Differential Geometry:

Stable Fluids

[*Stable, Circulation-Preserving, Simplicial Fluids*. Elcott *et al*., 2007]

Chains

<u>Recall</u>:

A *k-chain* of a simplicial complex K is linear combination of the *k*-simplices in K:

$$c = \sum_{\sigma \in \mathrm{K}^k} c(\sigma) \cdot \sigma$$

where *c* is a real-valued function.

The dual of a *k*-chain is a *k-cochain* which is a linear map taking a *k*-chain to a real value.

Chains and cochains related through evaluation:

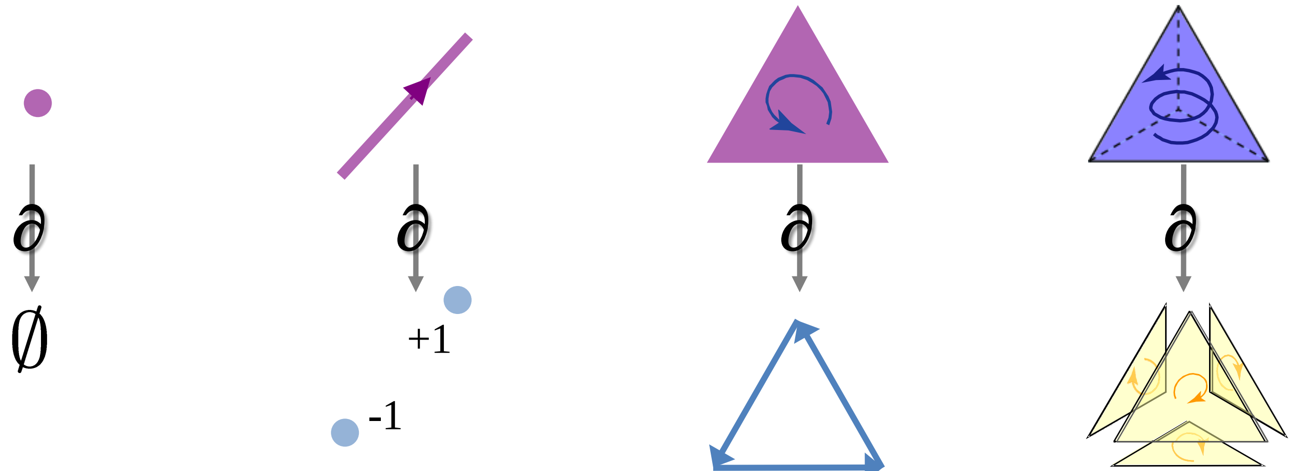
– <u>Cochains</u>: What we integrate

– <u>Chains</u>: What we integrate over

Boundary Operator

<u>Recall</u>:

The *boundary* $\partial\sigma$ of a *k*-simplex σ is the signed union of all (*k*-1)-faces.



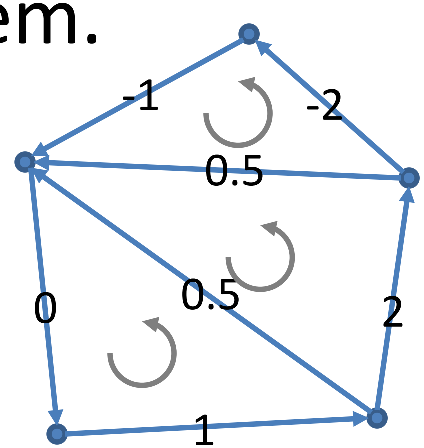
The boundary operator extends linearly to act on chains:

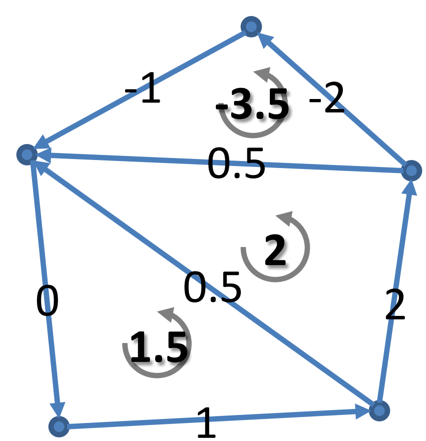
$$\partial c = \partial\left(\sum_{\sigma\in K^k} c(\sigma)\cdot\sigma\right) = \sum_{\sigma\in K^k} c(\sigma)\cdot\partial\sigma$$

Discrete Exterior Derivative

Recall:

The *exterior derivative* $d\!:\!\Omega^k \rightarrow \Omega^{k+1}$ is the operator on cochains that is the complement of the boundary operator, defined to satisfy Stokes' Theorem.





