Differential Geometry:

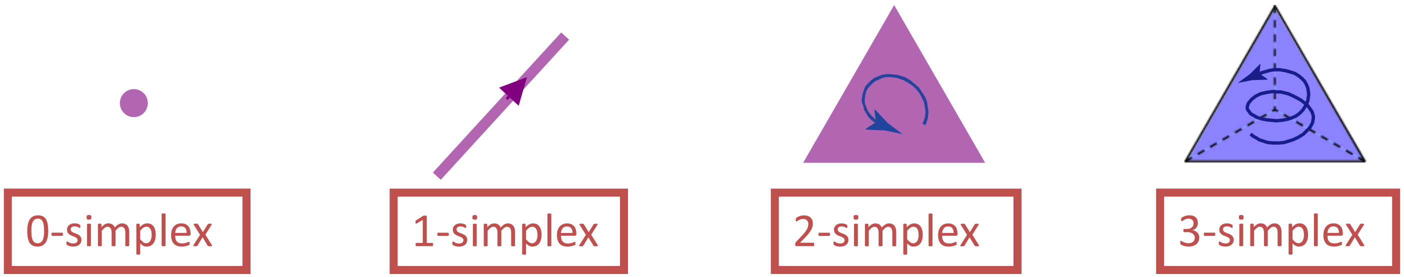
Discrete Exterior Calculus

[*Discrete Exterior Calculus* (Thesis). Hirani, 2003]
[*Discrete Differential Forms for Computational Modeling*. Desbrun *et al*., 2005]
[*Build Your Own DEC at Home*. Elcott *et al*., 2006]

Simplices

<u>Definition</u>:

A *k-simplex* σ is the non-degenerate convex hull of *k*+1 (ordered) geometrically distinct points.



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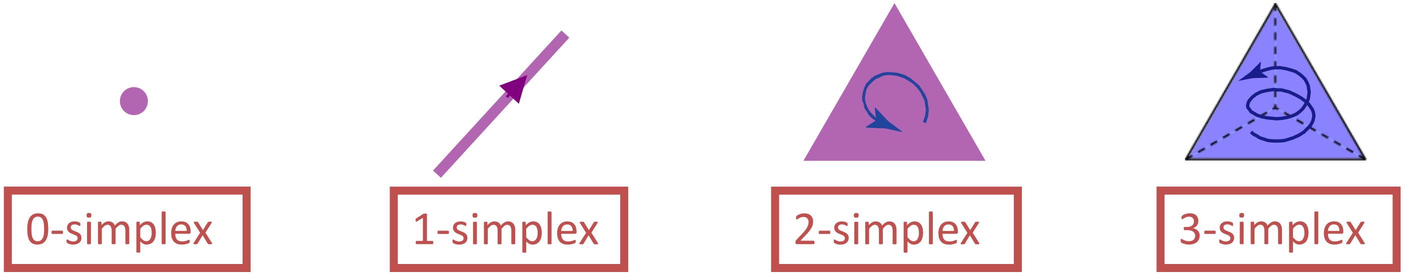

A (*k*-1)-*face* of a *k*-simplex σ is a (*k*-1)-simplex spanned by *k* of the *k*+1 vertices.

