Differential Geometry:

Discrete Exterior Calculus

[*Discrete Exterior Calculus* (Thesis). Hirani, 2003]
[*Discrete Differential Forms for Computational Modeling*. Desbrun *et al*., 2005]
[*Build Your Own DEC at Home*. Elcott *et al*., 2006]

Chains

Recall:

A *k-chain* of a simplicial complex K is linear combination of the *k*-simplices in K:

$$c = \sum_{\sigma \in \mathrm{K}^k} c(\sigma) \cdot \sigma$$

where *c* is a real-valued function.

The dual of a *k*-chain is a *k-cochain* which is a linear map taking a *k*-chain to a real value.

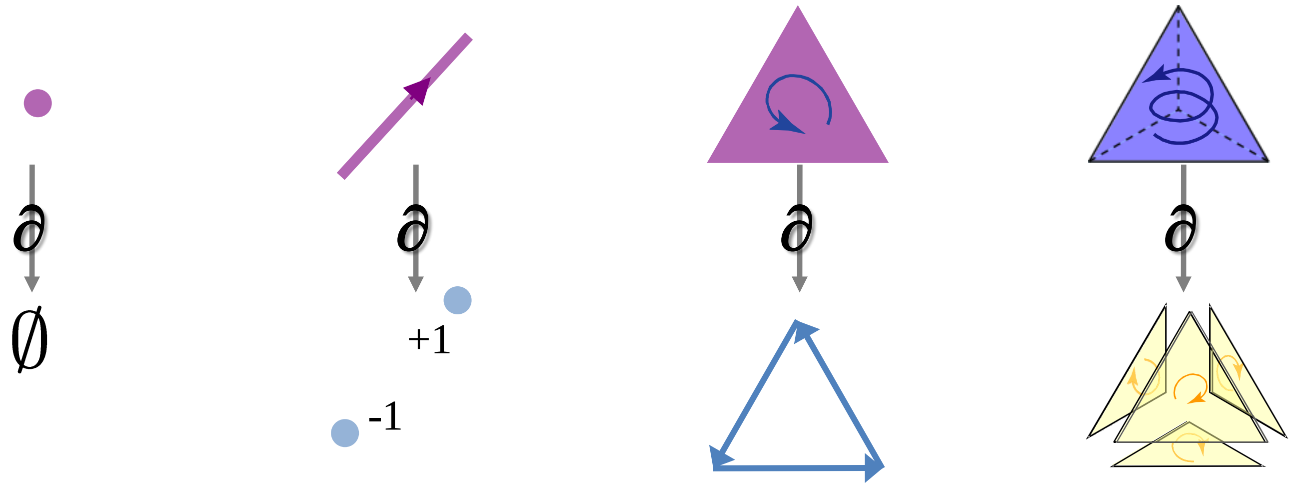
Chains and cochains related through evaluation:

– Cochains: What we integrate
– Chains: What we integrate over

Boundary Operator

<u>Recall</u>:

The *boundary* ∂σ of a *k*-simplex σ is the signed union of all (*k*-1)-faces.



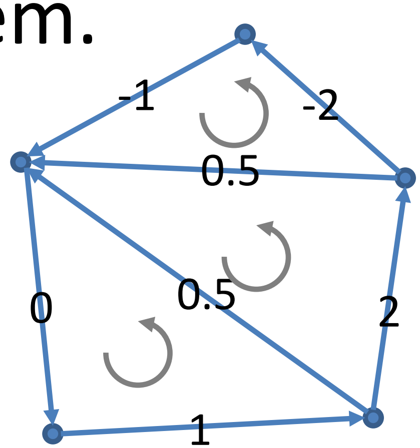
The boundary operator extends linearly to act on chains:

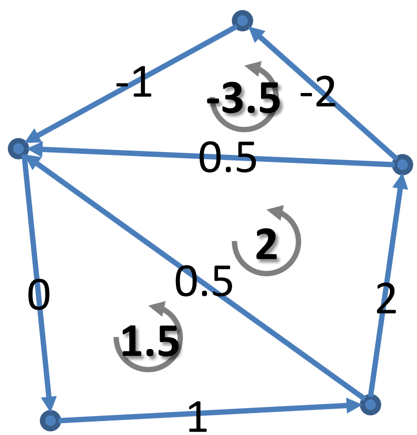
$$\partial c = \partial\left(\sum_{\sigma\in K^k} c(\sigma)\cdot\sigma\right) = \sum_{\sigma\in K^k} c(\sigma)\cdot\partial\sigma$$

Discrete Exterior Derivative

Recall:

The *exterior derivative* d:$\Omega^k \rightarrow \Omega^{k+1}$ is the operator on cochains that is the complement of the boundary operator, defined to satisfy Stokes' Theorem.





Since the boundary of the boundary is empty:

$$\partial\partial = 0 \qquad dd = 0$$

The Laplacian

Definition:

The *Laplacian* of a function *f* is a function defined as the divergence of the gradient of *f*:

$$\Delta f = \operatorname{div}(\nabla f) = \nabla \cdot \nabla f$$

The Laplacian

<u>Definition</u>:

The *Laplacian* of a function *f* is a function defined as the divergence of the gradient of *f*:

$$\Delta f = \mathrm{div}(\nabla f) = \nabla \cdot \nabla f$$

Q. What do the gradient and divergence operators act on?

The Gradient

<u>Definition</u>:

Given a function f:$\mathbf{R}^n \rightarrow \mathbf{R}$, the *gradient* of f at a point p is defined as:

$$\nabla f(p) = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)$$

The Gradient

<u>Definition</u>:

Given a function f:$\boldsymbol{R}^n \rightarrow \boldsymbol{R}$, the *gradient* of f at a point p is defined as:

$$\nabla f(p) = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)$$

Given a direction vector v, we have:

$$\begin{aligned} f\left(p+v\right) &\approx f(p) + v_1 \frac{\partial f}{\partial x_1} + \ldots + v_1 \frac{\partial f}{\partial x_1} \\ &= f(p) + \left\langle v, \nabla f(p) \right\rangle \end{aligned}$$

The Gradient

<u>Definition</u>:

Given a function f:$\boldsymbol{R}^n \rightarrow \boldsymbol{R}$, the *gradient* of f at a point p is defined as:

$$\nabla f(p) = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)$$

Given a direction vector v, we have:

$$f(p+v) \approx f(p) + \langle v, \nabla f(p) \rangle$$

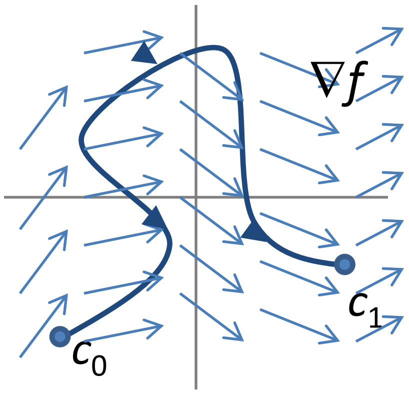
The gradient is an operator (not really a vector) telling us how moving in direction v will affect the value of the function.

The Gradient

Stokes' Theorem:

Given an oriented curve $c \subset \mathbf{R}^n$ with endpoints c_0 and c_1, integrating the gradient of a function over the curve, we get the difference between the values of the function at the endpoints:

$$\int_c \nabla f(p) = f(c_1) - f(c_0)$$



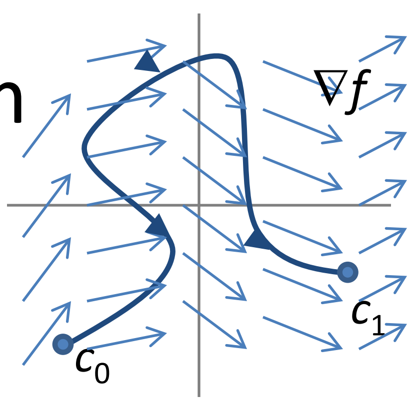
The Gradient

Stokes' Theorem:

Given an oriented curve $c \subset \boldsymbol{R}^n$ with endpoints c_0 and c_1, integrating the gradient of a function over the curve, we get the difference between the values of the function at the endpoints:

$$\int_c \nabla f(p) = f(c_1) - f(c_0)$$

The gradient satisfies Stokes' Theorem and takes something we evaluate at points to something we evaluate over curves.