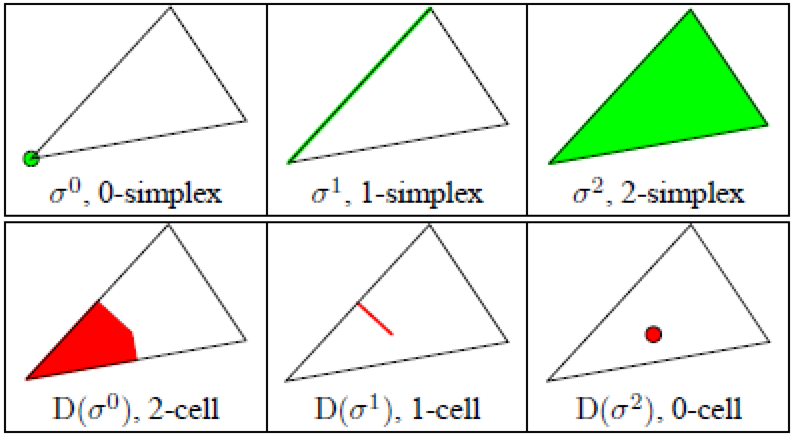
Since the boundary of the boundary is empty:

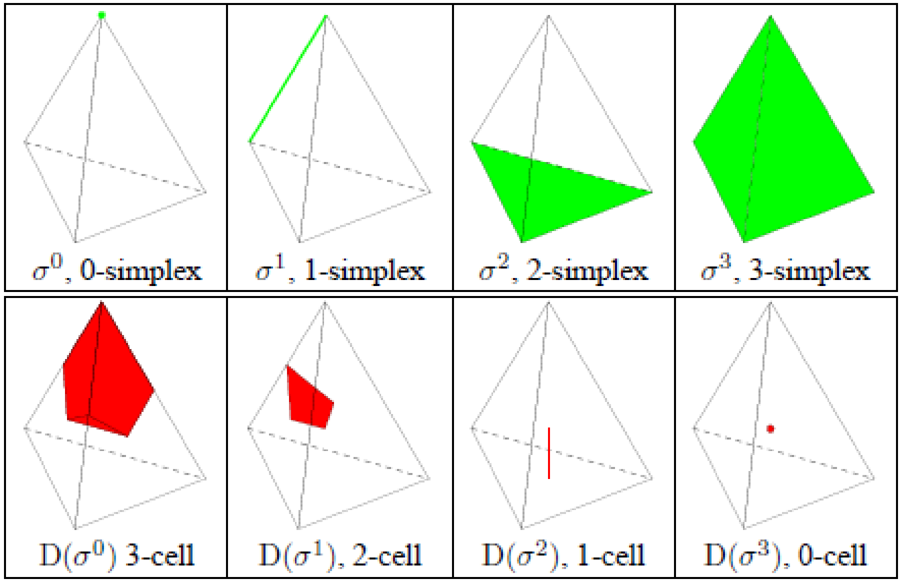
$$\partial\partial = 0 \qquad dd = 0$$

Dual Complexes

<u>Recall</u>:

Given a discrete manifold, we define the *Voronoi dual complex* **K* to be the (non-simplicial) dual complex obtained by associating simplices with their Voronoi duals.





Dual Complexes

Recall:

Using the association between primal simplices and dual cells, we define the discrete Hodge Star operator that takes *k*-cochains on the primal complex to (*n*-*k*)-cochains on the dual complex:

$$*:\Omega^k(\mathrm{K}) \to \Omega^{n-k}(*\mathrm{K})$$

A simple approximation results in a diagonal Hodge Star "redistributing" the average value:

$$*\omega(*\sigma) = \omega(\sigma)\frac{|*\sigma|}{|\sigma|}$$

Putting it All Together

In 3D:

– We have both primal and dual complexes.

– We have a boundary operator, and hence exterior derivative on both.

– We have *k*-cochains over both complexes.

– We have the Hodge Star operator to transition between the two.

$$
\begin{tikzcd}
\text{0-form} \arrow[r, "d"] \arrow[d, leftrightarrow, "*"] & \text{1-form} \arrow[r, "d"] \arrow[d, leftrightarrow, "*"] & \text{2-form} \arrow[r, "d"] \arrow[d, leftrightarrow, "*"] & \text{3-form} \arrow[d, leftrightarrow, "*"] \\
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\end{tikzcd}
$$

The Laplacian

<u>Recall</u>:

Using the exterior derivative, we can define the Laplacian of a 0-form:

$$\Delta f := -\left(*d*d\right)f$$

$$\begin{tikzcd}
\text{0-form} \arrow[r, "d"] \arrow[d, leftrightarrow, "*"] & \text{1-form} \arrow[r, "d"] \arrow[d, leftrightarrow, "*"] & \text{2-form} \arrow[r, "d"] \arrow[d, leftrightarrow, "*"] & \text{3-form} \arrow[d, leftrightarrow, "*"] \\
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The Laplacian

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Using the exterior derivative, we can define the Laplacian of a 0-form:

$$\Delta f := -\left(*\,d\,*\,d\right)f$$

And more generally, for a *k*-form ω, we have:

$$\Delta\omega := (-1)^{k+1}\left[\left(*\,d\,*\,d\right)-\left(d\,*\,d\,*\right)\right]\omega$$

$$\begin{tikzcd}
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\end{tikzcd}$$

Calculus

Definitions:

– *Flux*: The amount that flows through a unit area per unit time:

$$Flux = \iint_S \vec{F} \cdot \vec{n}\, ds$$

– *Vorticity*: The tendency for elements of a fluid to “spin” (the curl of a velocity field):

$$Vorticity = \nabla \times \vec{v}$$

– *Circulation*: The line integral around a closed curve of fluid velocity:

$$Circulation = \oint_C \vec{v} \cdot dl$$

Fluids

Euler Equations:

$\boxed{\rho = \text{const}}$

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p + f$$

$$\nabla \cdot u = 0$$

$$u \,||\, \partial D$$

pressure

momentum eq.

velocity

body forces

mass eq.