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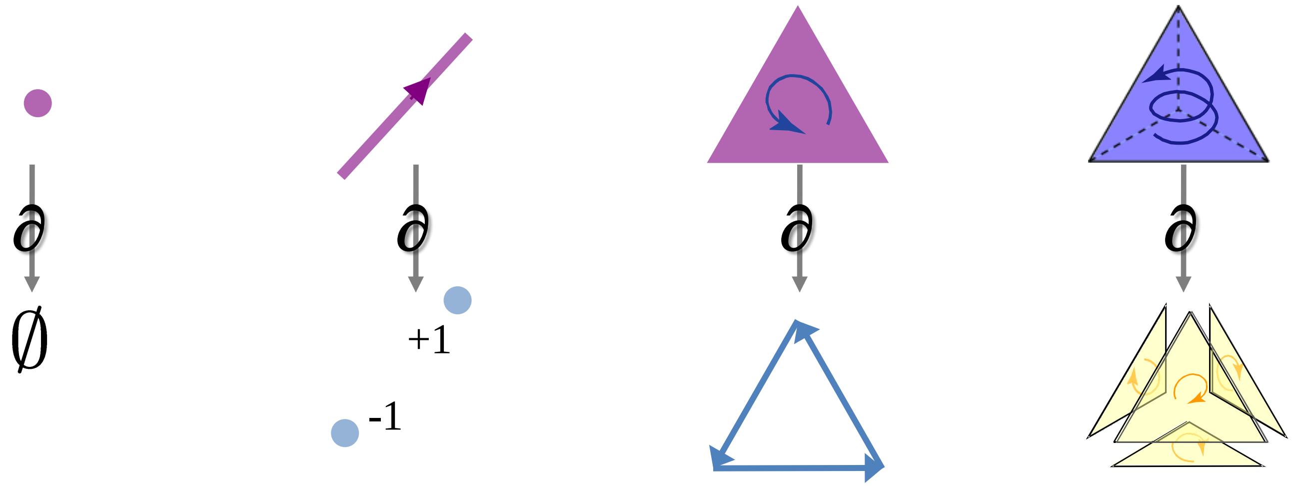
The *boundary* $\partial\sigma$ of a *k*-simplex σ is the signed union of all (*k*-1)-faces.



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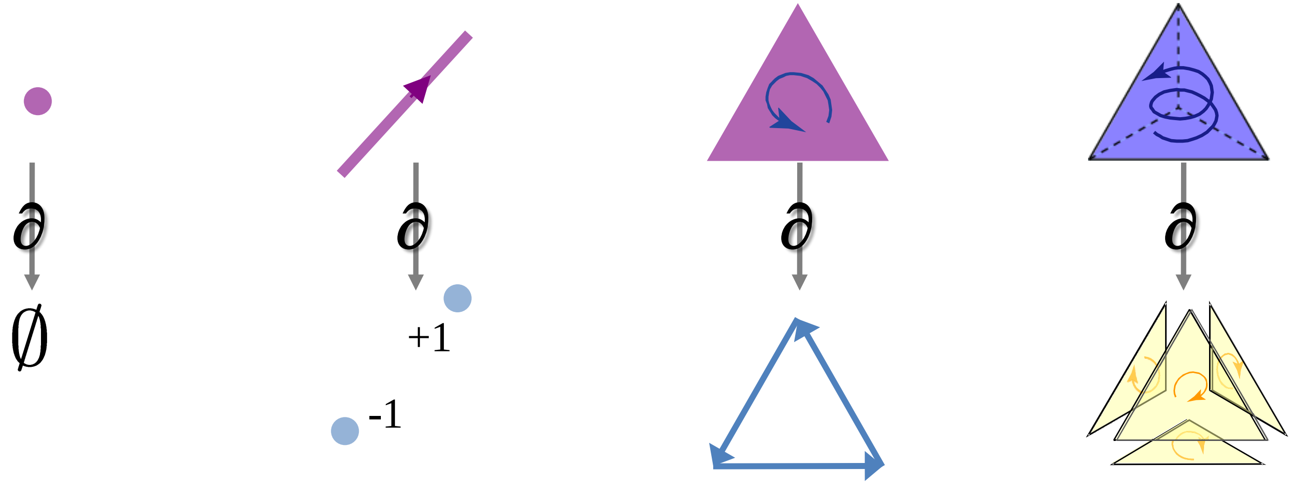

Formally, the boundary of $\sigma=\{v_0,\ldots,v_k\}$ is:

$$\partial\{v_0,\ldots,v_k\}=\sum_{j=0}^{k}(-1)^j\{v_0,\ldots,\hat{v}_j,\ldots,v_k\}$$

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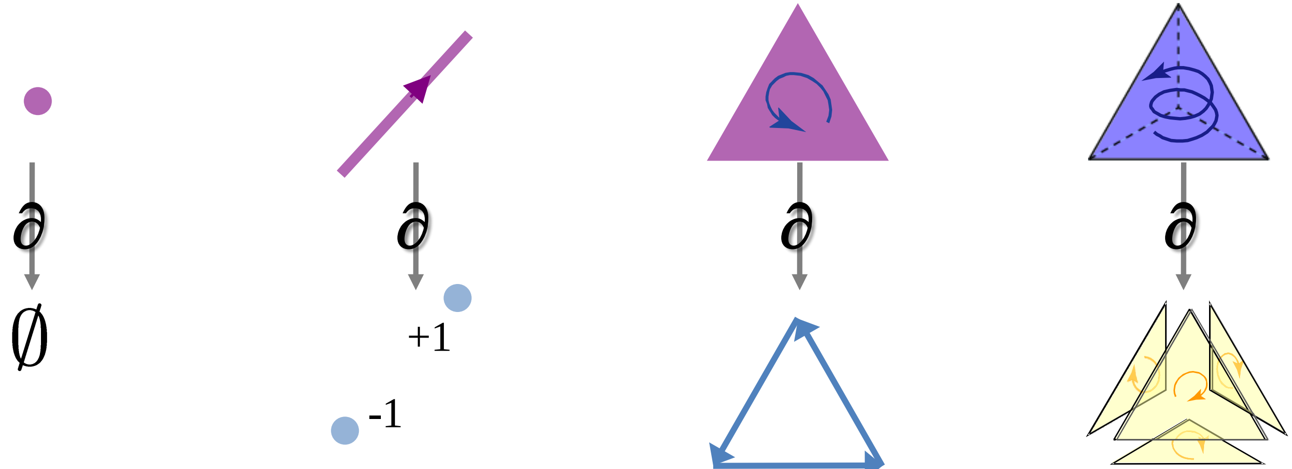


Note:

The boundary of a boundary is empty since each (*k*-2) appears twice, with opposite signs:

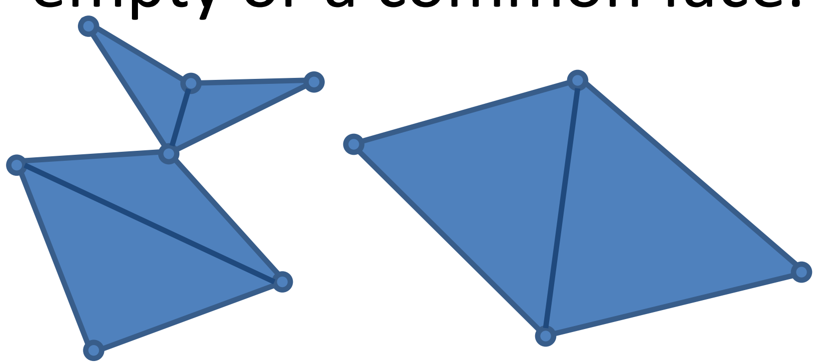
$$\partial\partial = \emptyset$$

Simplicial Complexes

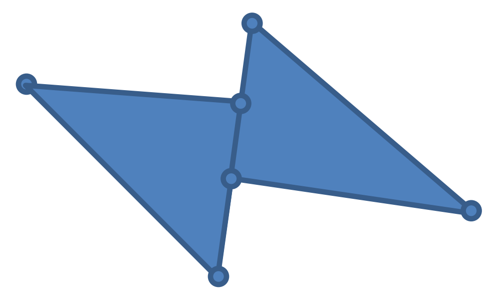
<u>Definition</u>:

A *simplicial complex* K is a collection of simplices satisfying the following conditions:

- Every face of each simplex $\sigma \in \mathrm{K}$ is also in K.
- For any σ_1, $\sigma_2 \in \mathrm{K}$, the intersection $\sigma_1 \cap \sigma_2$ is either empty or a common face.



Simplicial Complexes



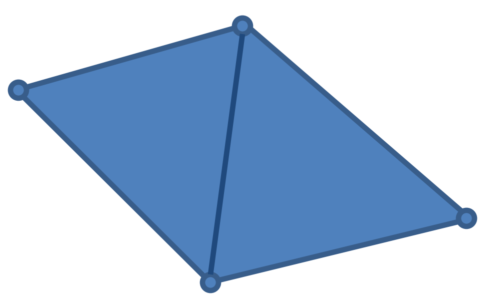
Not a Simplicial Complex

Discrete Manifolds

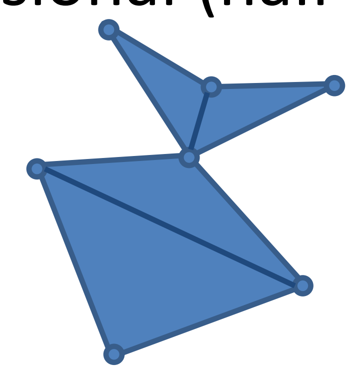
<u>Definition</u>:

An n-dimensional *discrete manifold M* is an n-dimensional simplicial complex with the property that:

- For each simplex $\sigma \in M$, the union of all incident n-simplices forms an n-dimensional (half-)ball.



A Discrete Manifold



Not a Discrete Manifold

Chains

<u>Definition</u>:

A *k-chain* of a simplicial complex K is linear combination of the *k*-simplices in K:

$$c = \sum_{\sigma \in \mathrm{K}^k} c(\sigma) \cdot \sigma$$

where *c* is a real-valued function.

The space of *k*-chains is denoted $C^k(\mathrm{K})$.

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The space of *k*-chains is denoted $C^k(\mathrm{K})$.

Informally, a *k*-chain is a weighted combination of *k*-dimensional volume elements.

Boundaries of Chains

Since *k*-chains are linear combinations of *k*-simplices and since we know how to compute the boundary of a *k*-simplex, we can extend the notion of the boundary operator to chains:

$$\partial c = \partial\left(\sum_{\sigma \in \mathrm{K}^k} c(\sigma)\cdot\sigma\right) = \sum_{\sigma \in \mathrm{K}^k} c(\sigma)\cdot\partial\sigma$$

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That is, the boundary operator $\partial : C^k(\mathrm{K}) \rightarrow C^{k-1}(\mathrm{K})$, takes weights assigned to *k*-simplices in K and transforms them into weights on their (*k*-1)-faces.

Cochains

Definition:

The dual of a *k*-chain is a *k-cochain* which is a linear map taking a *k*-chain to a real value.

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The space of *k*-cochains is denoted $\Omega^k(\mathrm{K})$.

Cochains

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- Cochain: What we would like to integrate.
- Chain: The domain over which we perform the integration.

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- Chain: The domain over which we perform the integration.

Given a *k*-chain and a *k*-cochain, we *evaluate* the cochain on the chain by integrating.

We denote the evaluation of $\omega \in \Omega^k(\mathrm{K})$ on $c \in C^k(\mathrm{K})$ by:

$$\int_c \omega = \int_{\sum_i c_i \sigma_i} \omega_i = \sum_i c_i \int_{\sigma_i} \omega_i$$

Discrete Exterior Derivative

Definition:

We define the *exterior derivative* d:$\Omega^k \rightarrow \Omega^{k+1}$ as the adjoint (complement) of the boundary operator.

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Given a $\omega \in \Omega^k(\mathrm{K})$ and $c \in C^{k+1}(\mathrm{K})$, we can evaluate ω on the boundary ∂c.

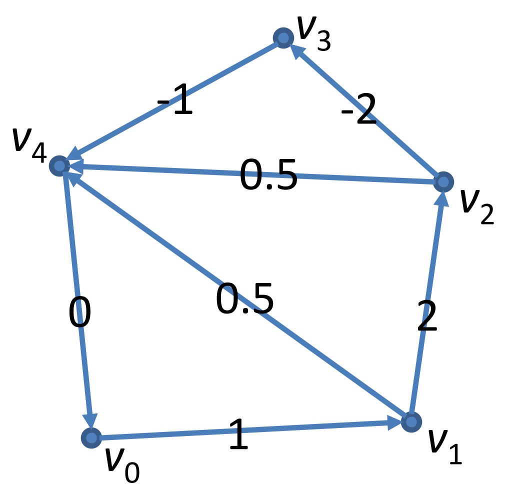
The derivative $d\omega$ is the (k+1)-cochain whose value on c is equal to the value of ω on ∂c:

$$\int_c d\omega = \int_{\partial c} \omega$$

Discrete Exterior Derivative

<u>Example 1</u>:

Given the 1-form ω which, when evaluated on the edges of the complex, gives:



$$\int_{\{v_0,v_1\}} \omega = 1 \quad \int_{\{v_1,v_2\}} \omega = 2 \quad \int_{\{v_2,v_3\}} \omega = -2 \quad \int_{\{v_3,v_4\}} \omega = -1$$

$$\int_{\{v_4,v_0\}} \omega = 0 \quad \int_{\{v_1,v_4\}} \omega = .5 \quad \int_{\{v_2,v_4\}} \omega = .5$$

we need to compute the evaluations of the 2-form $d\omega$ with the property:

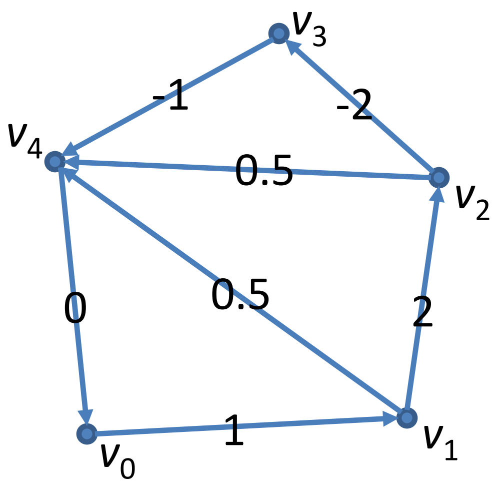
$$\int_c d\omega = \int_{\partial c} \omega$$

Discrete Exterior Derivative

<u>Example 1</u>:

Consider the evaluation of $d\omega$ on the 2-simples $\{v_0,v_1,v_4\}$. In this case we must have:

$$\int_{\{v_0,v_1,v_4\}} d\omega = \int_{\partial\{v_0,v_1,v_4\}} \omega$$



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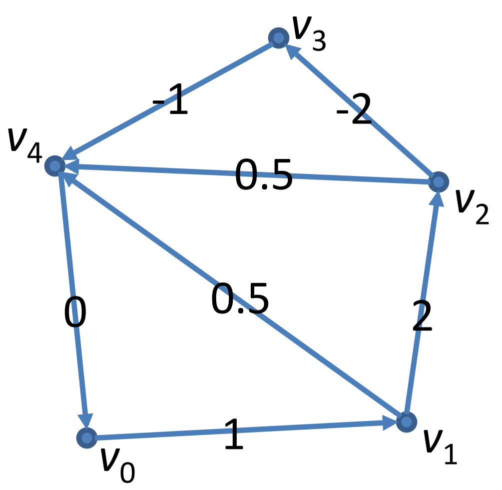

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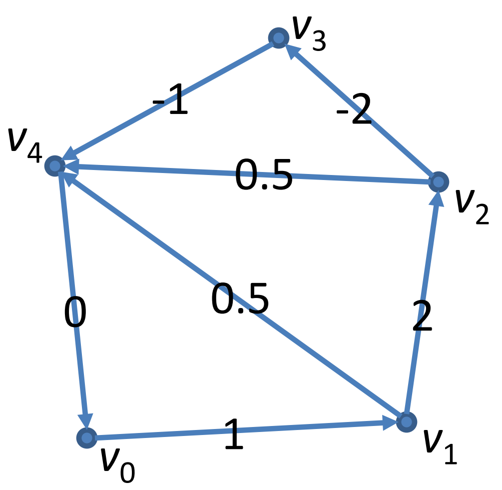

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$$= \int_{\{v_0,v_1\}} \omega + \int_{\{v_1,v_4\}} \omega + \int_{\{v_4,v_0\}} \omega$$

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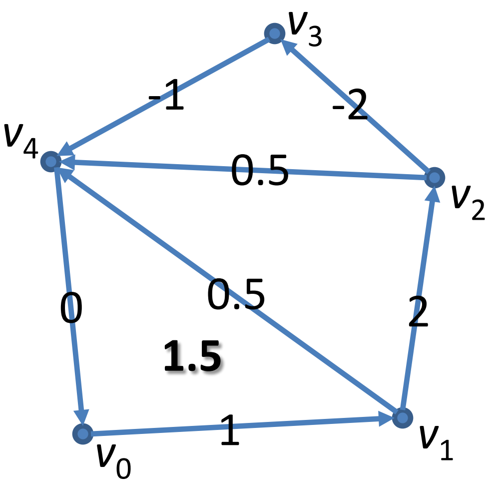

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$$= \int_{\{v_0,v_1\}} \omega + \int_{\{v_1,v_4\}} \omega + \int_{\{v_4,v_0\}} \omega$$

$$= 1 + 0.5 + 0$$

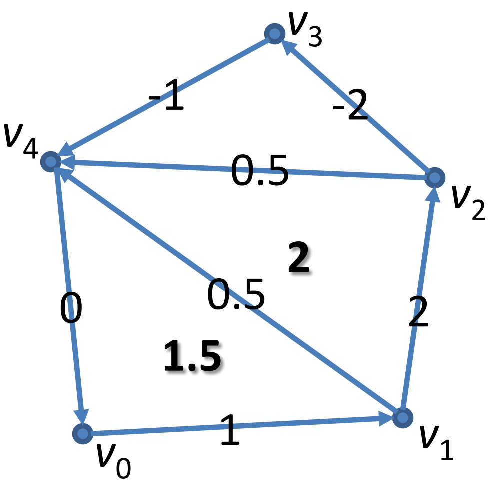
Discrete Exterior Derivative

Example 1:

Similarly, on the 2-simplex $\{v_1,v_2,v_4\}$ the evaluation of $d\omega$ becomes:

$$\int_{\{v_1,v_2,v_4\}} d\omega = \int_{\{v_1,v_2\}} \omega + \int_{\{v_2,v_4\}} \omega + \int_{\{v_4,v_1\}} \omega$$

$$= 2 + 0.5 - 0.5$$

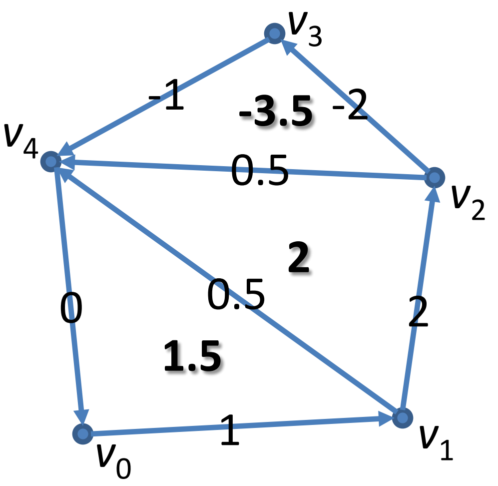


Discrete Exterior Derivative

Example 1:

And on the 2-simplex $\{v_2,v_3,v_4\}$ the evaluation of $d\omega$ becomes:

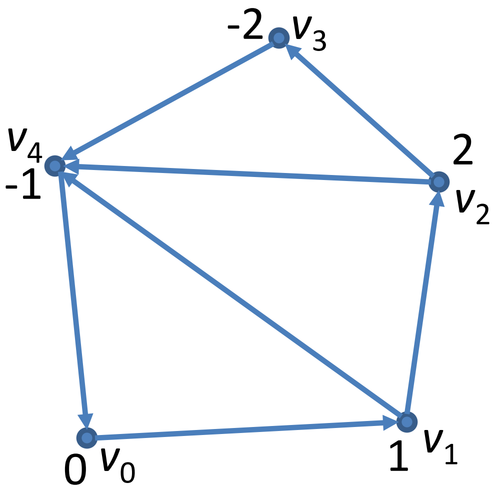
$$\int_{\{v_2,v_3,v_4\}} d\omega = -2-1-0.5$$



Discrete Exterior Derivative

Example 2:

Given the 0-form ω which, when evaluated on the vertices of the complex, gives:



$$\int_{\{v_0\}} \omega = 0 \quad \int_{\{v_1\}} \omega = 1 \quad \int_{\{v_2\}} \omega = 2 \quad \int_{\{v_3\}} \omega = -2 \quad \int_{\{v4\}} \omega = -2$$

we can compute the evaluation of *d*ω on the edges in an analogous manner.

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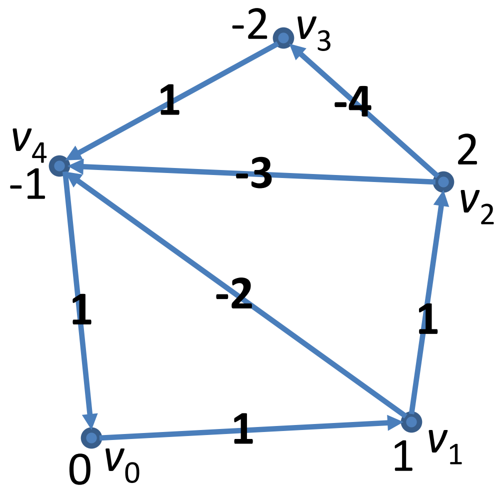

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$$\int_{\{v_0,v_1\}} d\omega = \int_{\{v_1\}} \omega - \int_{\{v_0\}} \omega \qquad \int_{\{v_1,v_2\}} d\omega = \int_{\{v_2\}} \omega - \int_{\{v_1\}} \omega \qquad \ldots$$

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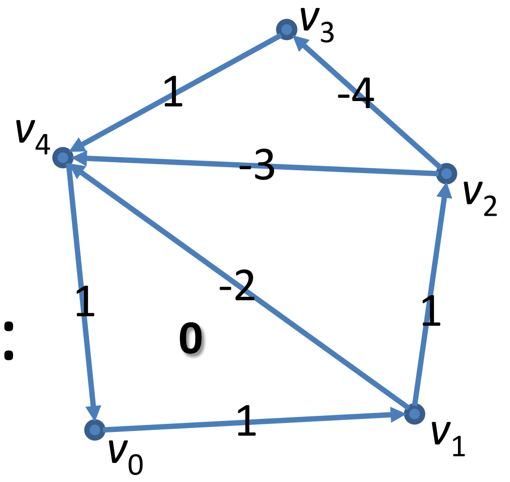
<u>Example 2</u>:

Using the evaluation of $d\omega$ on the edges, we can compute the evaluation of $d(d\omega)$ on the triangles:



$$\int_{\{v_0,v_1,v_4\}} d(d\omega) = \int_{\{v_0,v_1\}} d\omega + \int_{\{v_1,v_4\}} d\omega + \int_{\{v_4,v_0\}} d\omega = 0$$

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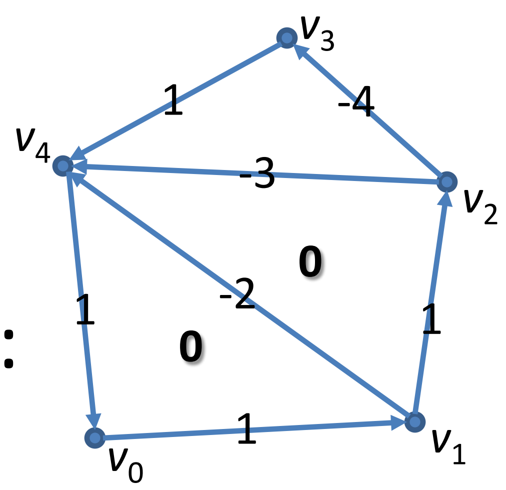

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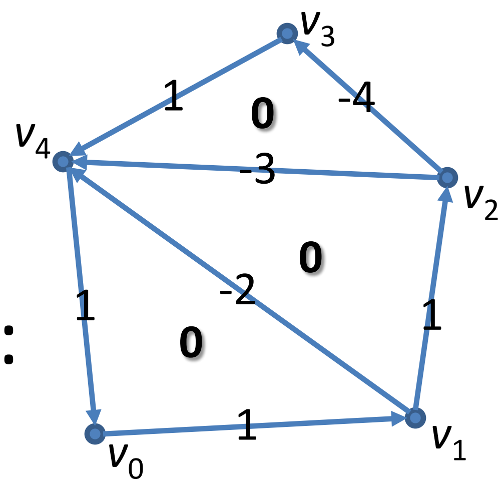

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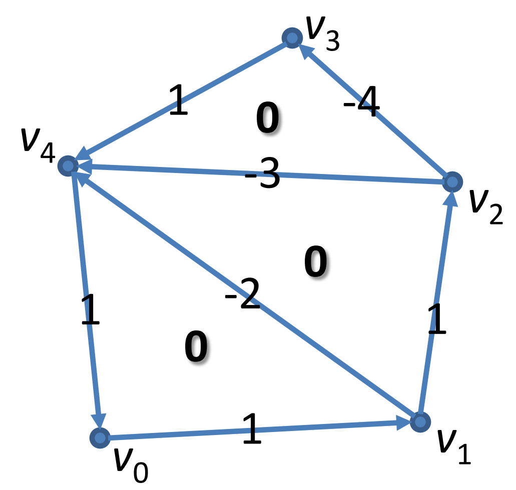
Discrete Exterior Derivative

Note:

By definition, for any $\omega \in \Omega^k(\mathrm{K})$ and any $c \in C^{k+2}(\mathrm{K})$, we must have:

$$\int_c d(d\omega) = \int_{\partial c} d\omega = \int_{\partial\partial c} \omega$$



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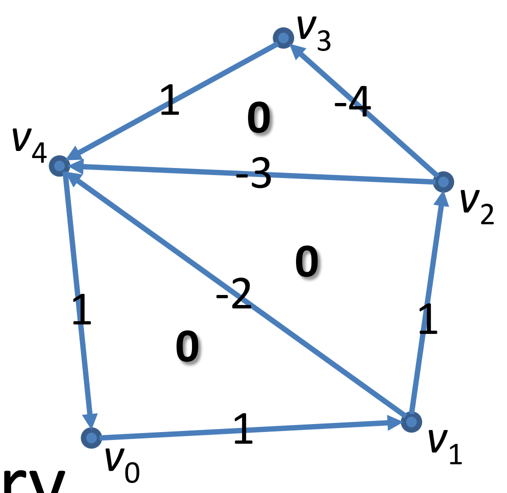

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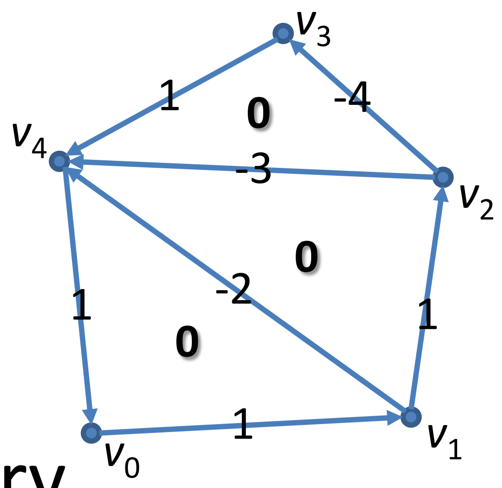

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Since this is true for all (k+2)-chains, we have:

$$d(d\omega) = 0$$