Differential Geometry:
Discrete Exterior Calculus

[Discrete Differential Forms for Computational Modeling. Desbrun et al., 2005]
[Build Your Own DEC at Home. Elcott et al., 2006]
Simplices

Definition:
A $k$-simplex $\sigma$ is the non-degenerate convex hull of $k+1$ (ordered) geometrically distinct points.
Simplices

Definition:
A $k$-simplex $\sigma$ is the non-degenerate convex hull of $k+1$ (ordered) geometrically distinct points.

A $(k-1)$-face of a $k$-simplex $\sigma$ is a $(k-1)$-simplex spanned by $k$ of the $k+1$ vertices.
Simplices

**Definition:**

The *boundary* $\partial \sigma$ of a $k$-simplex $\sigma$ is the signed union of all $(k-1)$-faces.
Simplices

Definition:
The boundary $\partial \sigma$ of a $k$-simplex $\sigma$ is the signed union of all $(k-1)$-faces.

Formally, the boundary of $\sigma = \{v_0, ..., v_k\}$ is:

$$\partial \{v_0, ..., v_k\} = \sum_{j=0}^{k} (-1)^j \{v_0, ..., \hat{v}_j, ..., v_k\}$$
Simplices

Definition:
The *boundary* \( \partial \sigma \) of a \( k \)-simplex \( \sigma \) is the signed union of all \((k-1)\)-faces.

Note:
The boundary of a boundary is empty since each \((k-2)\) appears twice, with opposite signs:

\[ \partial \partial = \emptyset \]
Simplicial Complexes

Definition:
A simplicial complex $K$ is a collection of simplices satisfying the following conditions:

– Every face of each simplex $\sigma \in K$ is also in $K$.
– For any $\sigma_1, \sigma_2 \in K$, the intersection $\sigma_1 \cap \sigma_2$ is either empty or a common face.
Discrete Manifolds

Definition:
An $n$-dimensional discrete manifold $M$ is an $n$-dimensional simplicial complex with the property that:

- For each simplex $\sigma \in M$, the union of all incident $n$-simplices forms an $n$-dimensional (half-)ball.

A Discrete Manifold

Not a Discrete Manifold
Chains

**Definition:**

A *k-chain* of a simplicial complex $\mathcal{K}$ is linear combination of the $k$-simplices in $\mathcal{K}$:

$$c = \sum_{\sigma \in \mathcal{K}^k} c(\sigma) \cdot \sigma$$

where $c$ is a real-valued function. The space of $k$-chains is denoted $C^k(\mathcal{K})$. 
Chains

Definition:
A $k$-chain of a simplicial complex $K$ is linear combination of the $k$-simplices in $K$:

$$c = \sum_{\sigma \in K^k} c(\sigma) \cdot \sigma$$

where $c$ is a real-valued function.

The space of $k$-chains is denoted $C^k(K)$.

Informally, a $k$-chain is a weighted combination of $k$-dimensional volume elements.
Boundaries of Chains

Since $k$-chains are linear combinations of $k$-simplices and since we know how to compute the boundary of a $k$-simplex, we can extend the notion of the boundary operator to chains:

$$\partial c = \partial \left( \sum_{\sigma \in K^k} c(\sigma) \cdot \sigma \right) = \sum_{\sigma \in K^k} c(\sigma) \cdot \partial \sigma$$
Boundaries of Chains

Since $k$-chains are linear combinations of $k$-simplices and since we know how to compute the boundary of a $k$-simplex, we can extend the notion of the boundary operator to chains:

$$
\partial c = \partial \left( \sum_{\sigma \in K^k} c(\sigma) \cdot \sigma \right) = \sum_{\sigma \in K^k} c(\sigma) \cdot \partial \sigma
$$

That is, the boundary operator $\partial: C^k(K) \to C^{k-1}(K)$, takes weights assigned to $k$-simplices in $K$ and transforms them into weights on their $(k-1)$-faces.
Cochains

Definition:
The dual of a $k$-chain is a $k$-cochain which is a linear map taking a $k$-chain to a real value.
Cochains

**Definition:**
The dual of a $k$-chain is a $k$-cochain which is a linear map taking a $k$-chain to a real value. The space of $k$-cochains is denoted $\Omega^k (K)$. 
Cochains

Informally, we can think of chains and cochains as being related through integration:

- **Cochain**: What we would like to integrate.
- **Chain**: The domain over which we perform the integration.
Cochains

Informally, we can think of chains and cochains as being related through integration:

– **Cochain**: What we would like to integrate.
– **Chain**: The domain over which we perform the integration.