– Incompressible

– Non-linear PDE, with linear constraint

– Inviscid fluids

Fluids

Navier-Stokes Equations:

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p + f + \nu \Delta u$$

$$\nabla \cdot u = 0$$

$$u\big|_{\partial D} = 0$$

– Incompressible
– Non-linear PDE, with linear constraint
– Now the fluid is viscous (ν=kinematic viscosity) so there is a loss of total energy during motion

Fluids

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p \qquad \nabla \cdot u = 0 \qquad u \parallel \partial D$$

<u>General Approach (Euler Equation)</u>:

At each time-step:

- Advect the vector field
- Project onto the divergence free component

Fluids

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<u>General Approach (Euler Equation)</u>:

At each time-step:

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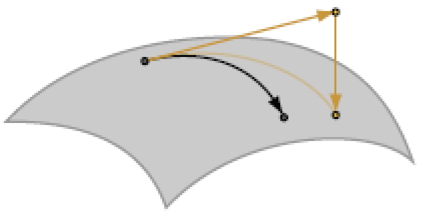
<u>Limitations</u>:

- Repeated projection causes energy loss
- Hard to do on arbitrary meshes



Fluids

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p \qquad \nabla \cdot u = 0 \qquad u \parallel \partial D$$

<u>Key Idea</u>:

Design a discrete formulation of fluid flow that preserves invariants (conservation of mass and conservation of momentum).

Fluids

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p \qquad \nabla \cdot u = 0 \qquad u \parallel \partial D$$

<u>Key Idea</u>:

Design a discrete formulation of fluid flow that preserves invariants (conservation of mass and conservation of momentum).

Instead of looking at how the vector field changes over time, we will look at the change in its *vorticity* (spin) over time.

Fluids

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p \qquad \nabla \cdot u = 0 \qquad u \,\|\, \partial D$$

<u>From Velocity to Vorticity</u>:

Setting ω to be the *vorticity* of the vector field:

$$\omega = \nabla \times u$$

taking the curl of the fluid-flow equation gives:

$$\nabla \times \left(\frac{\partial u}{\partial t} + u \cdot \nabla u \right) = -\nabla \times \nabla p$$

$$\Downarrow$$

$$\frac{\partial \omega}{\partial t} + \mathcal{L}_u \omega = 0$$

Fluids

$$\frac{\partial \omega}{\partial t} + \mathcal{L}_u \omega = 0 \qquad \omega = \nabla \times u \qquad \nabla \cdot u = 0 \qquad u \parallel \partial D$$

<u>From Velocity to Vorticity</u>:

In terms of the vorticity, the equation for the fluid flows states that the vorticity is advected with the flow.

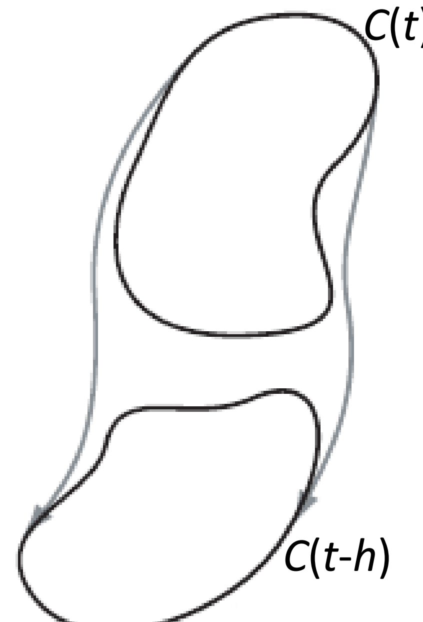
Fluids

$$\frac{\partial \omega}{\partial t} + \mathcal{L}_u \omega = 0 \qquad \omega = \nabla \times u \qquad \nabla \cdot u = 0 \qquad u \,\|\, \partial D$$

<u>Kelvin's Circulation Theorem</u>:

The circulation around any closed loop *C* is conserved throughout the motion of the loop in the fluid:

$$\Gamma(t) = \oint_{C(t)} u \cdot dl = \text{constant}$$



Fluids

$$\frac{\partial \omega}{\partial t} + \mathcal{L}_u \omega = 0 \qquad \omega = \nabla \times u \qquad \nabla \cdot u = 0 \qquad u \parallel \partial D$$

<u>Approach</u>:

1. Solve for the new vorticity at each time-step
2. Transform the vorticity into a vector field

Fluids

$$\frac{\partial \omega}{\partial t} + \mathcal{L}_u \omega = 0 \qquad \omega = \nabla \times u \qquad \nabla \cdot u = 0 \qquad u \parallel \partial D$$

<u>Approach</u>:

1. Solve for the new vorticity at each time-step
2. Transform the vorticity into a vector field

   Since *u* is assumed to be divergence-free (on a simple domain) this implies that *u* is the curl of some vector field:

   $$u = \nabla \times \phi$$

   so that the vorticity becomes:

   $$\omega = \Delta \phi$$

Fluids

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   so that the vorticity becomes:

   $$\omega = \Delta \phi$$

   Thus, given the vorticity, we can get the velocity field by solving the Poisson equation:

   $$u = \nabla \times \left(\Delta^{-1} \omega\right)$$

Fluids

$$\frac{\partial \omega}{\partial t} + \mathcal{L}_u \omega = 0 \qquad \omega = \nabla \times u \qquad \nabla \cdot u = 0 \qquad u \parallel \partial D$$

Discrete Flow:

In the discrete context, we have a simplicial-complex corresponding to the domain of the flow and a dual cell-complex.