The Gradient

<u>Stokes' Theorem</u>:

Given an oriented curve $c \subset \boldsymbol{R}^n$ with endpoints c_0 and c_1, integrating the gradient of a function over the curve, we get the difference between the values of the function at the endpoints:

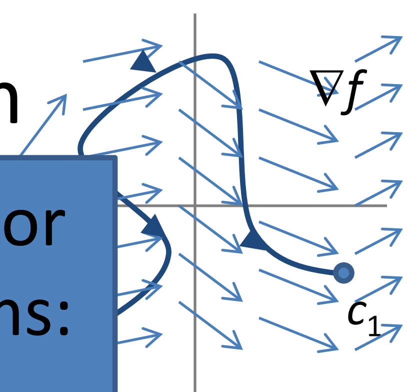
$$\int_c \nabla f(p) = f(c_1) - f(c_0)$$

The gradient satisfies Stokes' Theorem

and tak

points t

over cu



The gradient operator is the exterior derivative taking 0-forms to 1-forms:

$$\nabla = d : \Omega^0(\boldsymbol{R}^n) \rightarrow \Omega^1(\boldsymbol{R}^n)$$

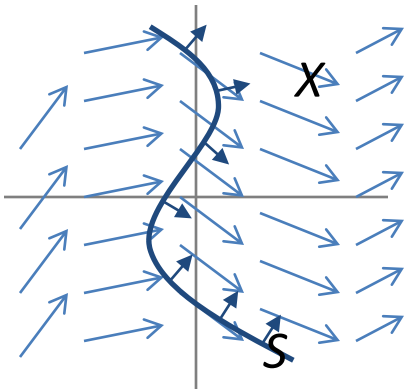
Divergence

<u>Definition</u>:

Given a vector-field X:$\mathbf{R}^n \rightarrow \mathbf{R}^n$, representing a flow, the *flux* of the vector field through a patch of hyper-surface S is the "amount" of flow that passes through the surface:

$$\text{Flux}_S(f) = \iint_S \langle X, n \rangle dS$$

where n is the surface normal.

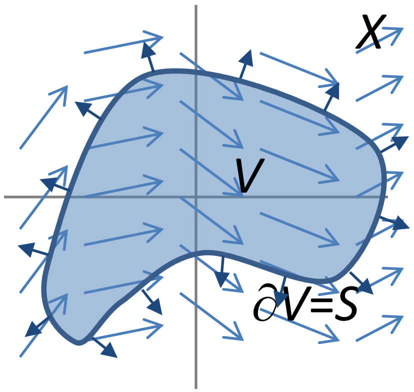


Divergence

Stokes' Theorem:

Given a volume $V \subset \boldsymbol{R}^n$ with boundary $\partial V = S$, integrating the divergence of a function over the volume, we get the flux across the boundary:

$$\iiint_V \nabla \cdot X = \iint_S \langle X, n \rangle dS$$



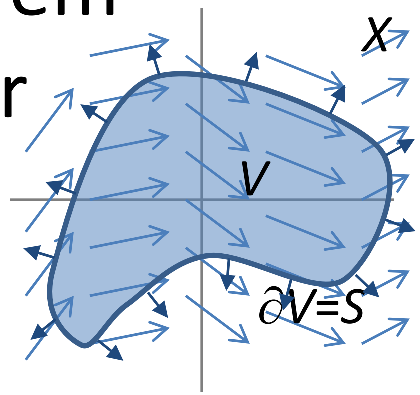
Divergence

<u>Stokes' Theorem</u>:

Given a volume $V\subset \mathbf{R}^n$ with boundary $\partial V=S$, integrating the divergence of a function over the volume, we get the flux across the boundary:

$$\iiint_V \nabla \cdot X = \iint_S \langle X, n \rangle dS$$

The divergence satisfies Stokes' Theorem and takes something we evaluate over hyper-surfaces to something we evaluate over volumes.



Divergence

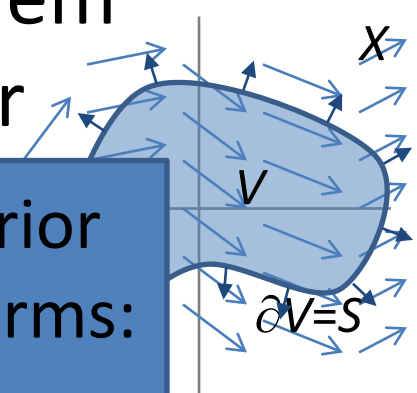
<u>Stokes' Theorem</u>:

Given a volume $V \subset \boldsymbol{R}^n$ with boundary $\partial V = S$, integrating the divergence of a function over the volume, we get the flux across the boundary:

$$\iiint_V \nabla \cdot X = \iint_S \langle X, n \rangle dS$$

The divergence satisfies Stokes' Theorem and takes something we evaluate over hyper

evalua



The divergence operator is the exterior derivative taking (n-1)-forms to n-forms:

$$\nabla \cdot = d : \Omega^{n-1}(\boldsymbol{R}^n) \to \Omega^n(\boldsymbol{R}^n)$$

The Laplacian

$$\Delta f = \operatorname{div}\left(\nabla f\right) = \nabla \cdot \nabla f$$

Q. What do the gradient and divergence operators act on?

A. The gradient and divergence operators are exterior derivatives that act on 0-forms and (n-1) forms respectively:

- Gradient $\nabla : \Omega^0(\boldsymbol{R}^n) \to \Omega^1(\boldsymbol{R}^n)$
- Divergence $\nabla \cdot : \Omega^{n-1}(\boldsymbol{R}^n) \to \Omega^n(\boldsymbol{R}^n)$

The Laplacian

$$\Delta f = \operatorname{div}(\nabla f) = \nabla \cdot \nabla f$$

– Gradient $\nabla : \Omega^0(\boldsymbol{R}^n) \rightarrow \Omega^1(\boldsymbol{R}^n)$
– Divergence $\nabla \cdot : \Omega^{n-1}(\boldsymbol{R}^n) \rightarrow \Omega^n(\boldsymbol{R}^n)$

Problem: The Laplacian is supposed to take functions (0-forms) to functions, but the divergence *doesn't* take 1-forms as input and it *doesn't* return 0-forms as output!!!

Poincaré Duality

Observation:

In an *n*-dimensional inner-product space, specifying a *k*-dimensional subspace is the same as specifying an (*n*-*k*)-dimensional subspace.

Poincaré Duality

Observation:

In an n-dimensional inner-product space, specifying a k-dimensional subspace is the same as specifying an (n-k)-dimensional subspace.

Moreover, if the space is oriented, specifying a k-dimensional oriented subspace is the same as specifying an (n-k)-dimensional oriented subspace.

Poincaré Duality

If we are given a *k*-dimensional subspace spanned by vectors $\{v_1,\ldots,v_k\}$, we can always find vectors $\{v_{k+1},\ldots,v_n\}$ such that the vectors $\{v_1,\ldots,v_n\}$ are a basis for the whole space and:

$$\left\langle v_i, v_j \right\rangle = 0$$

for all $1 \leq i \leq k$ and all $k+1 \leq j \leq n$.

Poincaré Duality

If we are given a k-dimensional subspace spanned by vectors $\{v_1,\ldots,v_k\}$, we can always find vectors $\{v_{k+1},\ldots,v_n\}$ such that the vectors $\{v_1,\ldots,v_n\}$ are a basis for the whole space and:

$$\left\langle v_i, v_j \right\rangle = 0$$

for all $1 \leq i \leq k$ and all $k+1 \leq j \leq n$.

If the space has an orientation, we can order the $\{v_{k+1},\ldots,v_n\}$ so that the vectors $\{v_1,\ldots,v_n\}$ have positive orientation.