Given a $k$-chain and a $k$-cochain, we evaluate the cochain on the chain by integrating.

We denote the evaluation of $\omega \in \Omega^k(K)$ on $c \in C^k(K)$ by:

$$\int_c \omega = \int_{\sum c_i \sigma_i} \omega = \sum c_i \int_{\sigma_i} \omega_i$$
Discrete Exterior Derivative

Definition:
We define the exterior derivative $d: \Omega^k \rightarrow \Omega^{k+1}$ as the adjoint (complement) of the boundary operator.
Discrete Exterior Derivative

**Definition:**
We define the *exterior derivative* $d: \Omega^k \rightarrow \Omega^{k+1}$ as the adjoint (complement) of the boundary operator.

Given a $\omega \in \Omega^k(K)$ and $c \in C^{k+1}(K)$, we can evaluate $\omega$ on the boundary $\partial c$. 
Discrete Exterior Derivative

Definition:
We define the *exterior derivative* \( d: \Omega^k \to \Omega^{k+1} \) as the adjoint (complement) of the boundary operator.

Given a \( \omega \in \Omega^k(K) \) and \( c \in C^{k+1}(K) \), we can evaluate \( \omega \) on the boundary \( \partial c \).

The derivative \( d\omega \) is the \((k+1)\)-cochain whose value on \( c \) is equal to the value of \( \omega \) on \( \partial c \):

\[
\int_c d\omega = \int_{\partial c} \omega
\]
Discrete Exterior Derivative

Example 1:
Given the 1-form $\omega$ which, when evaluated on the edges of the complex, gives:

$$\int \omega = \begin{cases} 1 & \text{on } \{v_0, v_1\} \\ 2 & \text{on } \{v_1, v_2\} \\ -2 & \text{on } \{v_2, v_3\} \\ -1 & \text{on } \{v_3, v_4\} \\ 0 & \text{on } \{v_4, v_0\} \\ .5 & \text{on } \{v_1, v_4\} \\ .5 & \text{on } \{v_2, v_4\} \end{cases}$$

we need to compute the evaluations of the 2-form $d\omega$ with the property:

$$\int_{c} d\omega = \int_{\partial c} \omega$$
Example 1:
Consider the evaluation of $d\omega$ on the 2-simples $\{v_0, v_1, v_4\}$. In this case we must have:

$$\int_{\{v_0, v_1, v_4\}} d\omega = \int_{\partial\{v_0, v_1, v_4\}} \omega$$
Discrete Exterior Derivative

Example 1:
Consider the evaluation of $d\omega$ on the 2-simplices $\{v_0, v_1, v_4\}$. In this case we must have:

$$\int_{\{v_0, v_1, v_4\}} d\omega = \int_{\partial\{v_0, v_1, v_4\}} \omega = \int_{\{v_0, v_1\} + \{v_1, v_4\} + \{v_4, v_0\}} \omega$$
Discrete Exterior Derivative

Example 1:
Consider the evaluation of $d\omega$ on the 2-simples $\{v_0,v_1,v_4\}$. In this case we must have:

$$\int d\omega = \int \omega_{\partial \{v_0,v_1,v_4\}}$$

$$= \int \omega_{\{v_0,v_1\} + \{v_1,v_4\} + \{v_4,v_0\}}$$

$$= \int \omega_{\{v_0,v_1\}} + \int \omega_{\{v_1,v_4\}} + \int \omega_{\{v_4,v_0\}}$$
Example 1:
Consider the evaluation of $d\omega$ on the 2-simplices $\{v_0, v_1, v_4\}$. In this case we must have:

$$
\int d\omega = \int \omega_{v_0,v_1,v_4} + \int \omega_{v_0,v_1,v_4} + \int \omega_{v_4,v_0} = \int \omega_{v_0,v_1} + \int \omega_{v_1,v_4} + \int \omega_{v_4,v_0} = 1 + 0.5 + 0
$$
Example 1:
Similarly, on the 2-simplex \( \{v_1, v_2, v_4\} \) the evaluation of \( d\omega \) becomes:

\[
\int d\omega = \int \omega + \int \omega + \int \omega \\
\{v_1, v_2, v_4\} \quad \{v_1, v_2\} \quad \{v_2, v_4\} \quad \{v_4, v_1\} \\
= 2 + 0.5 - 0.5
\]
Example 1:
And on the 2-simplex \( \{v_2, v_3, v_4\} \) the evaluation of \( d\omega \) becomes:

\[
\int d\omega = -2 - 1 - 0.5
\]