Fluids

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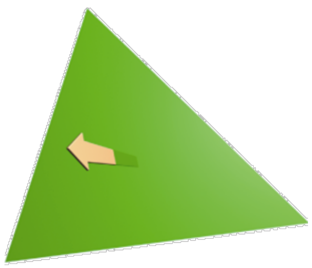
Discrete Flow:



In the discrete context, we have a simplicial-complex corresponding to the domain of the flow and a dual cell-complex.

- *Velocity Field/Flux*: The mass of fluid transported across a given surface area per unit time.

Fluids

$$\frac{\partial\omega}{\partial t}+\mathcal{L}_u\omega=0 \qquad \omega=\nabla\times u \qquad \nabla\cdot u=0 \qquad u\,\|\,\partial D$$

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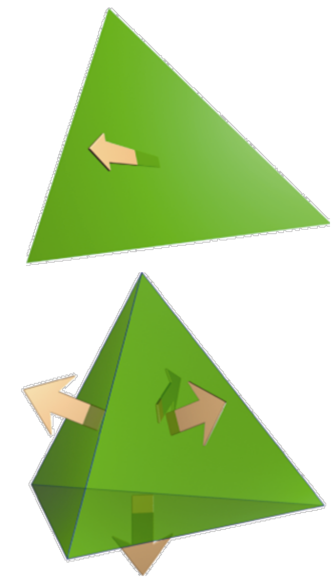


- *Velocity Field/Flux*
- *Divergence*: net flux through a tet.

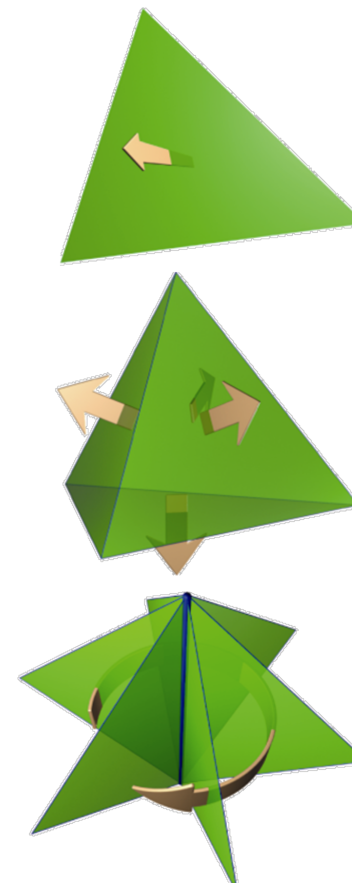
Fluids

$$\frac{\partial \omega}{\partial t} + \mathcal{L}_u \omega = 0 \qquad \omega = \nabla \times u \qquad \nabla \cdot u = 0 \qquad u \parallel \partial D$$

<u>Discrete Flow</u>:

In the discrete context, we have a simplicial-complex corresponding to the domain of the flow and a dual cell-complex.

- *Velocity Field/Flux*
- *Divergence*
- *Vorticity*: Torque on a "paddle wheel"



Fluids

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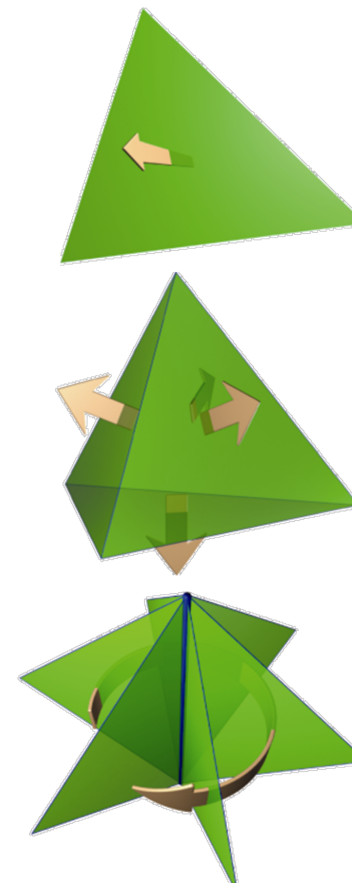
<u>Discrete Flow:</u>

- *Velocity Field/Flux*: A primal 2-form
- *Divergence*: The exterior derivative (*d*)
- *Vorticity*: The exterior co-derivative (**d**)



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Fluids

$$\frac{\partial \omega}{\partial t} + \mathcal{L}_u \omega = 0 \qquad \omega = \nabla \times u \qquad \nabla \cdot u = 0 \qquad u \parallel \partial D$$

<u>Solving for the Vorticity</u>:

At each time step, we want to compute the vorticity, ω, at the primal edge σ.

Fluids

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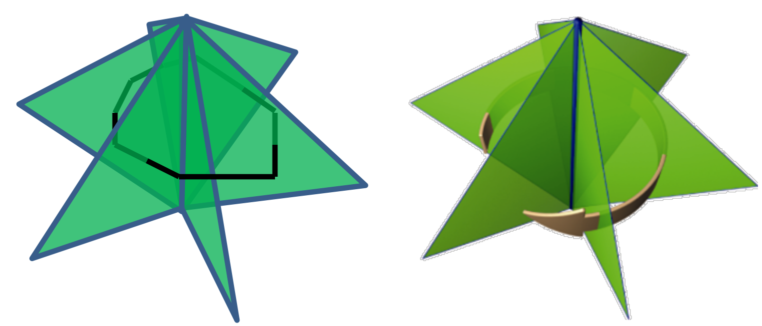
<u>Solving for the Vorticity</u>:

At each time step, we want to compute the new vorticity, ω, at the primal edge σ.