Poincaré Duality

Example:

Consider $\boldsymbol{R}^2$, with the standard inner product and orientation $x \wedge y$:

Poincaré Duality

Example:

Consider $\boldsymbol{R}^2$, with the standard inner product and orientation $x \wedge y$:

- $\boldsymbol{R}^0 \to \boldsymbol{R}^2$: $\varnothing \to x \wedge y$
- $\boldsymbol{R}^1 \to \boldsymbol{R}^1$: $x \to y$, $\qquad y \to -x$
- $\boldsymbol{R}^2 \to \boldsymbol{R}^0$: $x \wedge y \to \varnothing$

Poincaré Duality

Example:

Consider $\boldsymbol{R}^3$, with the standard inner product and orientation $x \wedge y \wedge z$:

Poincaré Duality

Example:

Consider $\boldsymbol{R}^3$, with the standard inner product and orientation $x \wedge y \wedge z$:

- $\boldsymbol{R}^0 \to \boldsymbol{R}^2$: $\varnothing \to x \wedge y \wedge z$
- $\boldsymbol{R}^1 \to \boldsymbol{R}^2$: $x \to y \wedge z$, $y \to -x \wedge z$, $z \to x \wedge y$
- $\boldsymbol{R}^2 \to \boldsymbol{R}^1$: $x \wedge y \to z$, $x \wedge z \to -y$, $y \wedge z \to x$
- $\boldsymbol{R}^3 \to \boldsymbol{R}^0$: $x \wedge y \wedge z \to \varnothing$

The Laplacian

$$\Delta f = \mathrm{div}(\nabla f) = \nabla \cdot \nabla f$$

- <u>Gradient</u> $\nabla : \Omega^0(\boldsymbol{R}^n) \rightarrow \Omega^1(\boldsymbol{R}^n)$
- <u>Divergence</u> $\nabla \cdot : \Omega^{n-1}(\boldsymbol{R}^n) \rightarrow \Omega^n(\boldsymbol{R}^n)$

<u>Problem</u>: The Laplacian is supposed to take functions (0-forms) to functions, but the divergence *doesn't* take 1-forms as input and it *doesn't* return 0-forms as output!!!

The Laplacian

$$\Delta f = \mathrm{div}(\nabla f) = \nabla \cdot \nabla f$$

- <u>Gradient</u> $\nabla : \Omega^0(\boldsymbol{R}^n) \rightarrow \Omega^1(\boldsymbol{R}^n)$
- <u>Divergence</u> $\nabla \cdot : \Omega^{n-1}(\boldsymbol{R}^n) \rightarrow \Omega^n(\boldsymbol{R}^n)$

<u>Problem</u>: The Laplacian is supposed to take functions (0-forms) to functions, but the divergence *doesn't* take 1-forms as input and it *doesn't* return 0-forms as output!!!

<u>Solution</u>: Use the duality between *k*-dimensional and (*n*-*k*)-dimensional subspaces to turn 1-forms into (*n*-1)-forms and *n*-forms into 0-forms.

Hodge Star

Locally, a *k*-form take *k*-dimensional, oriented, surface elements and return a value.

Q: The corresponding dual (*n*-*k*)-form should act on the (oriented) orthogonal complement, but what value should it return?

Hodge Star

Locally, a k-form take k-dimensional, oriented, surface elements and return a value.

Q: The corresponding dual $(n\text{-}k)$-form should act on the (oriented) orthogonal complement, but what value should it return?

A: Informally, we think of a k-form as assigning a density per unit k-volume (the "average"). We define the corresponding $(n\text{-}k)$-form with the same density per unit $(n\text{-}k)$-volume.

Hodge Star

Definition:

The map taking a *k*-form to the corresponding (*n*-*k*)-forms is called the *Hodge Star* operator:

$$* : \Omega^k(\boldsymbol{R}^n) \to \Omega^{n-k}(\boldsymbol{R}^n)$$

Hodge Star

For $\boldsymbol{R}^2$:

- $*: \Omega^0(\boldsymbol{R}^2) \rightarrow \Omega^2(\boldsymbol{R}^2)$: $\varnothing \rightarrow x \wedge y$
- $*: \Omega^1(\boldsymbol{R}^2) \rightarrow \Omega^1(\boldsymbol{R}^2)$: $x \rightarrow y, \quad y \rightarrow -x$
- $*: \Omega^2(\boldsymbol{R}^2) \rightarrow \Omega^0(\boldsymbol{R}^2)$: $x \wedge y \rightarrow \varnothing$

For $\boldsymbol{R}^3$:

- $*: \Omega^0(\boldsymbol{R}^3) \rightarrow \Omega^3(\boldsymbol{R}^3)$: $\varnothing \rightarrow x \wedge y \wedge z$
- $*: \Omega^1(\boldsymbol{R}^3) \rightarrow \Omega^2(\boldsymbol{R}^3)$: $x \rightarrow y \wedge z, \quad y \rightarrow -x \wedge z, \quad z \rightarrow x \wedge y$
- $*: \Omega^2(\boldsymbol{R}^3) \rightarrow \Omega^1(\boldsymbol{R}^3)$: $x \wedge y \rightarrow z, \quad x \wedge z \rightarrow -y, \quad y \wedge z \rightarrow x$
- $*: \Omega^3(\boldsymbol{R}^3) \rightarrow \Omega^0(\boldsymbol{R}^3)$: $x \wedge y \wedge z \rightarrow \varnothing$

Hodge Star

For $\boldsymbol{R}^2$:

- $*:\Omega^0(\boldsymbol{R}^2)\to \Omega^2(\boldsymbol{R}^2)$: $\varnothing\to x\wedge y$
- $*:\Omega^1(\boldsymbol{R}^2)\to \Omega^1(\boldsymbol{R}^2)$: $x\to y$, $y\to -x$
- $*:\Omega^2(\boldsymbol{R}^2)\to \Omega^0(\boldsymbol{R}^2)$: $x\wedge y\to\varnothing$

For $\boldsymbol{R}^3$:

- $*:\Omega^0(\boldsymbol{R}^3)\to \Omega^3(\boldsymbol{R}^3)$: $\varnothing\to x\wedge y\wedge z$
- $*:\Omega^1(\boldsymbol{R}^3)\to \Omega^2(\boldsymbol{R}^3)$: $x\to y\wedge z$, $y\to -x\wedge z$, $z\to x\wedge y$
- $*:\Omega^2(\boldsymbol{R}^3)\to \Omega^1(\boldsymbol{R}^3)$: $x\wedge y\to z$, $x\wedge z\to -y$, $y\wedge z\to x$

Note that the Hodge Star operator from *k*-forms to (*n*-*k*)-forms and the operator from (*n*-*k*)-forms to *k*-forms are almost inverses:

$$*_k *_{n-k} = (-1)^{k(n-k)}$$

Dual Complexes

In order to be able to define a discrete Laplacian, we need to have a discrete version of the Hodge Star.

Dual Complexes

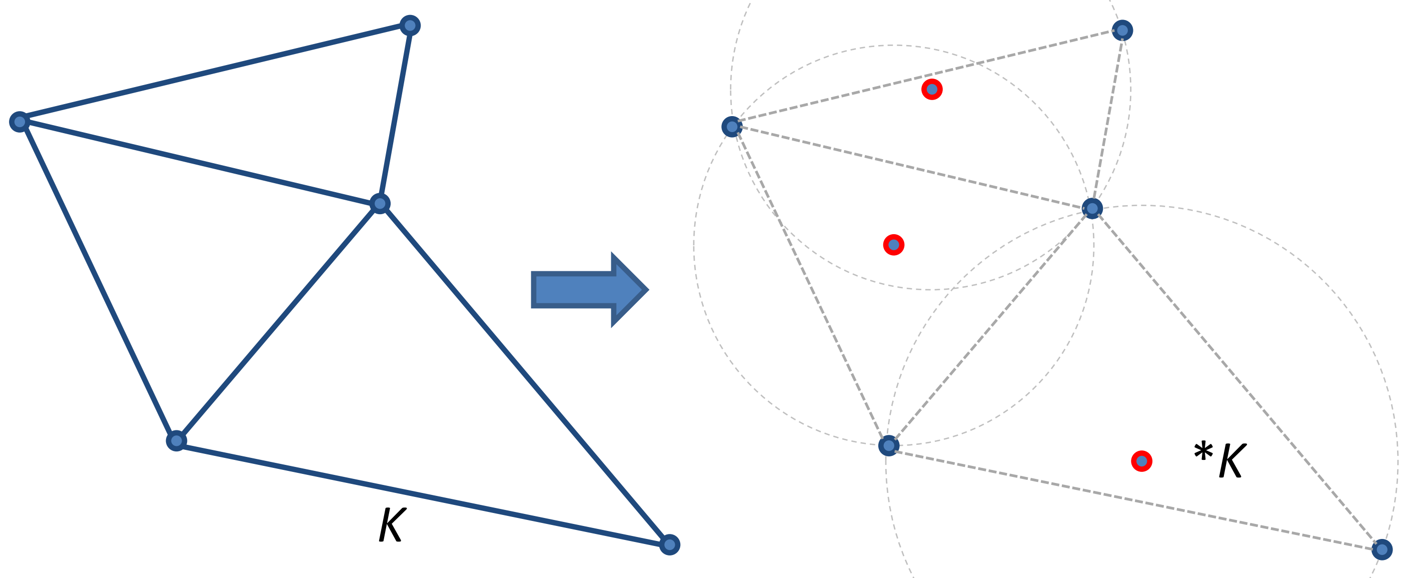
In order to be able to define a discrete Laplacian, we need to have a discrete version of the Hodge Star.