\( \{v_2, v_3, v_4\} \)
Example 2:
Given the 0-form $\omega$ which, when evaluated on the vertices of the complex, gives:

\[
\begin{align*}
\int_{\{v_0\}} \omega &= 0 \\
\int_{\{v_1\}} \omega &= 1 \\
\int_{\{v_2\}} \omega &= 2 \\
\int_{\{v_3\}} \omega &= -2 \\
\int_{\{v_4\}} \omega &= -2
\end{align*}
\]

we can compute the evaluation of $d\omega$ on the edges in an analogous manner.
Example 2:
Given the 0-form $\omega$ which, when evaluated on the vertices of the complex, gives:

$$
\int_{\{v_0\}} \omega = 0 \quad \int_{\{v_1\}} \omega = 1 \quad \int_{\{v_2\}} \omega = 2 \quad \int_{\{v_3\}} \omega = -2 \quad \int_{\{v_4\}} \omega = -2
$$

we can compute the evaluation of $d\omega$ on the edges in an analogous manner:

$$
\int_{\{v_0, v_1\}} d\omega = \int_{\{v_1\}} \omega - \int_{\{v_0\}} \omega \quad \int_{\{v_1, v_2\}} d\omega = \int_{\{v_2\}} \omega - \int_{\{v_1\}} \omega \quad \ldots
$$
Discrete Exterior Derivative

Example 2:
Using the evaluation of $d\omega$ on the edges, we can compute the evaluation of $d(d\omega)$ on the triangles:

$$
\int d(d\omega) = \int d\omega + \int d\omega + \int d\omega = 0
$$

\[
\int_{\{v_0,v_1,v_4\}} d\omega + \int_{\{v_0,v_1\}} d\omega + \int_{\{v_1,v_4\}} d\omega + \int_{\{v_4,v_0\}} d\omega = 0
\]
Example 2:
Using the evaluation of $d\omega$ on the edges, we can compute the evaluation of $d(d\omega)$ on the triangles:

$$
\int d(d\omega) = \int d\omega + \int d\omega + \int d\omega = 0
$$

$$
\int d(d\omega) = \int d\omega + \int d\omega + \int d\omega = 0
$$
Example 2:
Using the evaluation of $d\omega$ on the edges, we can compute the
evaluation of $d(d\omega)$ on the triangles:

\[
\int d(d\omega) = \int d\omega + \int d\omega + \int d\omega = 0
\]
\[
\{v_0,v_1,v_4\} \quad \{v_0,v_1\} \quad \{v_1,v_4\} \quad \{v_4,v_0\}
\]

\[
\int d(d\omega) = \int d\omega + \int d\omega + \int d\omega = 0
\]
\[
\{v_1,v_2,v_4\} \quad \{v_1,v_2\} \quad \{v_2,v_4\} \quad \{v_4,v_1\}
\]

\[
\int d(d\omega) = \int d\omega + \int d\omega + \int d\omega = 0
\]
\[
\{v_2,v_3,v_4\} \quad \{v_2,v_3\} \quad \{v_3,v_4\} \quad \{v_4,v_2\}
\]
Discrete Exterior Derivative

**Note:**

By definition, for any \( \omega \in \Omega^k(K) \) and any \( c \in C^{k+2}(K) \), we must have:

\[
\int_c d(d\omega) = \int_{\partial c} d\omega = \int_{\partial^2 c} \omega
\]
Discrete Exterior Derivative

**Note:**

By definition, for any $\omega \in \Omega^k(K)$ and any $c \in C^{k+2}(K)$, we must have:

$$\int_c d(d\omega) = \int d\omega = \int \omega$$

But since the boundary of a boundary is always empty, this implies that:

$$\int_c d(d\omega) = 0$$
Discrete Exterior Derivative

**Note:**

By definition, for any $\omega \in \Omega^k(K)$ and any $c \in C^{k+2}(K)$, we must have:

$$
\int_c d(d\omega) = \int_{\partial c} d\omega = \int_{\partial\partial c} \omega
$$

But since the boundary of a boundary is always empty, this implies that:

$$
\int_c d(d\omega) = 0
$$

Since this is true for all $(k+2)$-chains, we have:

$$
d(d\omega) = 0
$$