Using Hodge Star duality, knowing the primal 1-form ω is equivalent to knowing the dual 2-form $*\omega$ at $*\sigma$.



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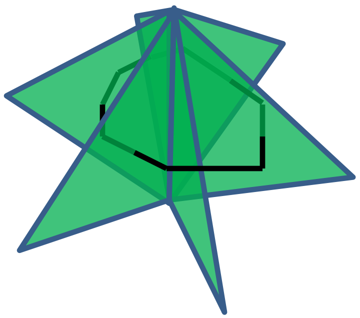
<u>Solving for the Vorticity</u>:

We know that dual 2-form $*\omega$ evaluated at the dual 2-face $*\sigma$ is a measure of the circulation about the boundary of $*\sigma$.



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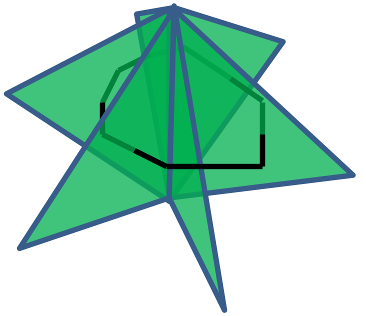
<u>Solving for the Vorticity</u>:

We know that dual 2-form $*\omega$ evaluated at the dual 2-face $*\sigma$ is a measure of the circulation about the boundary of $*\sigma$.

But by Kelvin's Theorem, this is equal to the old circulation over the pre-image of the boundary of $*\sigma$.



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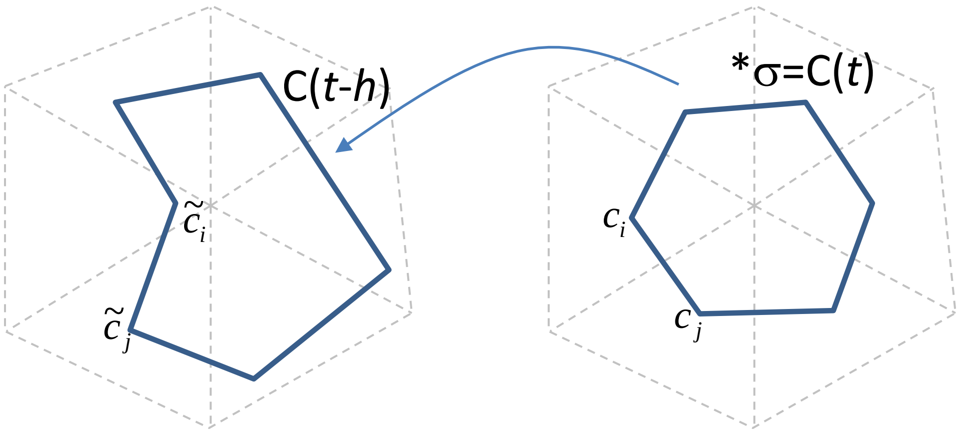
<u>Solving for the Vorticity</u>:

To get the new vorticity at an edge σ, we:

- Back-trace the boundary of $*\sigma$ to get the curve that is advected into $*\sigma$, and
- Evaluate the circulation over the back-traced curve.



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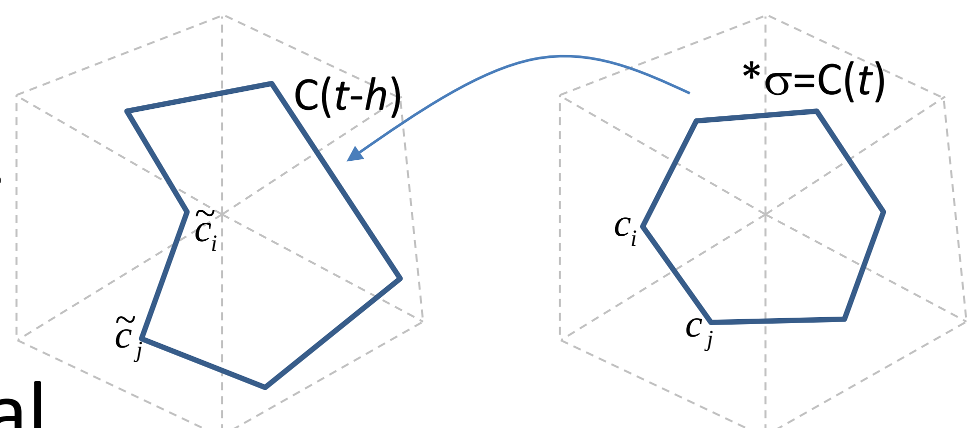
Solving for the Vorticity:

To get the new vorticity at an edge σ, we:

- Back-trace the boundary of $*\sigma$ to get the curve that is advected into $*\sigma$, and
- Evaluate the circulation over the back-traced curve.

Note:

In general, the edges of the back-traced curve are not edges in the dual.



