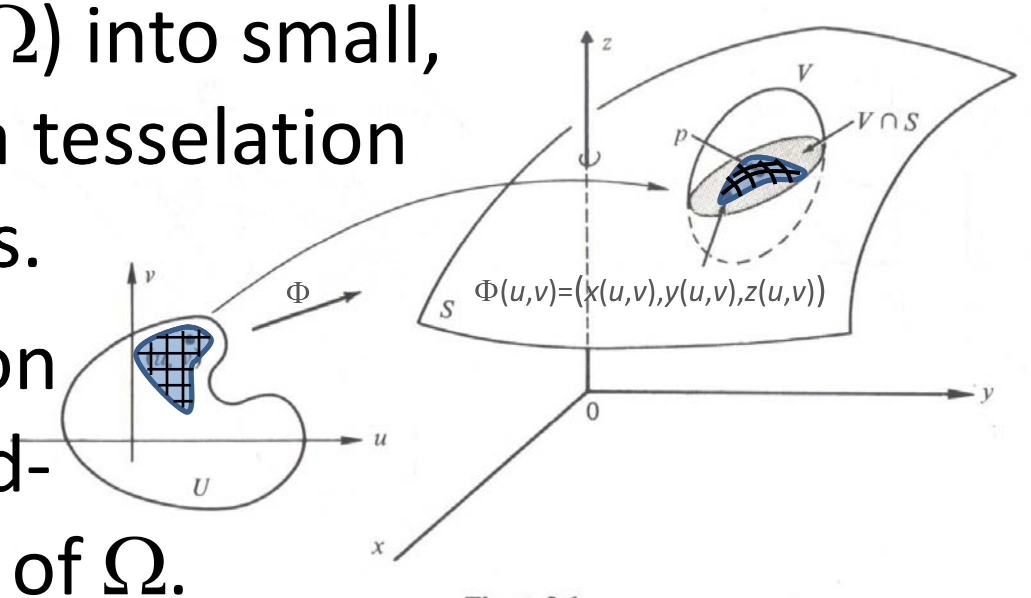
If we decompose $\Phi^{-1}(\Omega)$ into small, squares, this gives us a tesselation of Ω into quad patches.

Refining the tessellation and summing the quad-areas, we get the area of Ω.



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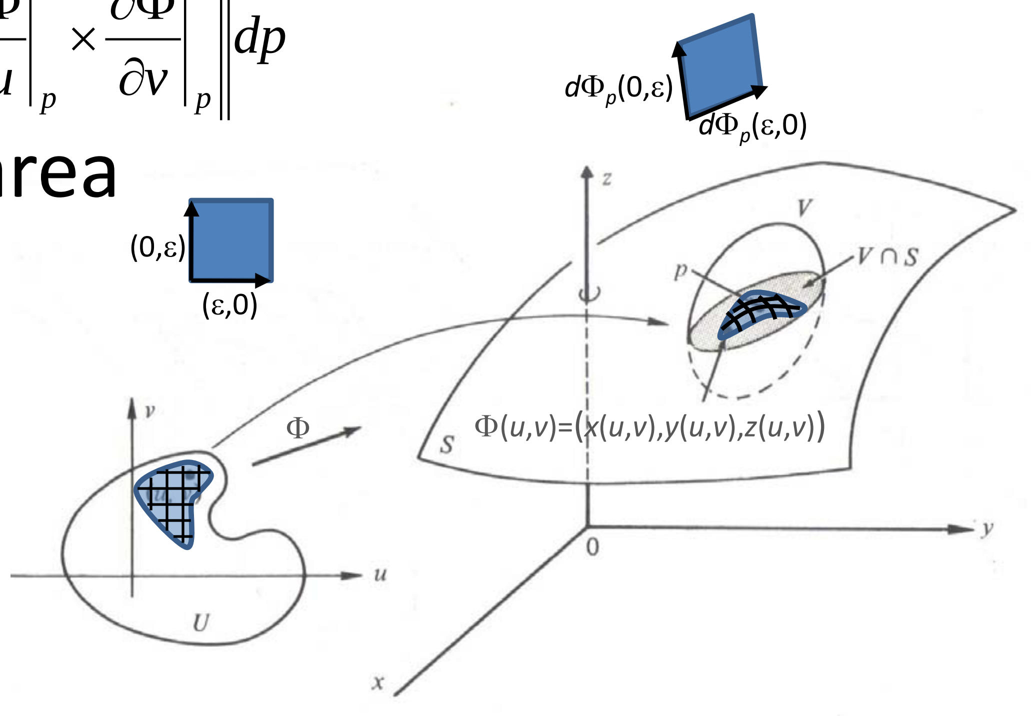
For small widths, the area of the patch on S is:

$$\varepsilon^2 \left\| \begin{pmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial u} \end{pmatrix} \times \begin{pmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial v} \end{pmatrix} \right\| = \varepsilon^2 \left\| \left.\frac{\partial \Phi}{\partial u}\right|_p \times \left.\frac{\partial \Phi}{\partial v}\right|_p \right\|$$



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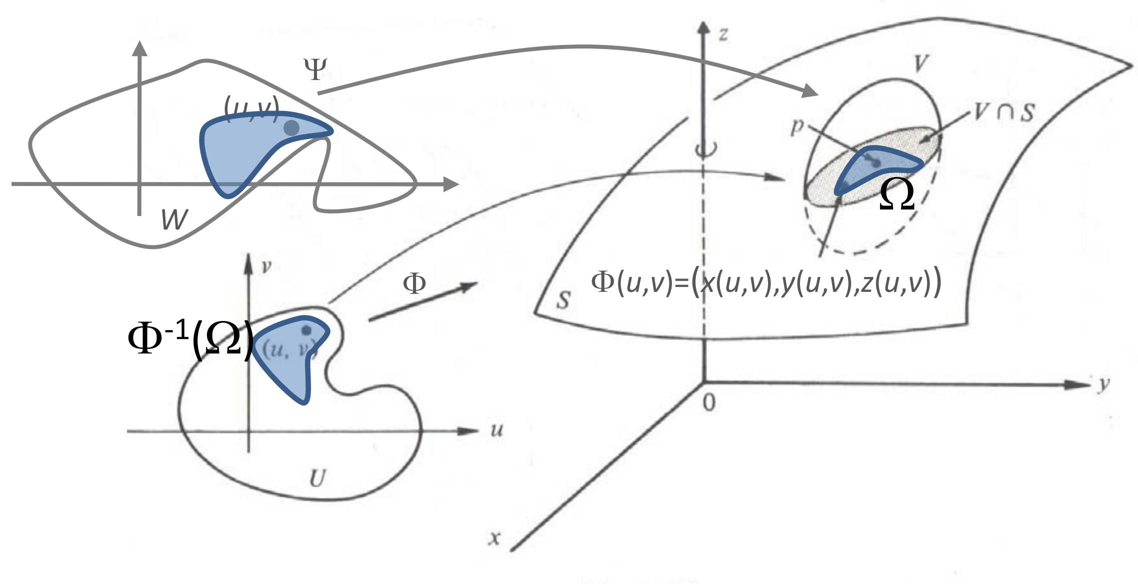
Areas of Patches on Surfaces

The definition of the area of $\Omega \subset V$ does not depend on the map.



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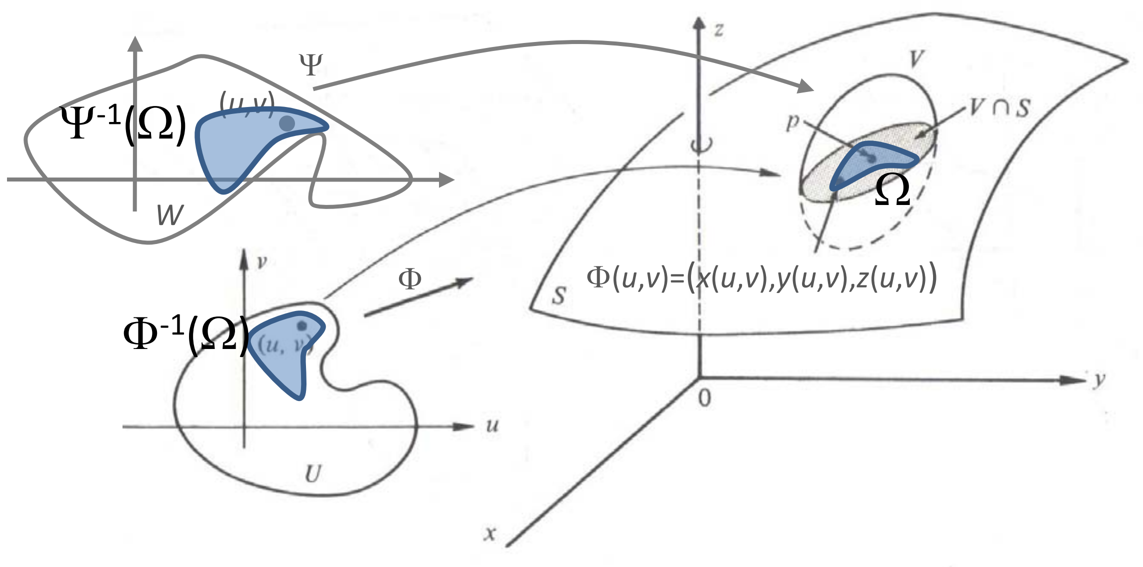
The definition of the area of $\Omega \subset V$ does not depend on the map.

That is, if $\Phi : U \to V \cap S$ and $\Psi : W \to V \cap S$ are two mappings into $V \cap S$, they define the same area.



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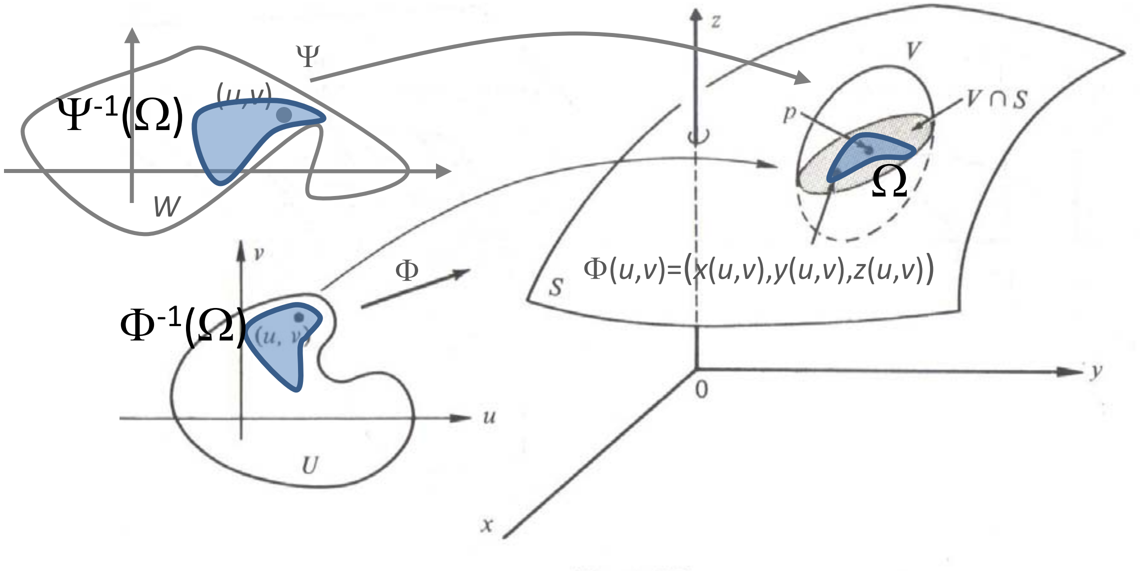
<u>Recall</u>:

$\Psi^{-1} \circ \Phi: V \to W$ is a differentiable map.



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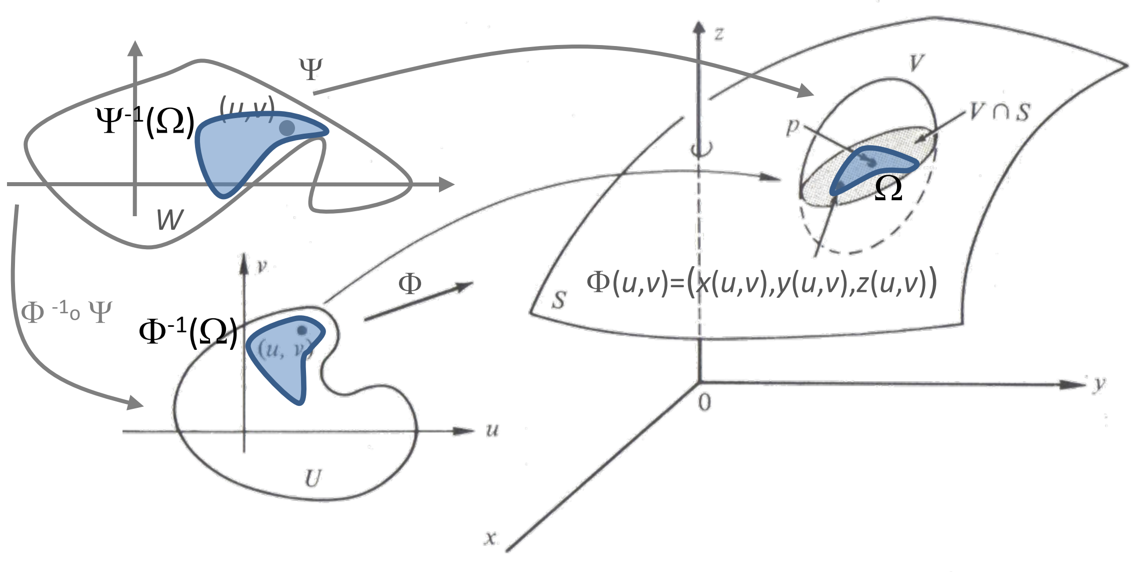


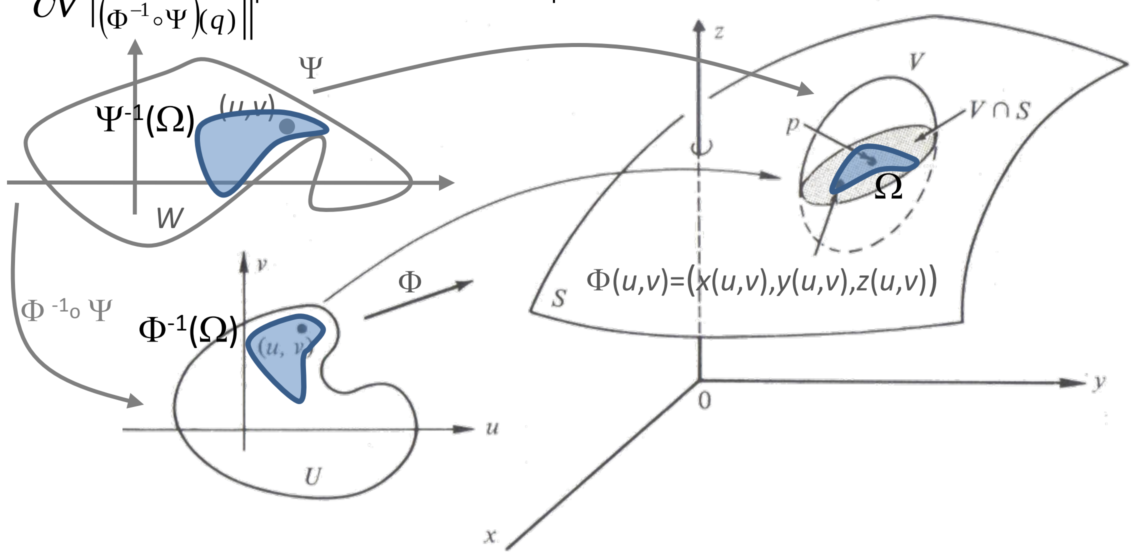
Figure 2-1

Areas of Patches on Surfaces

The definition of the area of $\Omega \subset V$ does not depend on the map.

$$A_\Phi(\Omega) = \int_{\Phi^{-1}(\Omega)} \left\| \left.\frac{\partial \Phi}{\partial u}\right|_p \times \left.\frac{\partial \Phi}{\partial v}\right|_p \right\| dp$$

$$= \int_{\Psi^{-1}(\Omega)} \left\| \left.\frac{\partial \Phi}{\partial u}\right|_{(\Phi^{-1}\circ\Psi)(q)} \times \left.\frac{\partial \Phi}{\partial v}\right|_{(\Phi^{-1}\circ\Psi)(q)} \right\| \left|\det d\left(\Phi^{-1}\circ\Psi\right)_q\right| dq$$



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$$A_\Phi(\Omega)=\int\limits_{\Phi^{-1}(\Omega)}\left\|\left.\frac{\partial\Phi}{\partial u}\right|_p\times\left.\frac{\partial\Phi}{\partial v}\right|_p\right\|dp$$

$$=\int\limits_{\Psi^{-1}(\Omega)}\left\|\left.\frac{\partial\Phi}{\partial u}\right|_{\left(\Phi^{-1}\circ\Psi\right)(q)}\times\left.\frac{\partial\Phi}{\partial v}\right|_{\left(\Phi^{-1}\circ\Psi\right)(q)}\right\|\left|\det d\left(\Phi^{-1}\circ\Psi\right)_q\right|dq$$

$$=\int\limits_{\Psi^{-1}(\Omega)}\left\|\left.\frac{\partial\left(\Psi\circ\Psi^{-1}\circ\Phi\right)}{\partial u}\right|_{\left(\Phi^{-1}\circ\Psi\right)(q)}\times\left.\frac{\partial\left(\Psi\circ\Psi^{-1}\circ\Phi\right)}{\partial v}\right|_{\left(\Phi^{-1}\circ\Psi\right)(q)}\right\|\left|\det d\left(\Phi^{-1}\circ\Psi\right)_q\right|dq$$

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$$
\begin{aligned}
A_\Phi(\Omega) &= \int_{\Phi^{-1}(\Omega)} \left\| \left.\frac{\partial \Phi}{\partial u}\right|_p \times \left.\frac{\partial \Phi}{\partial v}\right|_p \right\| dp \\
&= \int_{\Psi^{-1}(\Omega)} \left\| \left.\frac{\partial \Phi}{\partial u}\right|_{\left(\Phi^{-1}\circ\Psi\right)(q)} \times \left.\frac{\partial \Phi}{\partial v}\right|_{\left(\Phi^{-1}\circ\Psi\right)(q)} \right\| \left|\det d\left(\Phi^{-1}\circ\Psi\right)_q\right| dq \\
&= \int_{\Psi^{-1}(\Omega)} \left\| \left.\frac{\partial\left(\Psi\circ\Psi^{-1}\circ\Phi\right)}{\partial u}\right|_{\left(\Phi^{-1}\circ\Psi\right)(q)} \times \left.\frac{\partial\left(\Psi\circ\Psi^{-1}\circ\Phi\right)}{\partial v}\right|_{\left(\Phi^{-1}\circ\Psi\right)(q)} \right\| \left|\det d\left(\Phi^{-1}\circ\Psi\right)_q\right| dq \\
&= \int_{\Psi^{-1}(\Omega)} \left\| d\Psi_q \left.\frac{\partial\left(\Psi^{-1}\circ\Phi\right)}{\partial u}\right|_{\left(\Phi^{-1}\circ\Psi\right)(q)} \times d\Psi_q \left.\frac{\partial\left(\Psi^{-1}\circ\Phi\right)}{\partial v}\right|_{\left(\Phi^{-1}\circ\Psi\right)(q)} \right\| \left|\det d\left(\Phi^{-1}\circ\Psi\right)_q\right| dq
\end{aligned}
$$

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$$
\begin{aligned}
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&= \int_{\Psi^{-1}(\Omega)} \left\| \left.\frac{\partial\left(\Psi\circ\Psi^{-1}\circ\Phi\right)}{\partial u}\right|_{(\Phi^{-1}\circ\Psi)(q)} \times \left.\frac{\partial\left(\Psi\circ\Psi^{-1}\circ\Phi\right)}{\partial v}\right|_{(\Phi^{-1}\circ\Psi)(q)} \right\| \left|\det d\left(\Phi^{-1}\circ\Psi\right)_q\right| dq \\
&= \int_{\Psi^{-1}(\Omega)} \left\| d\Psi_q \left.\frac{\partial\left(\Psi^{-1}\circ\Phi\right)}{\partial u}\right|_{(\Phi^{-1}\circ\Psi)(q)} \times d\Psi_q \left.\frac{\partial\left(\Psi^{-1}\circ\Phi\right)}{\partial v}\right|_{(\Phi^{-1}\circ\Psi)(q)} \right\| \left|\det d\left(\Phi^{-1}\circ\Psi\right)_q\right| dq \\
&= \int_{\Psi^{-1}(\Omega)} \left\| \left.\frac{\partial \Psi}{\partial u}\right|_q \times \left.\frac{\partial \Psi}{\partial v}\right|_q \right\| dq
\end{aligned}
$$

Areas of Patches on Surfaces

The definition of the area of $\Omega \subset V$ does not depend on the map.

$$
\begin{aligned}
A_\Phi(\Omega) &= \int_{\Phi^{-1}(\Omega)} \left\| \left.\frac{\partial \Phi}{\partial u}\right|_p \times \left.\frac{\partial \Phi}{\partial v}\right|_p \right\| dp \\
&= \int_{\Psi^{-1}(\Omega)} \left\| \left.\frac{\partial \Phi}{\partial u}\right|_{(\Phi^{-1}\circ\Psi)(q)} \times \left.\frac{\partial \Phi}{\partial v}\right|_{(\Phi^{-1}\circ\Psi)(q)} \right\| \left|\det d\left(\Phi^{-1}\circ\Psi\right)_q\right| dq \\
&= \int_{\Psi^{-1}(\Omega)} \left\| \left.\frac{\partial \left(\Psi\circ\Psi^{-1}\circ\Phi\right)}{\partial u}\right|_{(\Phi^{-1}\circ\Psi)(q)} \times \left.\frac{\partial \left(\Psi\circ\Psi^{-1}\circ\Phi\right)}{\partial v}\right|_{(\Phi^{-1}\circ\Psi)(q)} \right\| \left|\det d\left(\Phi^{-1}\circ\Psi\right)_q\right| dq \\
&= \int_{\Psi^{-1}(\Omega)} \left\| d\Psi_q \left.\frac{\partial \left(\Psi^{-1}\circ\Phi\right)}{\partial u}\right|_{(\Phi^{-1}\circ\Psi)(q)} \times d\Psi_q \left.\frac{\partial \left(\Psi^{-1}\circ\Phi\right)}{\partial v}\right|_{(\Phi^{-1}\circ\Psi)(q)} \right\| \left|\det d\left(\Phi^{-1}\circ\Psi\right)_q\right| dq \\
&= \int_{\Psi^{-1}(\Omega)} \left\| \left.\frac{\partial \Psi}{\partial u}\right|_q \times \left.\frac{\partial \Psi}{\partial v}\right|_q \right\| dq = A_\Psi(\Omega)
\end{aligned}
$$

Integrals over Curves on Surfaces

Given a mapping Φ:$U\rightarrow V\cap S$, if we have a regular curve, α:[a,b] $\rightarrow U$, and a function F:$S\rightarrow \mathbf{R}$, then we define the integral of F over the curve α as:

$$\int_\alpha F(p)dp = \int_a^b F\left(\Phi\left(\alpha(t)\right)\right)\left|\left(\Phi\circ\alpha\right)'(t)\right|dt$$

$$= \int_a^b F\left(\Phi\left(\alpha(t)\right)\right)\left|d\Phi_{\alpha(t)}\alpha'(t)\right|dt$$

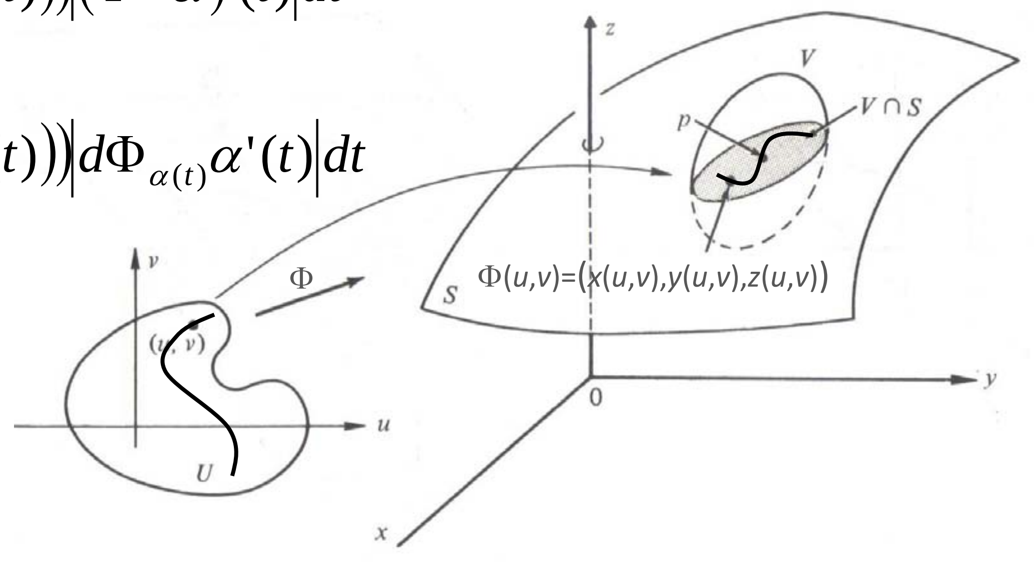


Figure 2-1

Integrals over Patches on Surfaces

Given a mapping $\Phi:U\to V\cap S$, if we have an open set $\Omega\subset S$ and a function $F:S\to\mathbf{R}$, then we define the integral of F over Ω as:

$$\int_{\Omega} F(p)dp = \int_{\Phi^{-1}(\Omega)} F(\Phi(p))\left\|\frac{\partial\Phi}{\partial u}\bigg|_p \times \frac{\partial\Phi}{\partial v}\bigg|_p\right\| dp$$

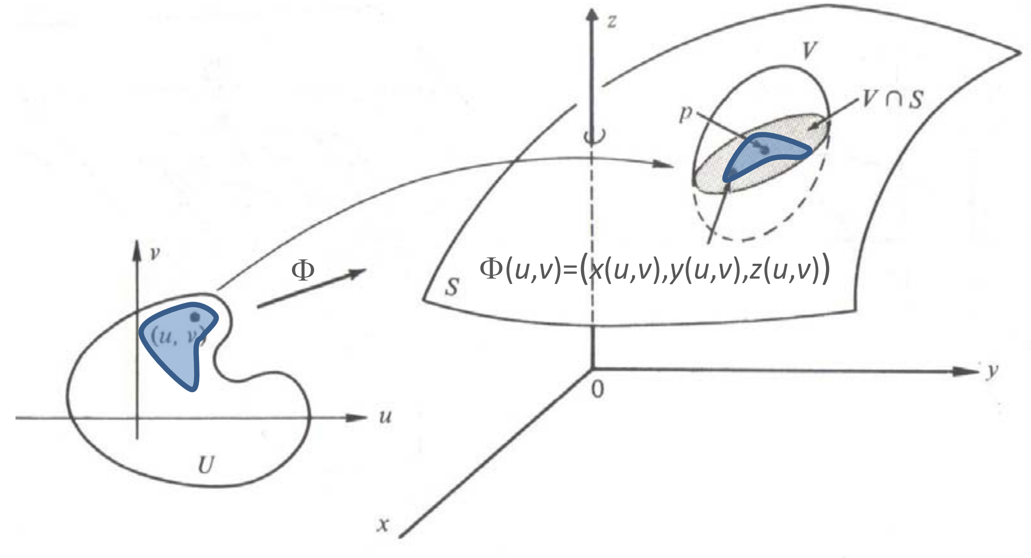
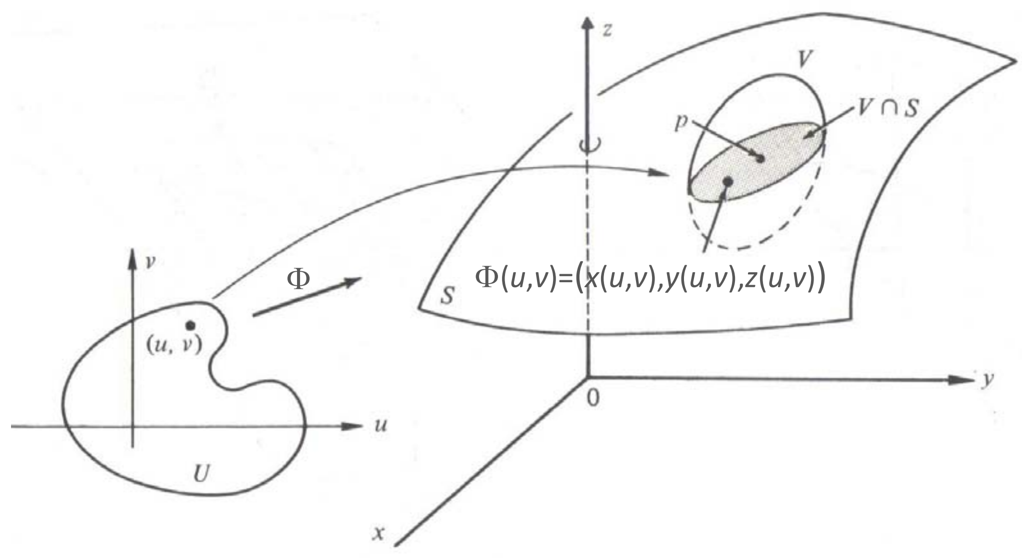


Figure 2-1

Integrals over Patches on Surfaces

Note that the definitions of integrals over curves and regions in *S* are independent of the parameterization.



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