In the same way as we defined dual/orthogonal subspaces in the continuous case, we would like to define dual/orthogonal simplexes in the discrete case.

Dual Complexes

Definition:

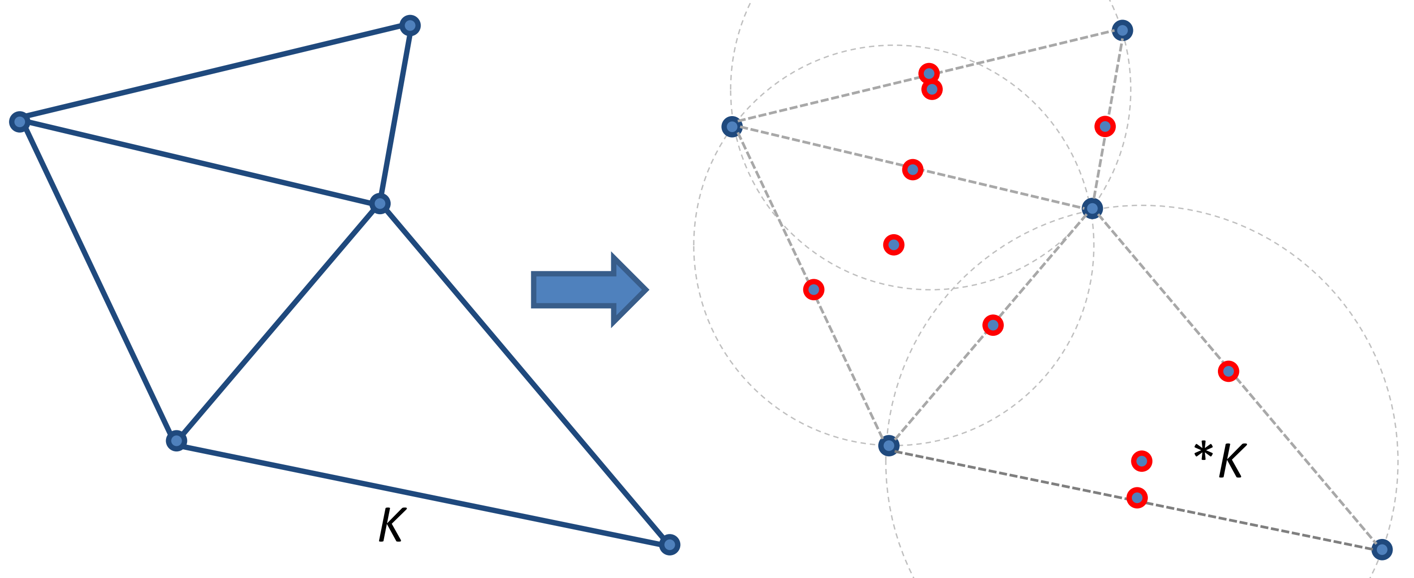
Given a discrete manifold, we define the *Voronoi dual complex* *K to be the (non-simplicial) dual complex obtained by associating simplices with their Voronoi duals.



Dual Complexes

<u>Definition</u>:

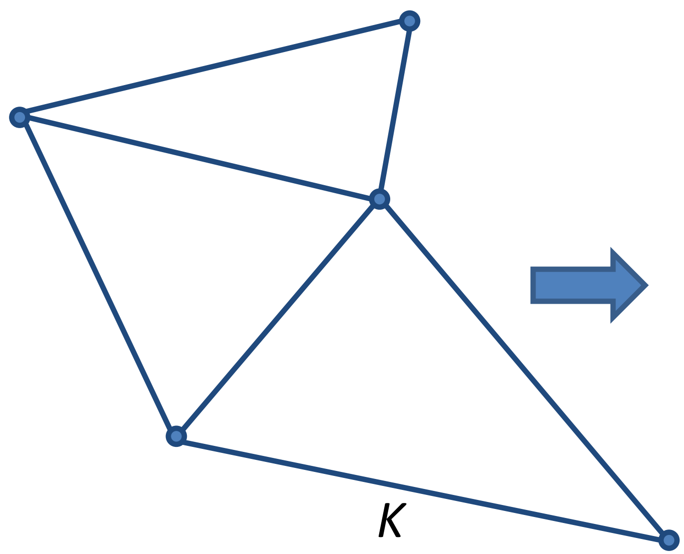
Given a discrete manifold, we define the *Voronoi dual complex* $*K$ to be the (non-simplicial) dual complex obtained by associating simplices with their Voronoi duals.

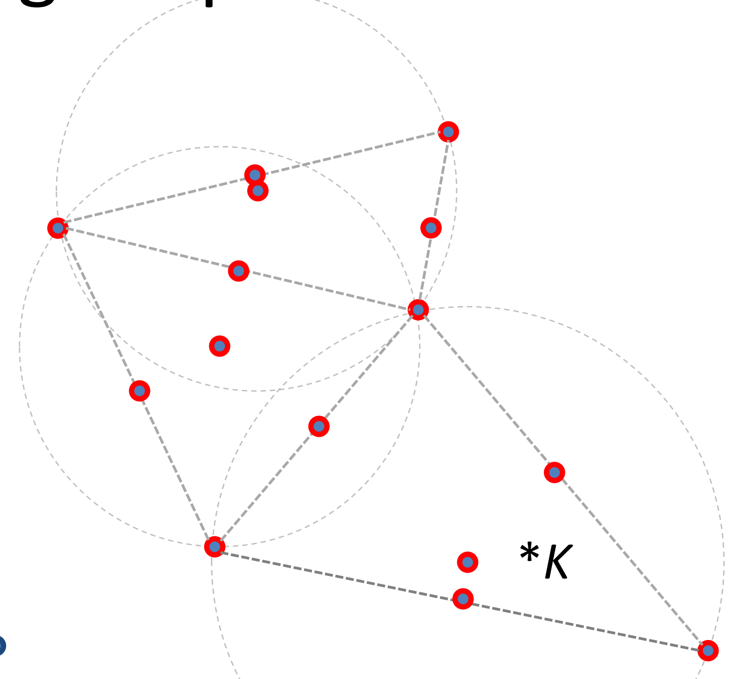


Dual Complexes

<u>Definition</u>:

Given a discrete manifold, we define the *Voronoi dual complex* **K* to be the (non-simplicial) dual complex obtained by associating simplices with their Voronoi duals.

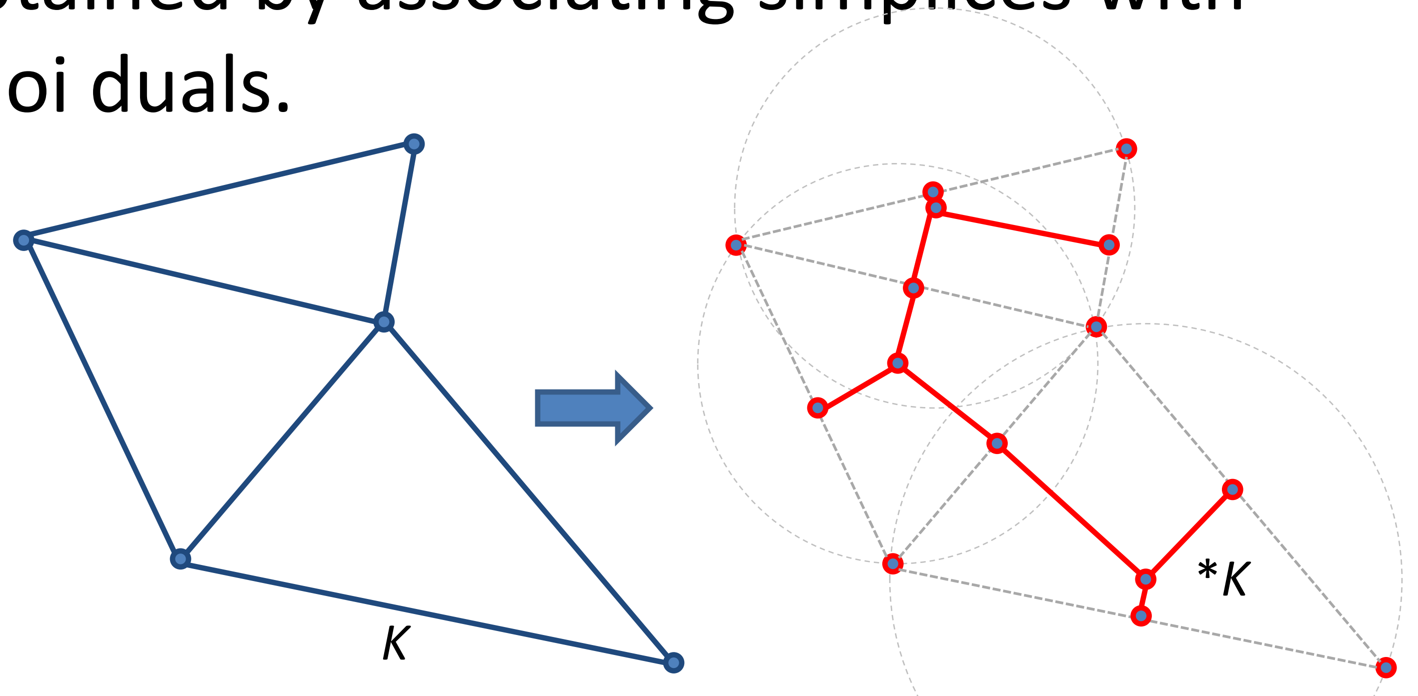




Dual Complexes

<u>Definition</u>:

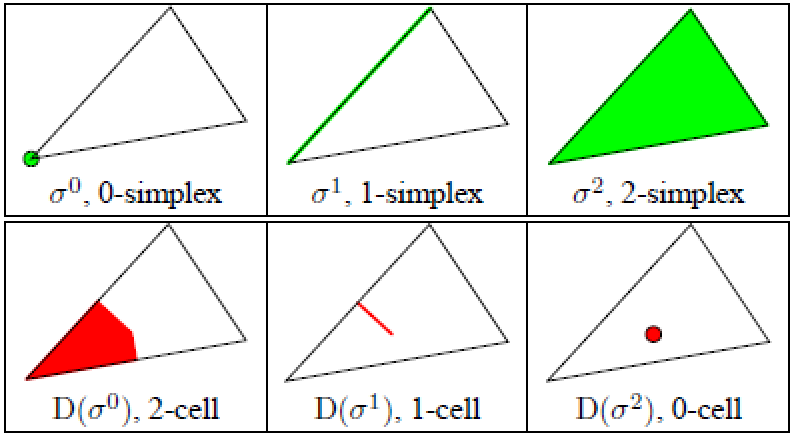
Given a discrete manifold, we define the *Voronoi dual complex* *$*K$* to be the (non-simplicial) dual complex obtained by associating simplices with their Voronoi duals.

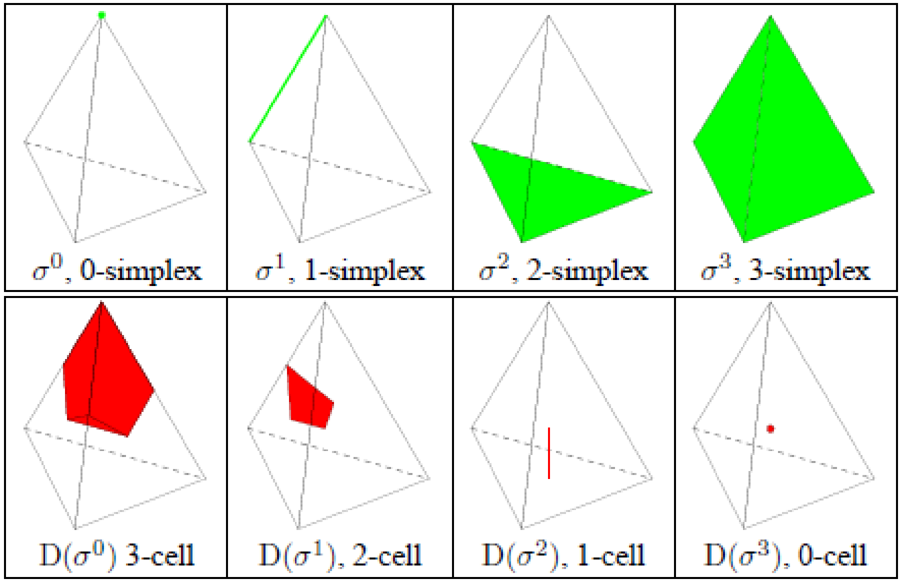


Dual Complexes

<u>Definition</u>:

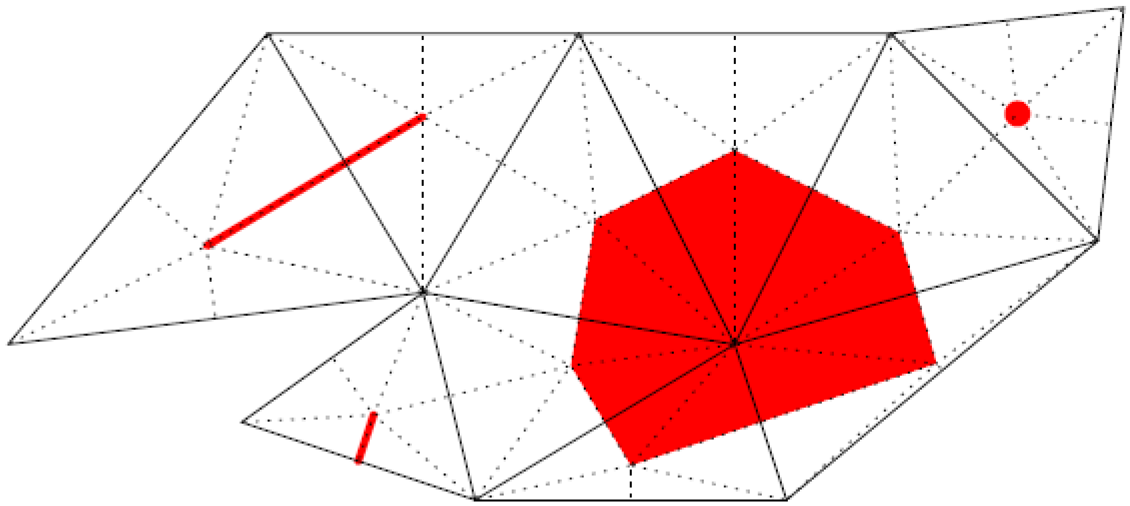
Given a discrete manifold, we define the *Voronoi dual complex* $*K$ to be the (non-simplicial) dual complex obtained by associating simplices with their Voronoi duals.





Dual Complexes

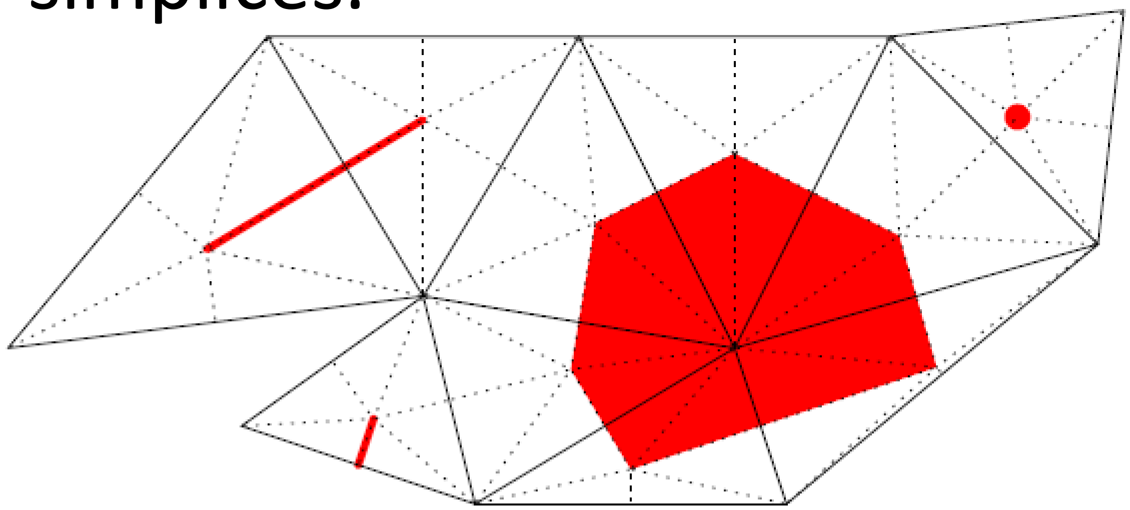
Using the dual complex, we can associate each (primal) *k*-simplex with a (dual) (*n*-*k*)-cell.



Dual Complexes

Using the dual complex, we can associate each (primal) *k*-simplex with a (dual) (*n*-*k*)-cell.

Furthermore, because we are using the Voronoi dual, the dual (*n*-*k*)-cells are orthogonal to the primal *k*-simplices.



Dual Complexes

Definition:

We denote by * the map that takes primal simplices to the corresponding (*n*-*k*)-cells:

$$* : \mathrm{K}^{(k)} \rightarrow *\mathrm{K}^{(n-k)}$$

Dual Complexes

Definition:

Using the association between primal simplices and dual cells, we define the discrete Hodge Star operator that takes *k*-cochains on the primal complex to (*n*-*k*)-cochains on the dual complex:

$$*:\Omega^k(\mathrm{K}) \rightarrow \Omega^{n-k}(*\mathrm{K})$$

Dual Complexes

<u>Definition</u>:

If ω is a *k*-cochain with value $\omega(\sigma)$ on σ, we treat ω as having a density of $\omega(\sigma)/|\sigma|$ per unit *k*-volume on σ (the "average" value of ω on σ).

Dual Complexes

Definition:

If ω is a *k*-cochain with value $\omega(\sigma)$ on σ, we treat ω as having a density of $\omega(\sigma)/|\sigma|$ per unit *k*-volume on σ (the "average" value of ω on σ).

Thus, we define $*\omega$ so it has the same density $\omega(\sigma)/|\sigma|$ per unit $(n-k)$-volume on $*\sigma$.

Dual Complexes

<u>Definition</u>:

If ω is a *k*-cochain with value $\omega(\sigma)$ on σ, we treat ω as having a density of $\omega(\sigma)/|\sigma|$ per unit *k*-volume on σ (the "average" value of ω on σ).

Thus, we define $*\omega$ so it has the same density $\omega(\sigma)/|\sigma|$ per unit (*n*-*k*)-volume on $*\sigma$.

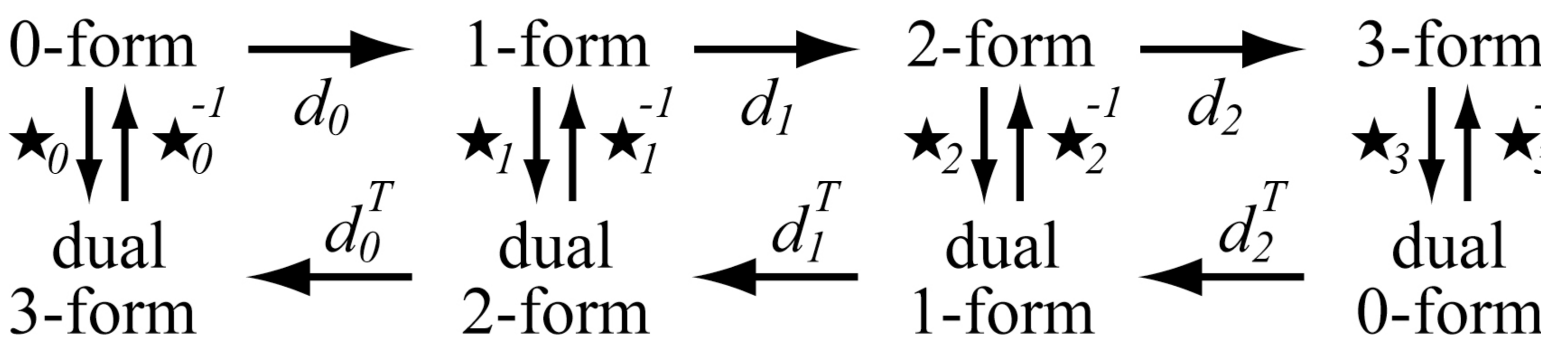
Or in other words:

$$*\omega(*\sigma) = \omega(\sigma)\frac{|*\sigma|}{|\sigma|}$$

Putting it All Together

<u>In 3D</u>:

– We have both primal and dual complexes.
– We have a boundary operator, and hence exterior derivative on both.
– We have *k*-cochains over both complexes.
– We have the Hodge Star operator to transition between the two.



Putting it All Together

Using this structure, we define the Laplacian:

1. Compute the gradient using the ext. derivative:

   primal 0-cochain $\rightarrow$ primal 1-cochain

Putting it All Together

Using this structure, we define the Laplacian:

1. Compute the gradient using the ext. derivative:
   primal 0-cochain $\rightarrow$ primal 1-cochain
2. Apply the Hodge star operator:
   primal 1-cochain $\rightarrow$ dual (n-1)-cochain

Putting it All Together

Using this structure, we define the Laplacian:

1. Compute the gradient using the ext. derivative:

   primal 0-cochain $\rightarrow$ primal 1-cochain

2. Apply the Hodge star operator:

   primal 1-cochain $\rightarrow$ dual (n-1)-cochain

3. Compute the divergence using the ext. derivative:

   dual (n-1)-cochain $\rightarrow$ dual n-cochain

Putting it All Together

Using this structure, we define the Laplacian:

1. Compute the gradient using the ext. derivative:
   primal 0-cochain $\rightarrow$ primal 1-cochain
2. Apply the Hodge star operator:
   primal 1-cochain $\rightarrow$ dual (n-1)-cochain
3. Compute the divergence using the ext. derivative:
   dual (n-1)-cochain $\rightarrow$ dual n-cochain
4. Apply the Hodge star operator:
   dual n-cochain $\rightarrow$ primal 0-cochain

$$\Delta := (-1)^{n(k-1)+1} * d * d$$

Putting it All Together

In Practice:

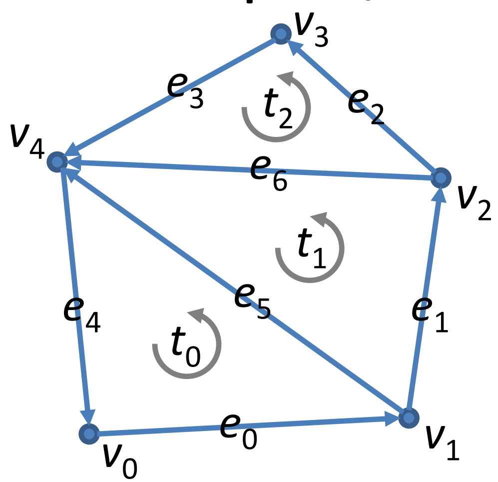
To work with DEC in practice, we only need to define two types of matrices:

Putting it All Together

<u>In Practice</u>:

To work with DEC in practice, we only need to define two types of matrices:

1. <u>The boundary operators ∂</u>:
   Specify the faces and orientations of each simplex/cell in the primal/dual complex.



$$\partial_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

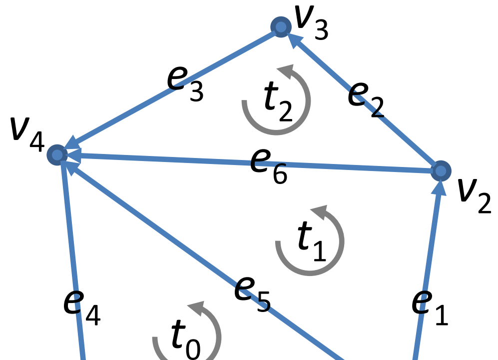
$$\partial_1 = \begin{pmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 1 \end{pmatrix}$$

Putting it All Together

<u>In Practice</u>:

To work with DEC in practice, we only need to define two types of matrices:

1. <u>The boundary operators ∂</u>:
   Specify the faces and orientations of each simplex/cell in the primal/dual complex.



$$\partial_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ \text{[illegible]} & \text{[illegible]} & \text{[illegible]} \end{pmatrix} \qquad \partial_1 = \begin{pmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ \text{[illegible]} & \text{[illegible]} & \text{[illegible]} & \text{[illegible]} & \text{[illegible]} & \text{[illegible]} & 1 \end{pmatrix}$$

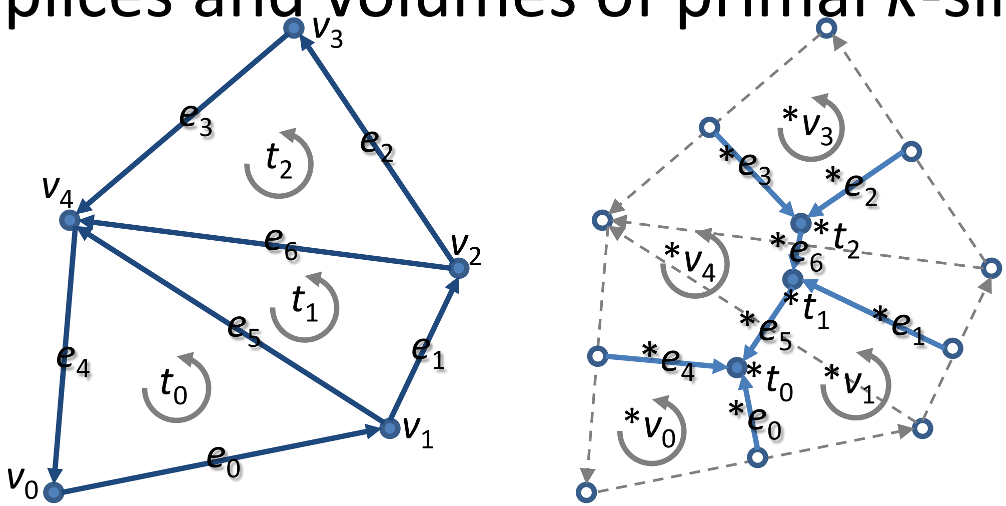
The exterior derivative is just the transpose matrix.

Putting it All Together

<u>In Practice</u>:

To work with DEC in practice, we only need to define two types of matrices:

2. <u>The discrete Hodge Star operators</u>:
   Specify the ratio of between volumes of dual (*n*-*k*)-simplices and volumes of primal *k*-simplices.

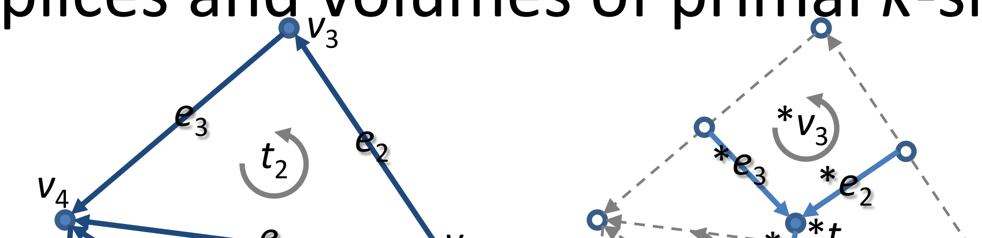


Putting it All Together

<u>In Practice</u>:

To work with DEC in practice, we only need to define two types of matrices:

2. <u>The discrete Hodge Star operators</u>:
   Specify the ratio of between volumes of dual (*n*-*k*)-simplices and volumes of primal *k*-simplices.

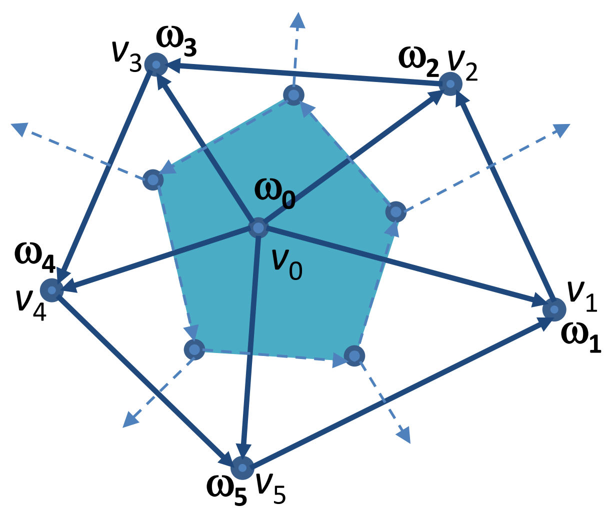


This is just a diagonal matrix whose entries are the ratios of the volumes of the dual (*n*-*k*)-cells and the volumes of the corresponding primal *k*-simplices.

Putting it All Together

Example (2D Laplacian):

Starting with a 0-cochain (function) ω, we evaluate the cochain to get a value ω_i at each vertex.



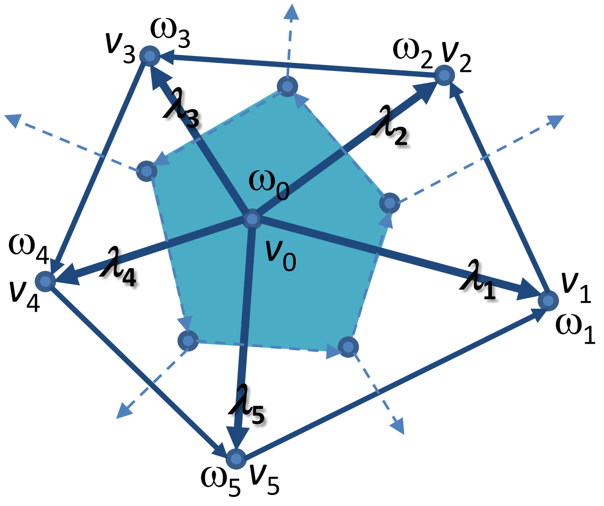
Putting it All Together

<u>Example (2D Laplacian)</u>:

Applying the exterior derivative, we get a 1-cochain that evaluates to:

$$\lambda_i = \omega_i - \omega_0$$

on primal edges from v_0 to v_i.



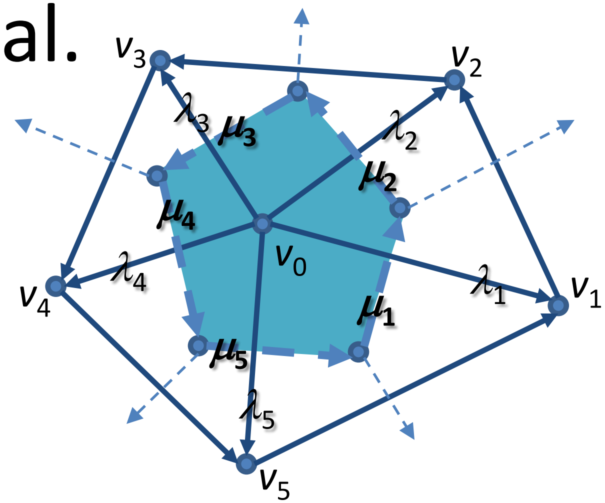
Putting it All Together

Example (2D Laplacian):

Applying the Hodge Star, we get a 1-cochain that evaluates to μ_i on corresponding dual edges:

$$\mu_i = \lambda_i \left(l_i^* / l_i \right)$$

where l_i is the length of the primal edge from v_0 to v_i, and l_i^* is the length of its dual.

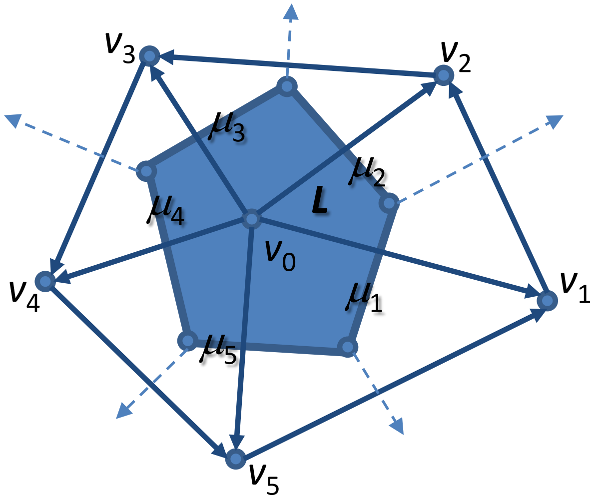


Putting it All Together

Example (2D Laplacian):

Applying the exterior derivative again, we get a 2-cochain that evaluates to L on the 2-cell dual to v_0:

$$L = \sum_i \mu_i$$



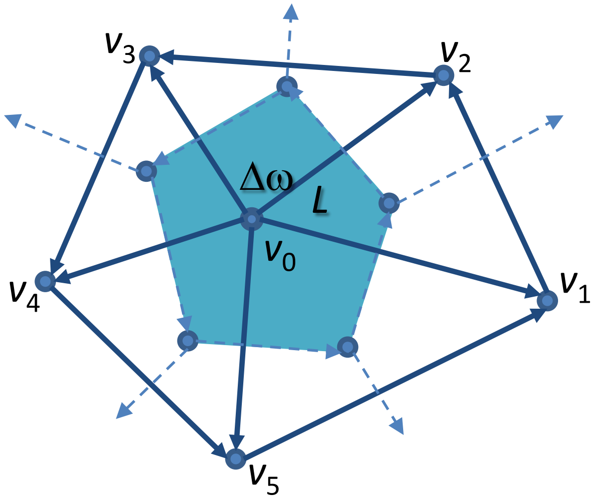
Putting it All Together

<u>Example (2D Laplacian)</u>:

Finally, applying the Hodge Star again, we get a 0-chain whose value at vertex v_0 is the Laplacian:

$$(\Delta\omega)(v_0) = L/A$$

where A is the area of the 2-cell dual to v_0.

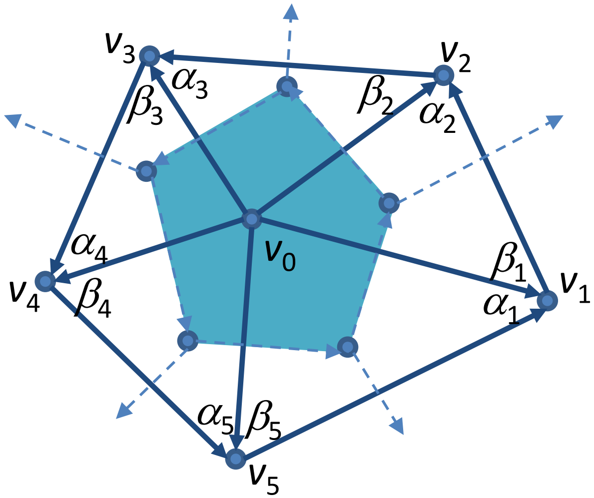


Putting it All Together

<u>Example (2D Laplacian)</u>:

$$\mu_i = \lambda_i\left(l_i^* / l_i\right)$$

If we denote by α_i and β_i the two angles adjacent to the primal edge from v_0 to v_i…

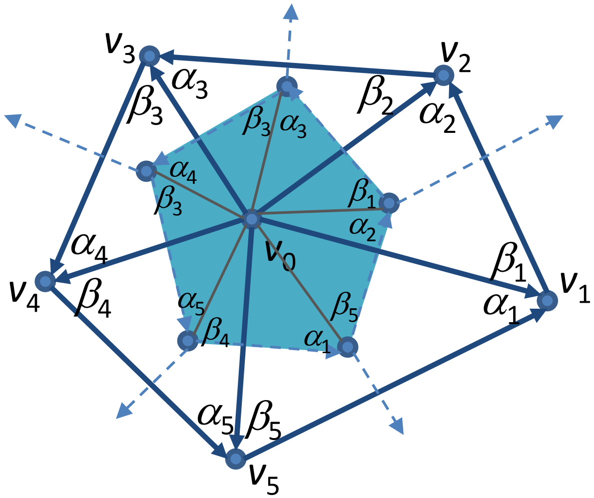


Putting it All Together

<u>Example (2D Laplacian)</u>:

$$\mu_i = \lambda_i \left(l_i^* / l_i \right)$$

If we denote by α_i and β_i the two angles adjacent to the primal edge from v_0 to v_i, the angles between the edges from v_0 to the dual vertices and the dual edges themselves become α_i and β_i.



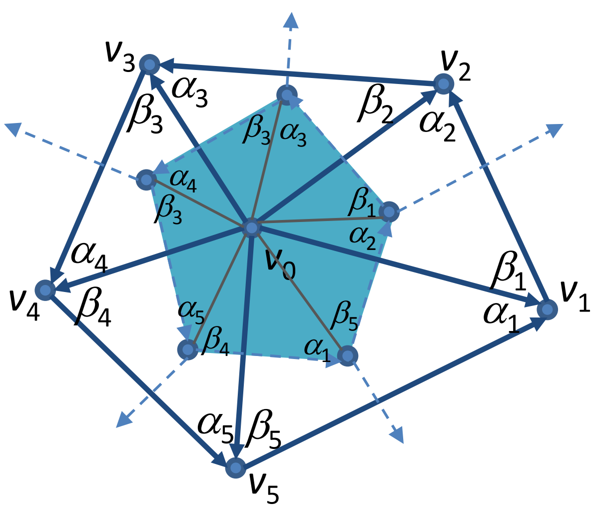
Putting it All Together

<u>Example (2D Laplacian)</u>:

$$\mu_i = \lambda_i\left(l_i^* / l_i\right)$$

Thus, for the edge from v_0 to v_i, we have:

$$l_i = 2\sin\beta_{i-1} = 2\sin\alpha_{i+1} \qquad l_i^* = \cos\beta_{i-1} + \cos\alpha_{i+1}$$



Putting it All Together

Example (2D Laplacian):

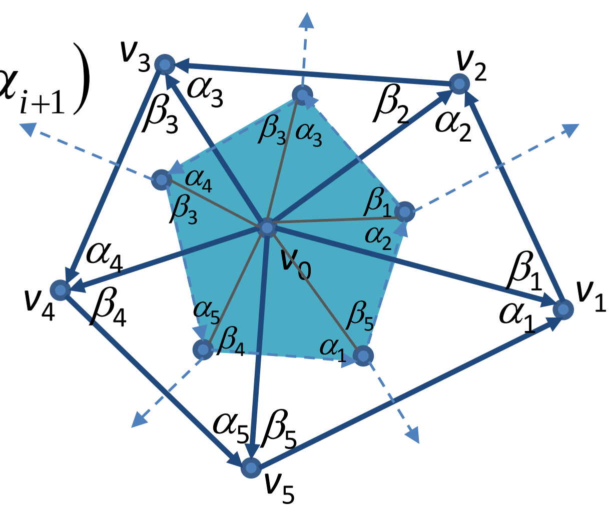
$$\mu_i = \lambda_i\left(l_i^* / l_i\right)$$

Thus, for the edge from v_0 to v_i, we have:

$$l_i = 2\sin\beta_{i-1} = 2\sin\alpha_{i+1} \qquad l_i^* = \cos\beta_{i-1} + \cos\alpha_{i+1}$$

and the ratio of dual to primal edge-lengths is:

$$\frac{l_i^*}{l_i} = \frac{1}{2}\left(\frac{\cos\beta_{i-1}}{\sin\beta_{i-1}} + \frac{\cos\alpha_{i+1}}{\sin\alpha_{i+1}}\right) = \frac{1}{2}\left(\cot\beta_{i-1} + \cot\alpha_{i+1}\right)$$



Thanks Barry.