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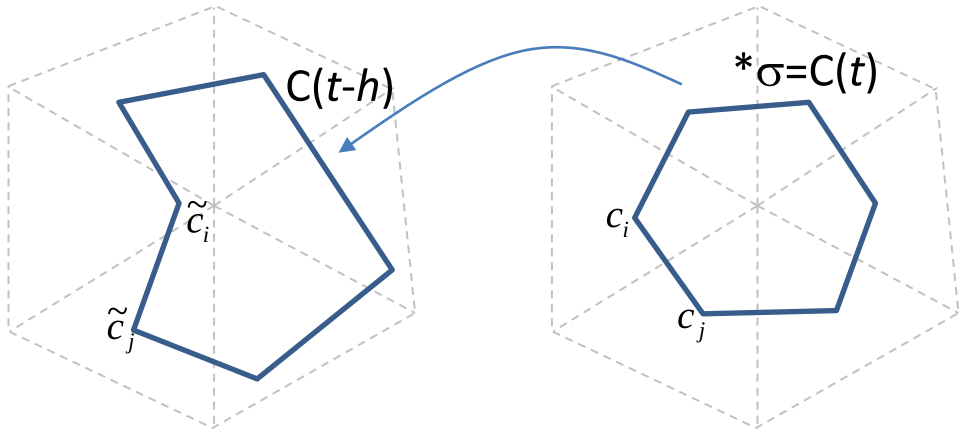
<u>Evaluating the Circulation on the Back-Trace</u>:

Although the flux is defined as a scalar value on primal-faces/dual-edges we can define a "corresponding" vector on primal-tets./dual-vertices.



Fluids

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<u>Evaluating the Circulation on the Back-Trace</u>:

Specifically, we can define a vector associated to each tet. whose dot-product with the (area-weighted) normals of the tet. faces is equal to the flux.

Fluids

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<u>Evaluating the Circulation on the Back-Trace</u>:

Specifically, we can define a vector associated to each tet. whose dot-product with the (area-weighted) normals of the tet. faces is equal to the flux.

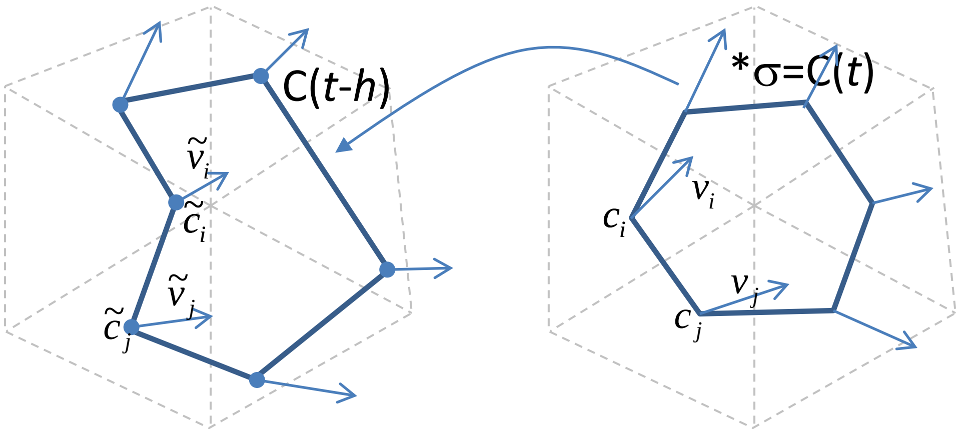
(Note that though there are four faces, there are only three constraints, since the divergence-free property of the field implies that one of the constraints is redundant.)

Fluids

$$\frac{\partial \omega}{\partial t} + \mathcal{L}_u \omega = 0 \qquad \omega = \nabla \times u \qquad \nabla \cdot u = 0 \qquad u \parallel \partial D$$

<u>Evaluating the Circulation on the Back-Trace</u>:

Using the "corresponding" normals, we can extrapolate the vector field and sample it at each of the back-traced vertices.



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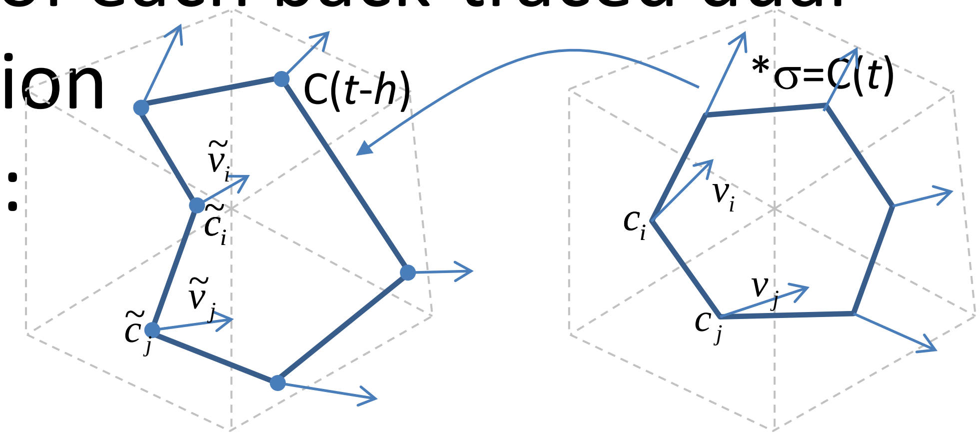
<u>Evaluating the Circulation on the Back-Trace</u>:

Using the "corresponding" normals, we can extrapolate the vector field and sample it at each of the back-traced vertices.

Then, the contribution of each back-traced dual-edge (*i*,*j*) to the circulation around a face becomes:

$$\frac{1}{2}\left(\tilde{v}_i + \tilde{v}_j\right) \cdot \left(\tilde{c}_j - \tilde{c}_i\right)$$



Implementation

Pre-Computation:

Set the Curl and Laplacian matrices:

– $C = *d*$

– $L = *d*d + d*d*$

Warning: Equations correct up to sign.

Implementation

Time-Stepping (h):

// Advect Vortices

For each dual vertex c_i:

$\tilde{c}_i$=PathTraceBackwards(c_i , h)

$\tilde{v}_i$=InterpolateVelocityField($\tilde{c}_i$)

For each dual face f:

$\Omega_f = 0$;

For each dual edge (i,j) on f:

$$\Omega_f += \frac{1}{2}\left(\tilde{v}_i + \tilde{v}_j\right)\cdot\left(\tilde{c}_j - \tilde{c}_i\right)$$

// Convert Vortices to Fluxes

Φ = Solve($L\Phi=\Omega$);

$U = d\Phi$;

Implementation

<u>More Complex Flows</u>:

// Advect Vortices

…

// Add Forces

Ω += *hCF*;

// Add Diffusion

Ψ = Solve((*-ν*hL*)Ψ =Ω);

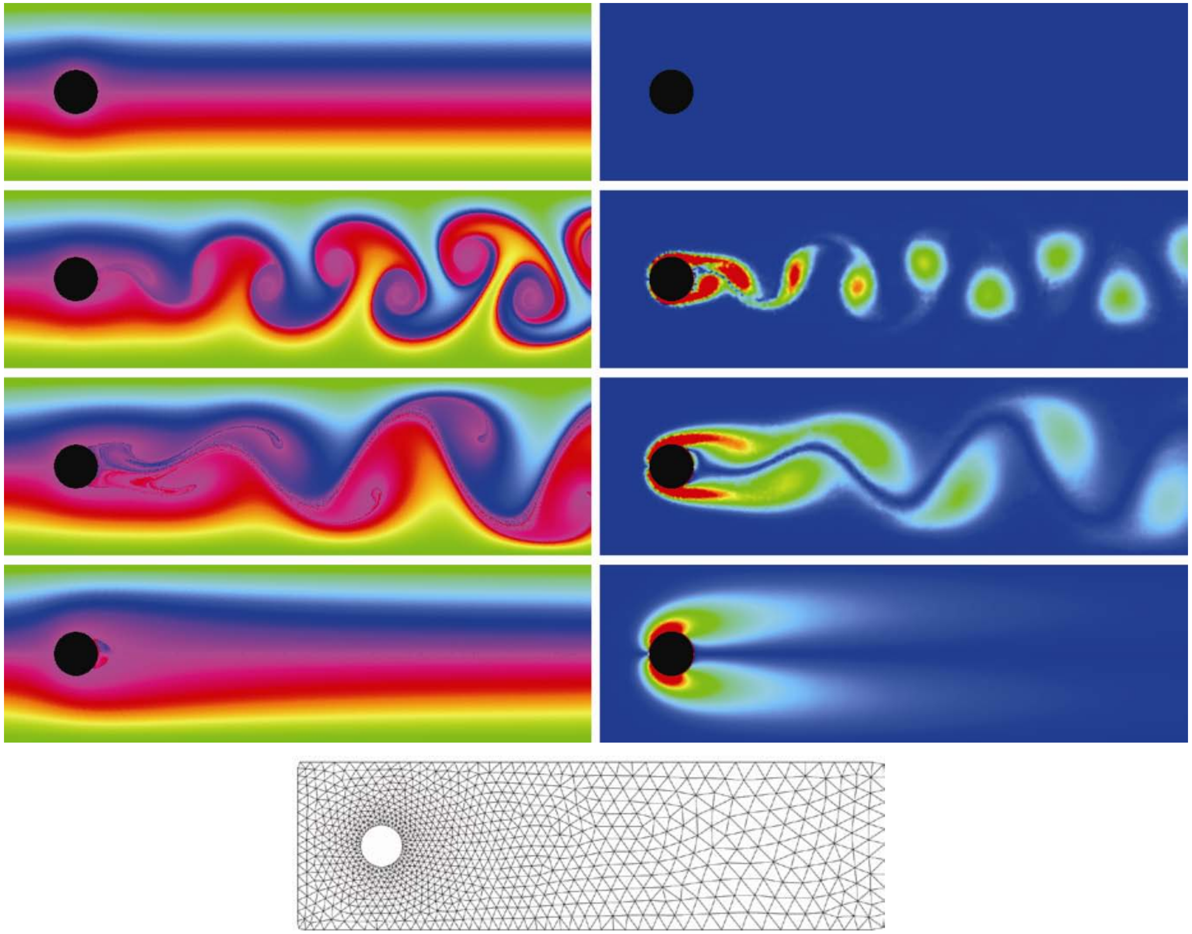
Ω = *Ψ;

// Convert Vortices to Fluxes

…

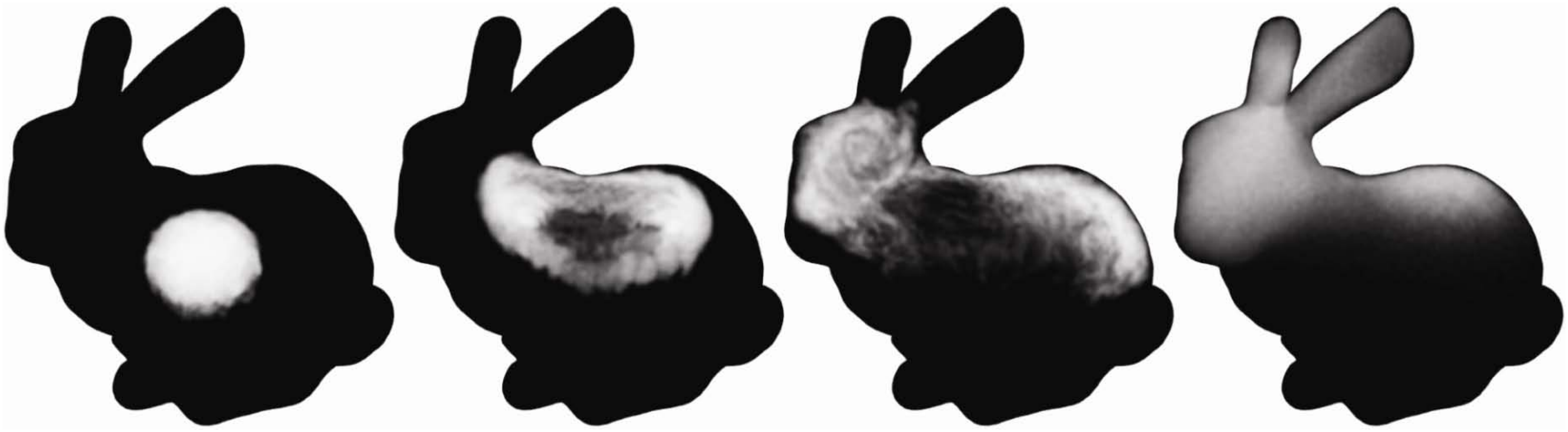
Examples

<u>Channel</u>:



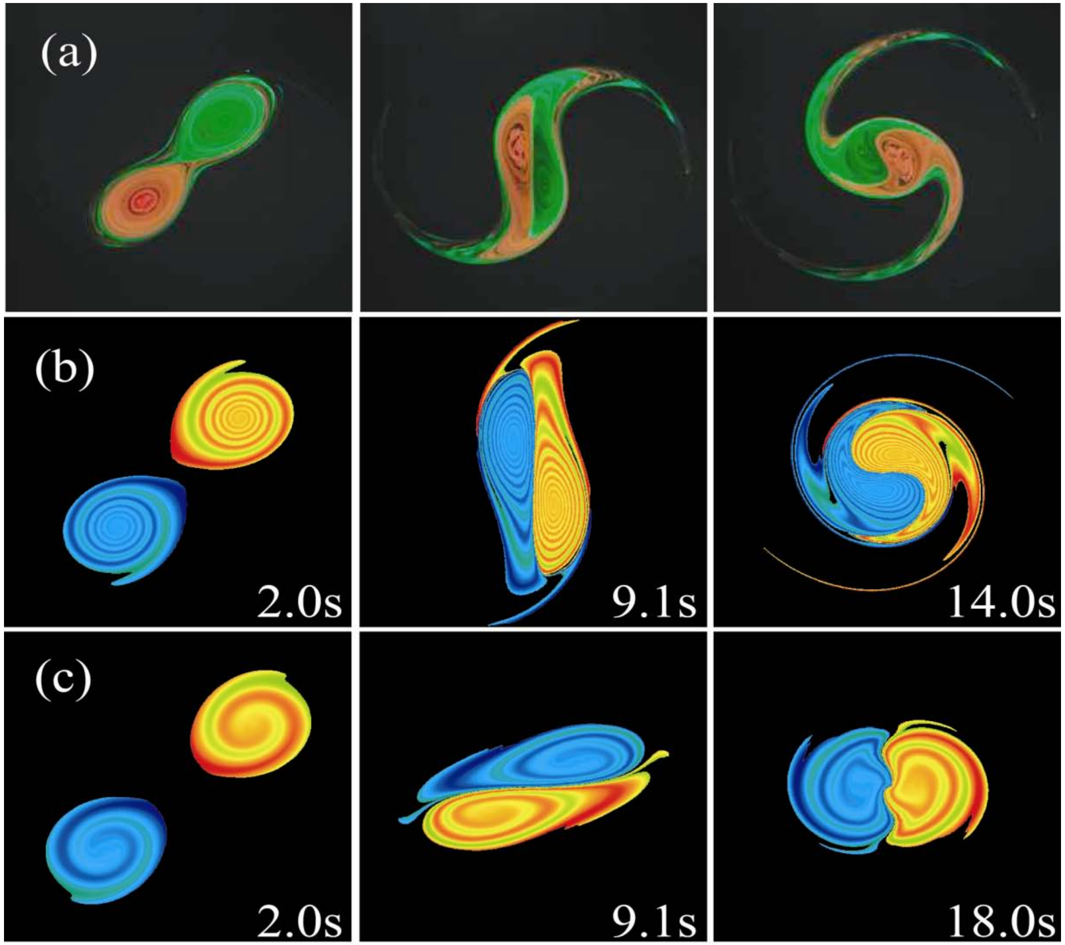
Examples

<u>Smoking Bunny</u>:



Examples

Merging Vortices:



Examples

Video: