Differential Geometry:
Willmore Flow

[*Discrete Willmore Flow*. Bobenko and Schröder, 2005]

Quaternions

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Like the complex numbers, we can add quaternions together by summing the individual components:

$$\begin{array}{r} (a_1 + ib_1 + jc_1 + kd_1) \\ + \; (a_2 + ib_2 + jc_2 + kd_2) \\ \hline = (a_1 + a_2) + i(b_1 + b_2) + j(c_1 + c_2) + k(d_1 + d_2) \end{array}$$

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However, the multiplication rules are more complex:

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Note that multiplication of quaternions is not commutative.

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And, we define the <u>reciprocal</u> by dividing the conjugate by the square norm:

$$\frac{1}{q} = \frac{\bar{q}}{\|q\|^2}$$

Quaternions

One way to express a quaternion is as a pair consisting of the real value and the 3D vector consisting of the imaginary components:

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The advantage of this representation is that it is easier to express quaternion multiplication:

$$\begin{aligned} q_1 q_2 &= (\alpha_1, w_1)(\alpha_2, w_2) \\ &= \left(\alpha_1\alpha_2 - \langle w_1, w_2\rangle, \alpha_1 w_2 + \alpha_2 w_1 + w_1 \times w_2\right) \end{aligned}$$

Möbius Transformations

If we think of the plane as the set of complex numbers, any Möbius transformation can be expressed as a *fractional linear transformation*:

$$f(z) = \frac{az + b}{cz + d}$$

with $ad - bc \neq 0$.

Cross Ratio

A more useful description of a conformal map is in terms of the map:

$$S(z) = \frac{z - z_3}{z - z_4} \frac{z_2 - z_4}{z_2 - z_3}$$

for points z_2, z_3, z_4 in the complex plane.

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for points z_2, z_3, z_4 in the complex plane.
The map is a Möbius transformation and has the property that:

$$S(z_2) = 1 \qquad S(z_3) = 0 \qquad S(z_4) = \infty$$

Cross Ratio

<u>Claim</u>:

Every Möbius transformation can be written in this form.

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Proof:

Given a Möbius transformation *T*, we can define the cross-ratio:

$$S(z)=\frac{z-T^{-1}(0)}{z-T^{-1}(\infty)}\frac{T^{-1}(1)-T^{-1}(\infty)}{T^{-1}(1)-T^{-1}(0)}$$

which takes the same points to $(0,1,\infty)$ as *T*.

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$$S(z) = \frac{z - T^{-1}(0)}{z - T^{-1}(\infty)} \frac{T^{-1}(1) - T^{-1}(\infty)}{T^{-1}(1) - T^{-1}(0)}$$

But then ST^{-1} is a transformation that takes $(0,1,\infty)$ back to $(0,1,\infty)$.

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But if we consider the expression for the Möbius transformation:

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we see that the only way to have $(0,1,\infty)$ map back to itself is to have $f(z)$ be the identity.

Thus ST^{-1} is the identity so $S=T$.

Cross Ratio

<u>Definition</u>:

Given four (distinct) points in the complex plane $z_1, z_2, z_3, z_4 \in \boldsymbol{C}$, the *cross-ratio* is value:

$$\left(z_1, z_2, z_3, z_4\right) = \frac{z_1 - z_3}{z_1 - z_4} \frac{z_2 - z_4}{z_2 - z_3}$$

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That is, the cross-ratio is the image of z_1 under the Möbius transformation sending (z_2, z_3, z_4) to $(0, 1, \infty)$.

Cross Ratio

Claim 1:

The cross ratio of four points $z_1, z_2, z_3, z_4 \in \boldsymbol{C}$ is invariant under Möbius transformations:

$$\left(z_1, z_2, z_3, z_4\right) = \left(Tz_1, Tz_2, Tz_3, Tz_4\right)$$

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Proof:

If S is the Möbius transformation sending z to (z, z_2, z_3, z_4) then ST^{-1} takes (Tz_2, Tz_3, Tz_4) to $(0, 1, \infty)$.

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But that means that:

$$ST^{-1}(Tz_1) = \left(Tz_1, Tz_2, Tz_3, Tz_4\right)$$

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But that means that:

$$ST^{-1}(Tz_1) = (Tz_1, Tz_2, Tz_3, Tz_4)$$

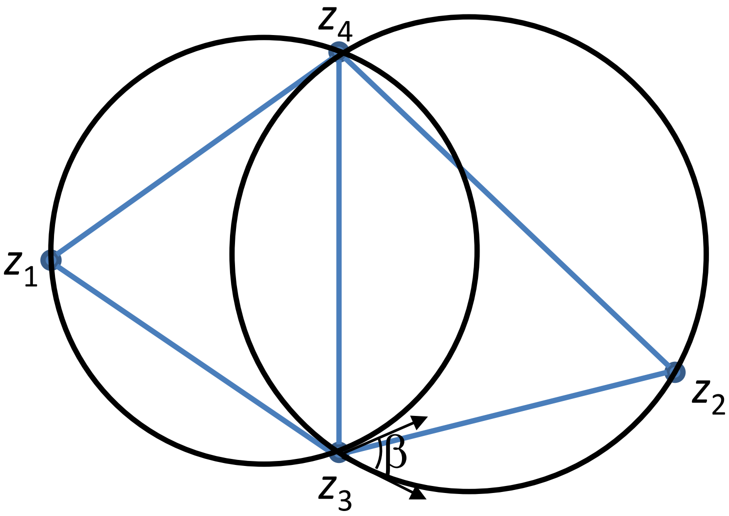
$$\Downarrow$$

$$(z_1, z_2, z_3, z_4) = (Tz_1, Tz_2, Tz_3, Tz_4)$$

Cross Ratio

Claim 2:

Given the four points $z_1, z_2, z_3, z_4 \in \boldsymbol{C}$, the angle of the cross ratio is π-β, where β is the angle between the circum-circles.



Cross Ratio

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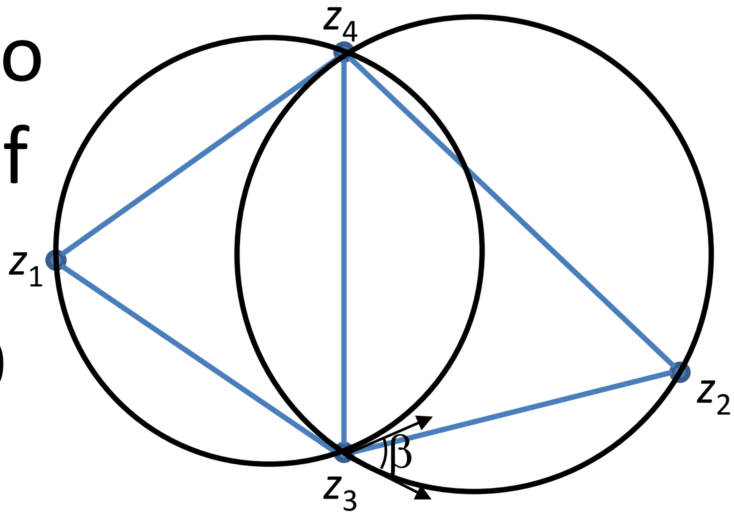
Given the four points z_1, z_2, z_3, $z_4 \in \boldsymbol{C}$, the angle of the cross ratio is π-β, where β is the angle between the circum-circles.

Proof:

The angle of the product of two complex numbers is the sum of their angles:



$$\arg(c_1 \cdot c_2) = \arg(c_1) + \arg(c_2)$$

$$r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

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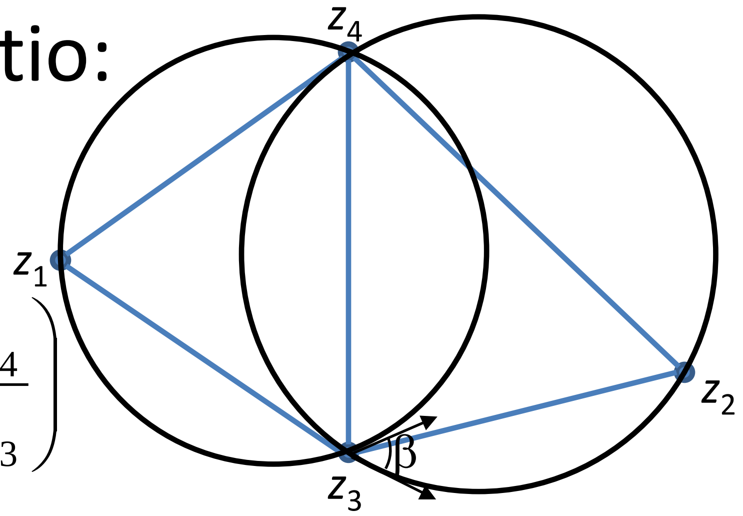
Proof:

Thus, the angle of the cross-ratio:

$$(z_1, z_2, z_3, z_4) = \frac{z_1 - z_3}{z_1 - z_4} \frac{z_2 - z_4}{z_2 - z_3}$$

is the sum:

$$\arg(z_1, z_2, z_3, z_4) = \arg\left(\frac{z_1 - z_3}{z_1 - z_4}\right) + \arg\left(\frac{z_2 - z_4}{z_2 - z_3}\right)$$



Cross Ratio

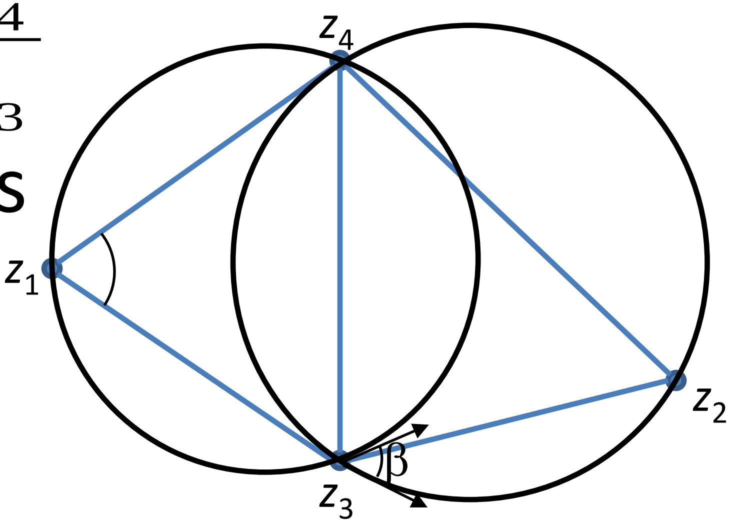
<u>Claim 2</u>:

Given the four points z_1, z_2, z_3, $z_4 \in \boldsymbol{C}$, the angle of the cross ratio is π-β, where β is the angle between the circum-circles.

<u>Proof</u>:

$$(z_1, z_2, z_3, z_4) = \frac{z_1 - z_3}{z_1 - z_4} \frac{z_2 - z_4}{z_2 - z_3}$$

But the angle of $(z_1-z_3)/(z_1-z_4)$ is the difference between the angles of (z_3-z_1) and (z_4-z_1).



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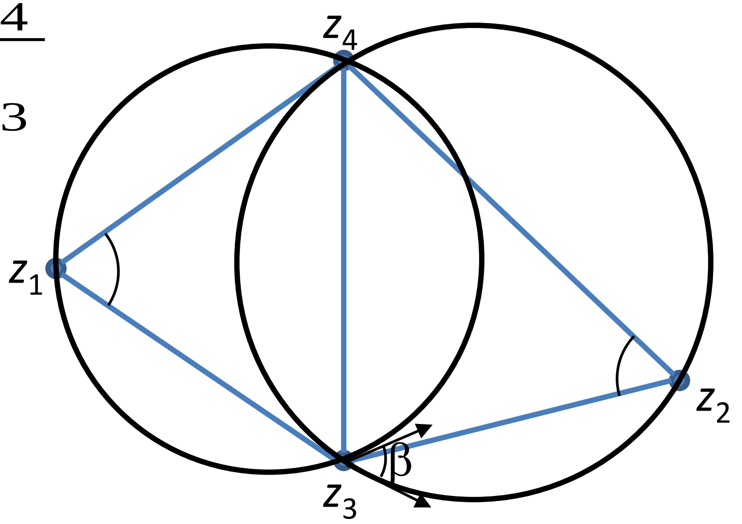
Proof:

$$(z_1, z_2, z_3, z_4) = \frac{z_1 - z_3}{z_1 - z_4} \frac{z_2 - z_4}{z_2 - z_3}$$

Similarly, we know that the angle of $(z_2$-$z_4)/(z_2$-$z_3)$ is…



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Proof:

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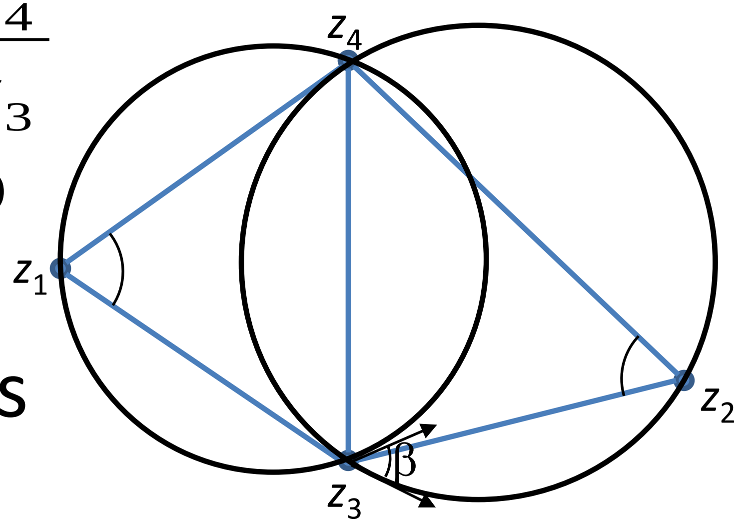

So, the angle of the cross-ratio is the sum of the angles that are opposite the edge, which is exactly π-β.

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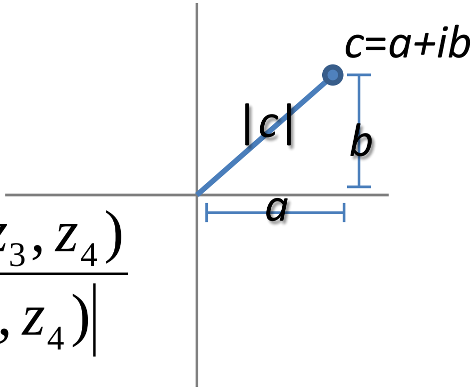
<u>Note</u>:

For a complex number $c \in \mathbf{C}$, the cosine of the angle of c is just the real part of c divided by its length, so we have:



$$\cos(\pi-\beta)=\frac{\mathrm{Re}(z_1,z_2,z_3,z_4)}{\left|(z_1,z_2,z_3,z_4)\right|} \iff \cos\beta=-\frac{\mathrm{Re}(z_1,z_2,z_3,z_4)}{\left|(z_1,z_2,z_3,z_4)\right|}$$

“Extended” Cross Ratio

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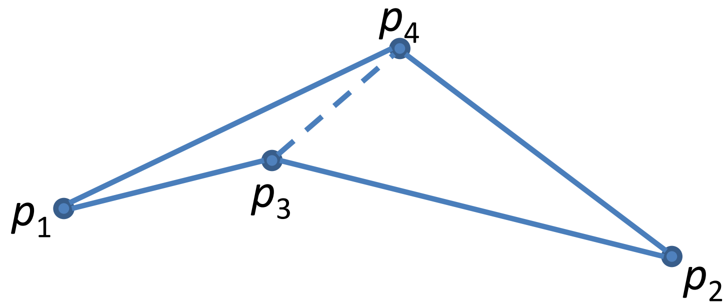
Specifically, given a point $p=(x,y,z)\in\mathbf{R}^3$, we can associate it with the quaternion:

$$q_p = ix + jy + kz$$

“Extended” Cross Ratio

Then, given four points p_1, p_2, p_3, $p_4 \in \mathbf{R}^3$ associated with four imaginary quaternions $q_1, q_2, q_3, q_4 \in \mathbf{Q}$, we can define the quaternionic cross-ratio:

$$(q_1, q_2, q_3, q_4) = (q_1 - q_3)\frac{1}{(q_1 - q_4)}(q_2 - q_4)\frac{1}{(q_2 - q_3)}$$



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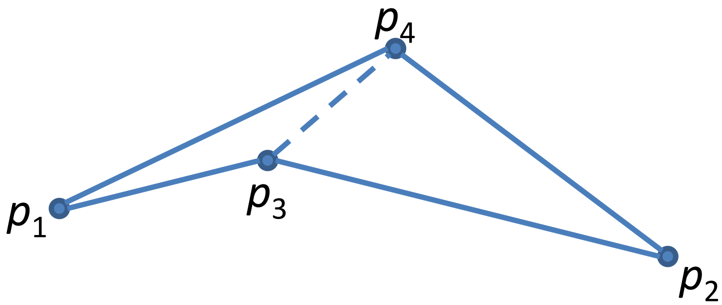
While it is <u>not</u> true that the quaternionic cross-product is Möbius-invariant, the real part and the norm of the cross-product are.



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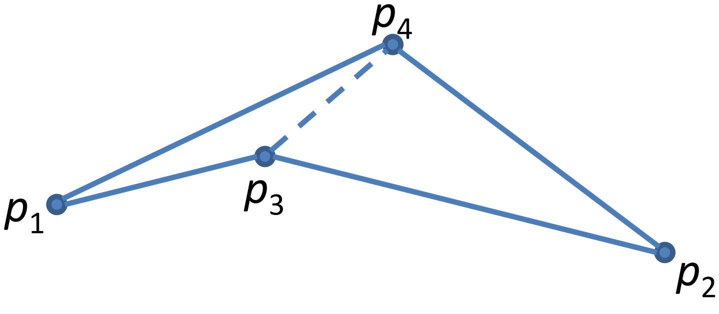
While it is <u>not</u> true that the quaternionic cross-product is Möbius-invariant, the real part and the norm of the cross-product are.

Thus, the ratio of the real part to the norm is also Möbius-invariant.

$$\frac{\mathrm{Re}(q_1, q_2, q_3, q_4)}{\left|(q_1, q_2, q_3, q_4)\right|}$$



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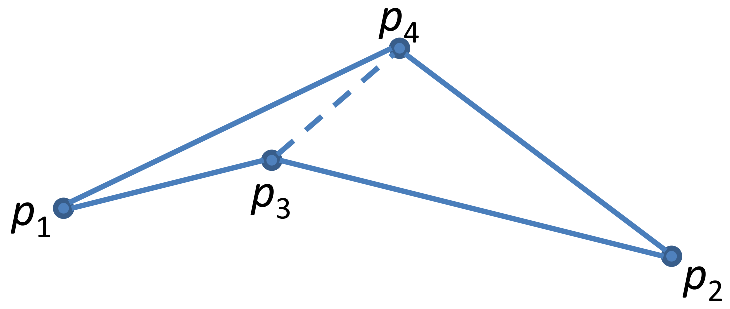
Since we can fit a sphere to the four points, we can find a Möbius transformation that takes a point on the sphere (not one of the four) to ∞.



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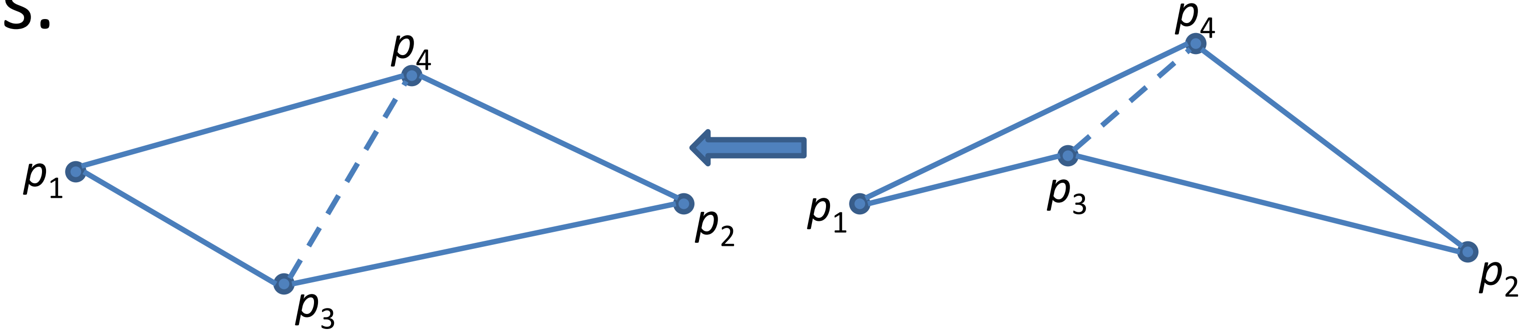
Since we can fit a sphere to the four points, we can find a Möbius transformation that takes a point on the sphere (not one of the four) to ∞.

This would map the sphere to a plane, so that the two triangles in 3D would become planar triangles.



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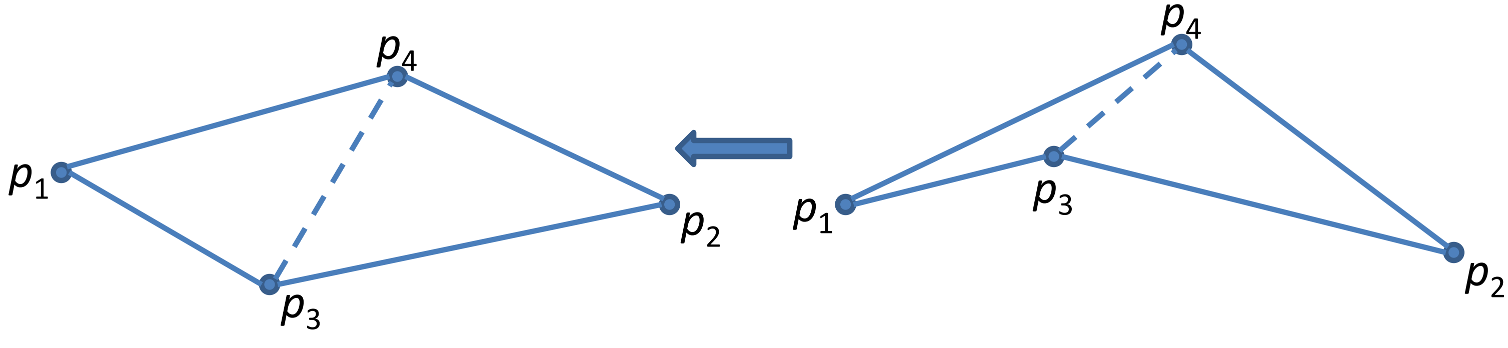
In the planar case, the real-part of the quaternionic cross-product equals the real part of the complex cross-product, and the norm of the quaternionic cross-product equals the norm of the complex cross-product.



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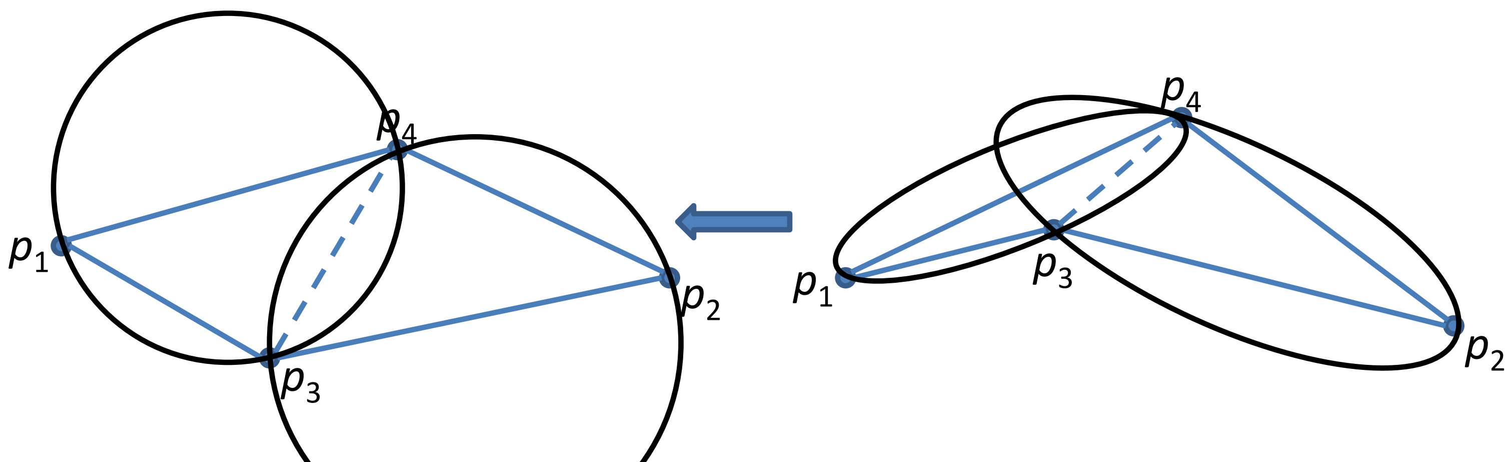
On the other hand, since the map was conformal it had to preserve the angles between the circum-circles.



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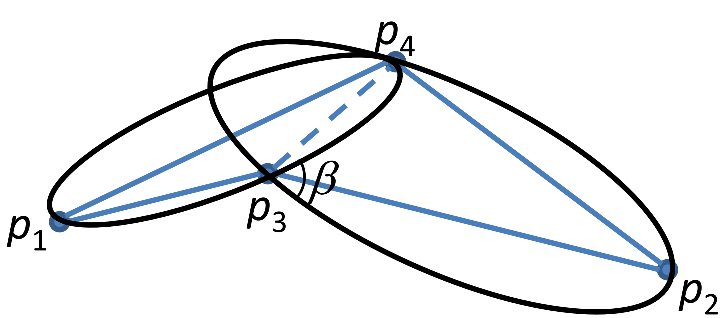
On the other hand, since the map was conformal it had to preserve the angles between the circum-circles.

Thus, we can express the intersection angle between the circum-circles as:

$$\cos\beta = -\frac{\mathrm{Re}(q_1, q_2, q_3, q_4)}{|(q_1, q_2, q_3, q_4)|}$$



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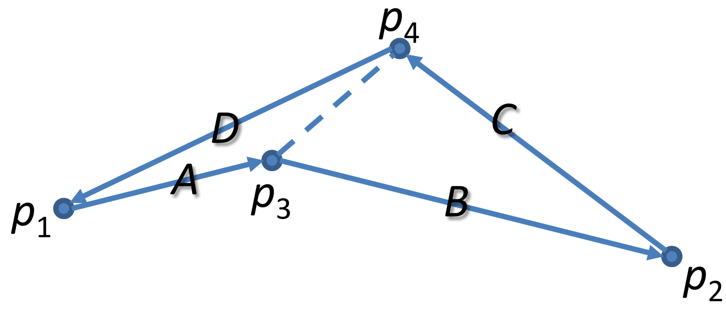
$$(q_1,q_2,q_3,q_4)=(q_1-q_3)\frac{1}{(q_1-q_4)}(q_2-q_4)\frac{1}{(q_2-q_3)}$$

If we set *A*, *B*, *C*, *D* to be the difference vectors and *a*=*A*/|*A*|, *b*=*B*/|*B*|, *c*=*C*/|*C*|, *d*=*D*/|*D*| to be the normalized difference vectors, we get:

$$\cos\beta=-\frac{\mathrm{Re}(q_1,q_2,q_3,q_4)}{|(q_1,q_2,q_3,q_4)|}=-\frac{\mathrm{Re}\left(AD^{-1}CB^{-1}\right)}{\left|AD^{-1}CB^{-1}\right|}$$



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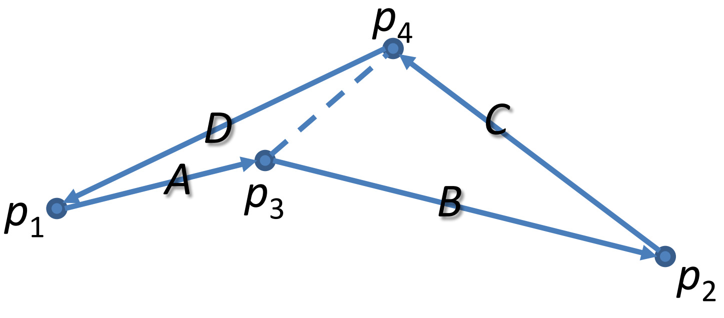

“Extended” Cross Ratio

$$(q_1,q_2,q_3,q_4)=(q_1-q_3)\frac{1}{(q_1-q_4)}(q_2-q_4)\frac{1}{(q_2-q_3)}$$

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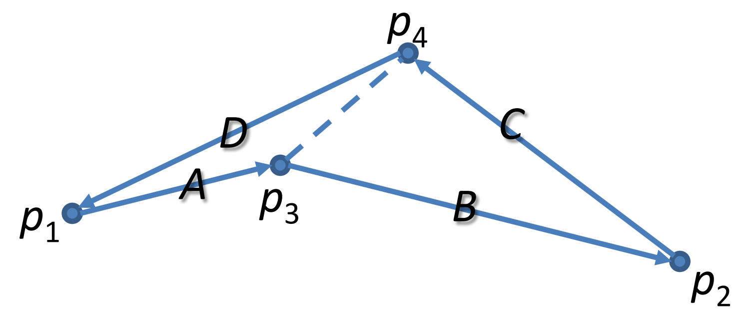
$$=-\mathrm{Re}(ad^{-1}cb^{-1})$$

$$=-\mathrm{Re}(a\overline{d}c\overline{b})$$



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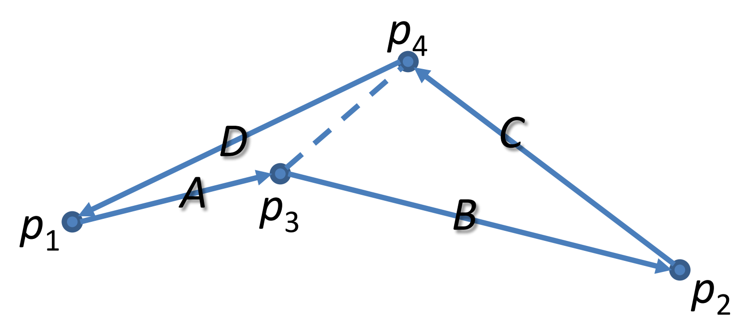
$$= -\mathrm{Re}\left(ad^{-1}cb^{-1}\right)$$

$$= -\mathrm{Re}\left(a\overline{d}c\overline{b}\right)$$

$$= -\mathrm{Re}(adcb)$$



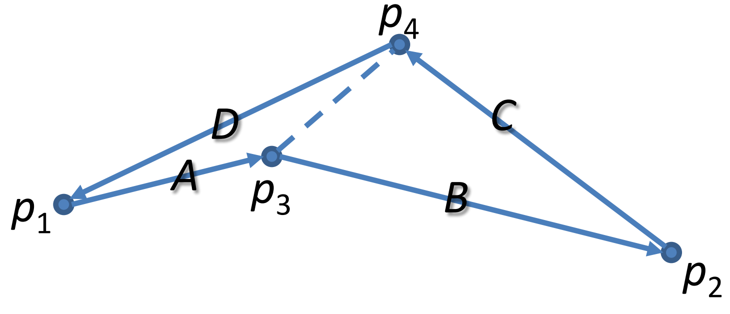
“Extended” Cross Ratio

Putting it all together, we get:

$$\cos\beta = -\langle a,b\rangle\langle c,d\rangle - \langle b,c\rangle\langle d,a\rangle + \langle a,c\rangle\langle b,d\rangle$$

$$a = \frac{A}{|A|} \quad b = \frac{B}{|B|}$$

$$c = \frac{C}{|C|} \quad d = \frac{D}{|D|}$$



Willmore Energy

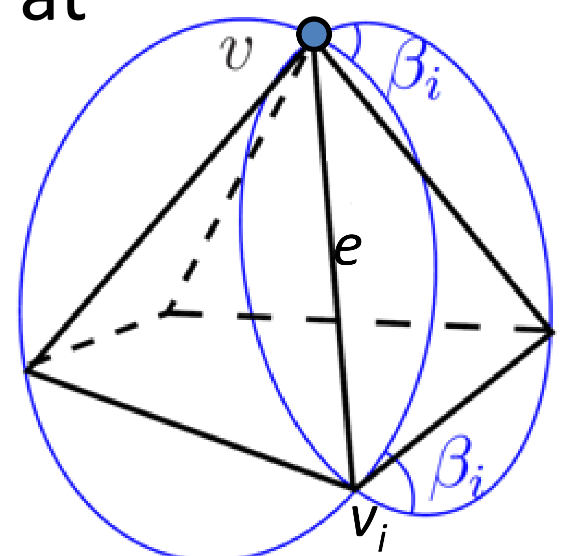
Putting it all together, we get:

$$\cos\beta = -\langle a,b\rangle\langle c,d\rangle - \langle b,c\rangle\langle d,a\rangle + \langle a,c\rangle\langle b,d\rangle$$

Thus, we get a closed-form expression for the component of the Willmore energy at vertex *v* that comes from edge *e*:

$$E(v,e) = \cos^{-1}\left(-\frac{\mathrm{Re}(Q)}{|Q|}\right)$$

where *Q* is the cross-ratio.



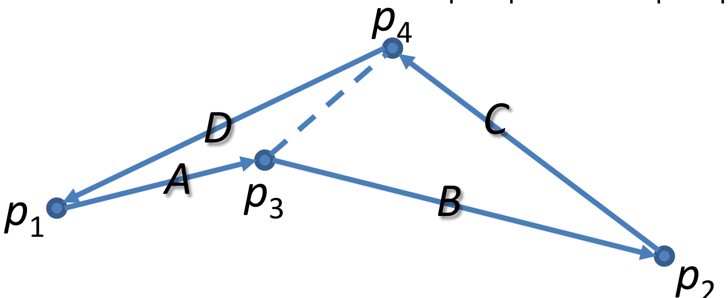
Willmore Energy

$$\cos\beta = -\langle a,b\rangle\langle c,d\rangle - \langle b,c\rangle\langle d,a\rangle + \langle a,c\rangle\langle b,d\rangle$$

In order to flow the surface to minimize the Willmore energy, we need to compute the gradient of the energy.

$$a = \frac{A}{|A|} \quad b = \frac{B}{|B|}$$

$$c = \frac{C}{|C|} \quad d = \frac{D}{|D|}$$



Willmore Energy

$$\cos\beta = -\langle a,b\rangle\langle c,d\rangle - \langle b,c\rangle\langle d,a\rangle + \langle a,c\rangle\langle b,d\rangle$$

In order to flow the surface to minimize the Willmore energy, we need to compute the gradient of the energy.

Taking the gradient with respect to p_1, we get:

$$\frac{\partial\cos\beta}{\partial p_1} = -\frac{\partial\langle a,b\rangle\langle c,d\rangle}{\partial p_1} - \frac{\partial\langle b,c\rangle\langle d,a\rangle}{\partial p_1} + \frac{\partial\langle a,c\rangle\langle b,d\rangle}{\partial p_1}$$

$$a = \frac{A}{|A|} \qquad b = \frac{B}{|B|}$$

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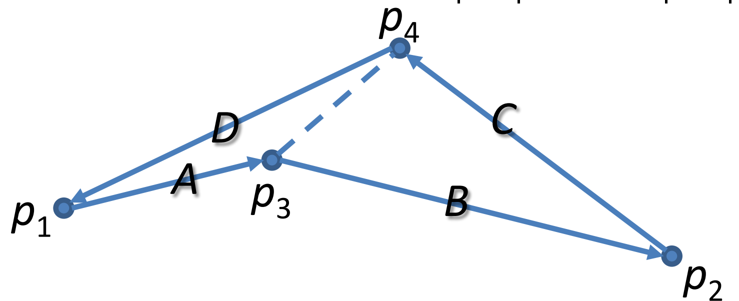


Willmore Energy

$$\frac{\partial \cos\beta}{\partial p_1} = -\frac{\partial \langle a,b\rangle\langle c,d\rangle}{\partial p_1} - \frac{\partial \langle b,c\rangle\langle d,a\rangle}{\partial p_1} + \frac{\partial \langle a,c\rangle\langle b,d\rangle}{\partial p_1}$$

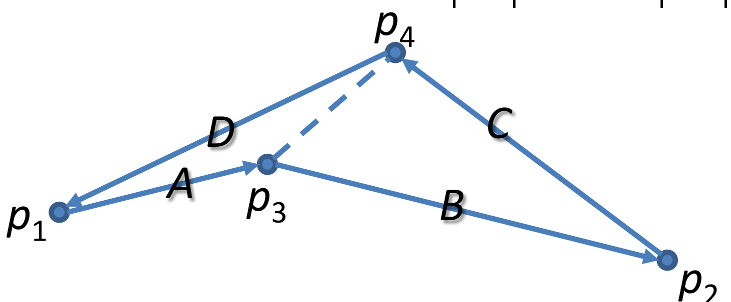
With a bit of manipulation we get:

$$\frac{\partial \cos\beta}{\partial p_1} = \left(\frac{\cos\beta}{|A|^2} - \frac{\langle b,c\rangle}{|A||D|}\right)A + \left(\frac{\langle a,c\rangle}{|B||D|} - \frac{\langle c,d\rangle}{|A||B|}\right)B$$

$$-\left(\frac{\langle a,b\rangle}{|C||D|} - \frac{\langle b,d\rangle}{|A||C|}\right)C - \left(\frac{\cos\beta}{|D|^2} - \frac{\langle b,c\rangle}{|A||D|}\right)D$$

$$a = \frac{A}{|A|} \qquad b = \frac{B}{|B|}$$

$$c = \frac{C}{|C|} \qquad d = \frac{D}{|D|}$$



Willmore Energy

$$\frac{\partial \cos\beta}{\partial p_1} = -\frac{\partial \langle a,b\rangle\langle c,d\rangle}{\partial p_1} - \frac{\partial \langle b,c\rangle\langle d,a\rangle}{\partial p_1} + \frac{\partial \langle a,c\rangle\langle b,d\rangle}{\partial p_1}$$

With a bit of manipulation we get:

$$\frac{\partial \cos\beta}{\partial p_1} = \left(\frac{\cos\beta}{|A|^2} - \frac{\langle b,c\rangle}{|A||D|}\right)A + \left(\frac{\langle a,c\rangle}{|B||D|} - \frac{\langle c,d\rangle}{|A||B|}\right)B$$
$$-\left(\frac{\langle a,b\rangle}{|C||D|} - \frac{\langle b,d\rangle}{|A||C|}\right)C - \left(\frac{\cos\beta}{|D|^2} - \frac{\langle b,c\rangle}{|A||D|}\right)D$$

Thus, we get:

$$\frac{\partial \beta}{\partial p_1} = -\frac{1}{\sin\beta}\left[\left(\frac{\cos\beta}{|A|^2} - \frac{\langle b,c\rangle}{|A||D|}\right)A + \left(\frac{\langle a,c\rangle}{|B||D|} - \frac{\langle c,d\rangle}{|A||B|}\right)B\right.$$
$$\left.-\left(\frac{\langle a,b\rangle}{|C||D|} - \frac{\langle b,d\rangle}{|A||C|}\right)C - \left(\frac{\cos\beta}{|D|^2} - \frac{\langle b,c\rangle}{|A||D|}\right)D\right]$$

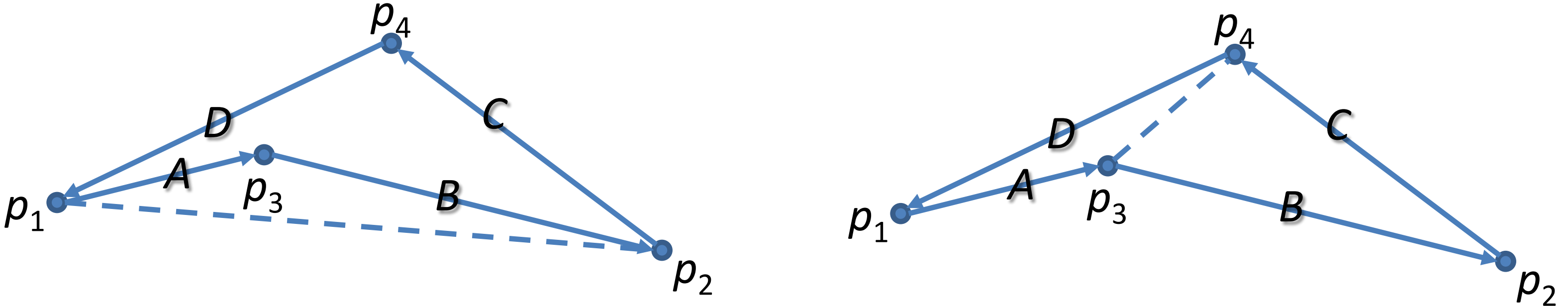
Willmore Energy

$$\cos\beta(a,b,c,d) = -\langle a,b\rangle\langle c,d\rangle - \langle b,c\rangle\langle d,a\rangle + \langle a,c\rangle\langle b,d\rangle$$

To compute the other gradients, we can either work through the math again, or we can observe that β doesn't change if we shift the indices:

$$\cos\beta(a,b,c,d) = \cos\beta(b,c,d,a)$$

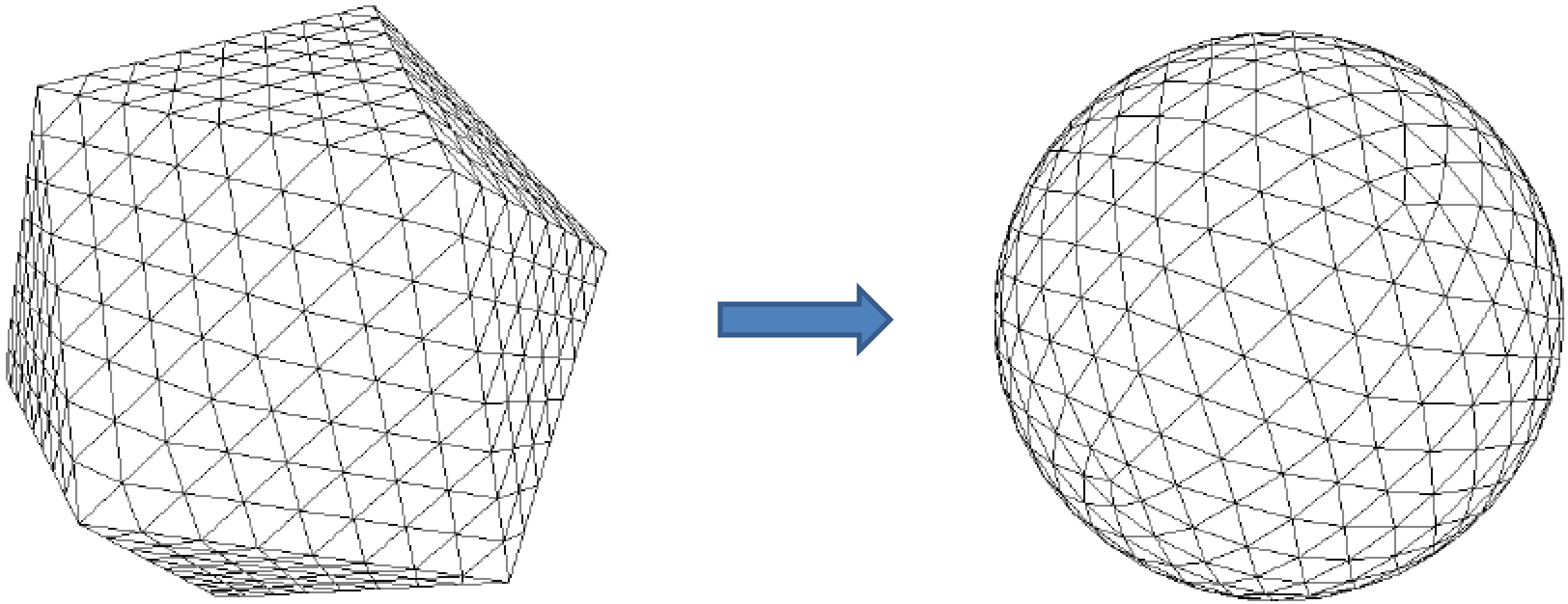
Thus, to compute the gradient w.r.t. p_3, we can just permute and use the equation from p_1.



Willmore Energy

$$\frac{\partial \beta}{\partial p_1} = -\frac{1}{\sin\beta}\left[\left(\frac{\cos\beta}{|A|^2} - \frac{\langle b,c\rangle}{|A||D|}\right)A + \left(\frac{\langle a,c\rangle}{|B||D|} - \frac{\langle c,d\rangle}{|A||B|}\right)B - \left(\frac{\langle a,b\rangle}{|C||D|} - \frac{\langle b,d\rangle}{|A||C|}\right)C - \left(\frac{\cos\beta}{|D|^2} - \frac{\langle b,c\rangle}{|A||D|}\right)D\right]$$

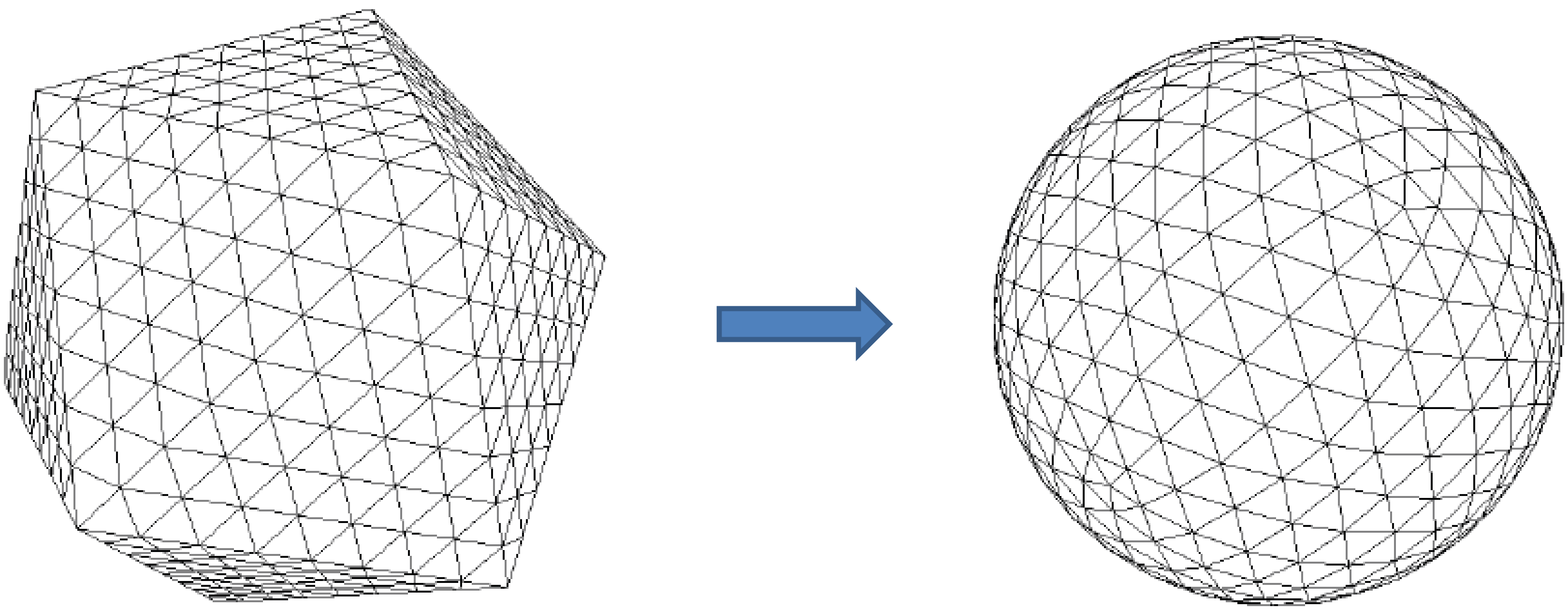
Using the equation for the gradient of the Willmore energy, we can flow the surface in order to minimize the total energy.



Willmore Energy

$$\frac{\partial\beta}{\partial p_1} = -\frac{1}{\sin\beta}\left[\left(\frac{\cos\beta}{|A|^2} - \frac{\langle b,c\rangle}{|A||D|}\right)\boxed{A} + \left(\frac{\langle a,c\rangle}{|B||D|} - \frac{\langle c,d\rangle}{|A||B|}\right)\boxed{B} - \left(\frac{\langle a,b\rangle}{|C||D|} - \frac{\langle b,d\rangle}{|A||C|}\right)\boxed{C} - \left(\frac{\cos\beta}{|D|^2} - \frac{\langle b,c\rangle}{|A||D|}\right)\boxed{D}\right]$$

As with the harmonic energy, we can linearize the system, treating everything but the difference vectors *A*, *B*, *C*, *D* as constant.



Willmore Energy

$$\frac{\partial \beta}{\partial p_1} = -\frac{1}{\sin\beta}\left[\left(\frac{\cos\beta}{|A|^2} - \frac{\langle b,c\rangle}{|A||D|}\right)\boxed{A} + \left(\frac{\langle a,c\rangle}{|B||D|} - \frac{\langle c,d\rangle}{|A||B|}\right)\boxed{B} - \left(\frac{\langle a,b\rangle}{|C||D|} - \frac{\langle b,d\rangle}{|A||C|}\right)\boxed{C} - \left(\frac{\cos\beta}{|D|^2} - \frac{\langle b,c\rangle}{|A||D|}\right)\boxed{D}\right]$$

As with the harmonic energy, we can linearize the system, treating everything but the difference vectors *A*, *B*, *C*, *D* as constant.

This gives us the Hessian and allows us to use semi-implicit integration to flow with more stability.

Willmore Energy

$$\frac{\partial\beta}{\partial p_1} = -\frac{1}{\sin\beta}\left[\left(\frac{\cos\beta}{|A|^2} - \frac{\langle b,c\rangle}{|A||D|}\right)\boxed{A} + \left(\frac{\langle a,c\rangle}{|B||D|} - \frac{\langle c,d\rangle}{|A||B|}\right)\boxed{B} - \left(\frac{\langle a,b\rangle}{|C||D|} - \frac{\langle b,d\rangle}{|A||C|}\right)\boxed{C} - \left(\frac{\cos\beta}{|D|^2} - \frac{\langle b,c\rangle}{|A||D|}\right)\boxed{D}\right]$$

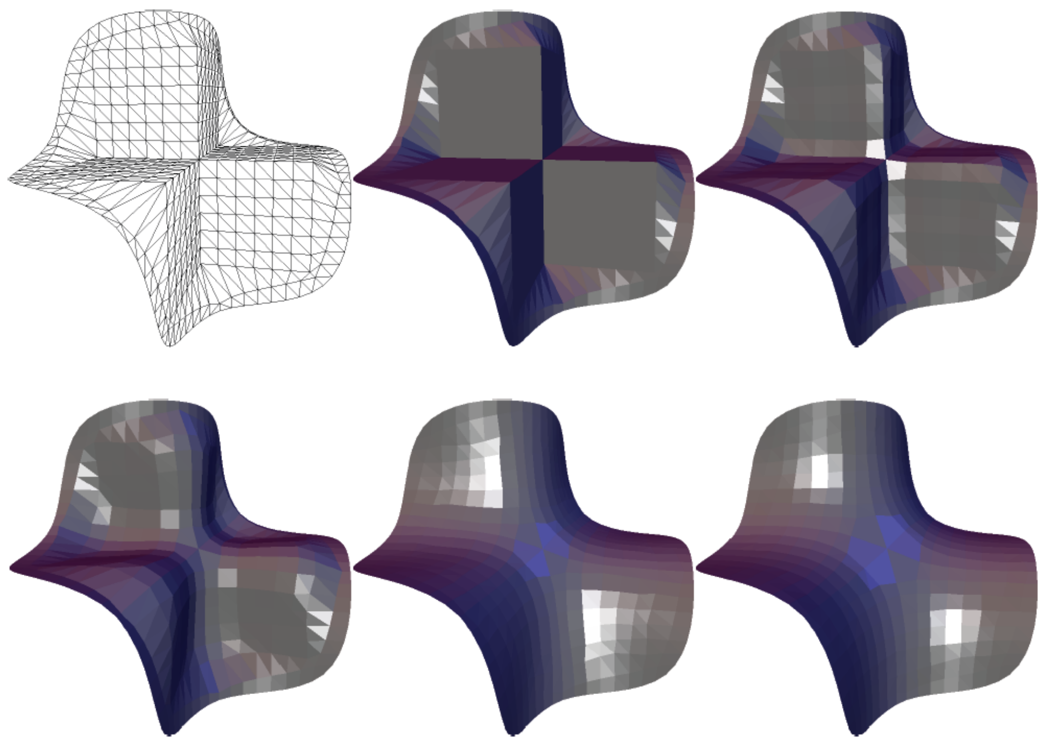
As with the harmonic energy, we can linearize the system, treating everything but the difference vectors *A*, *B*, *C*, *D* as constant.

This gives us the Hessian and allows us to use semi-implicit integration to flow with more stability.

Note that, as with the harmonic energy, this has the nice property that the system we need to solve is *n*x*n*, not 3*n*x3*n*.

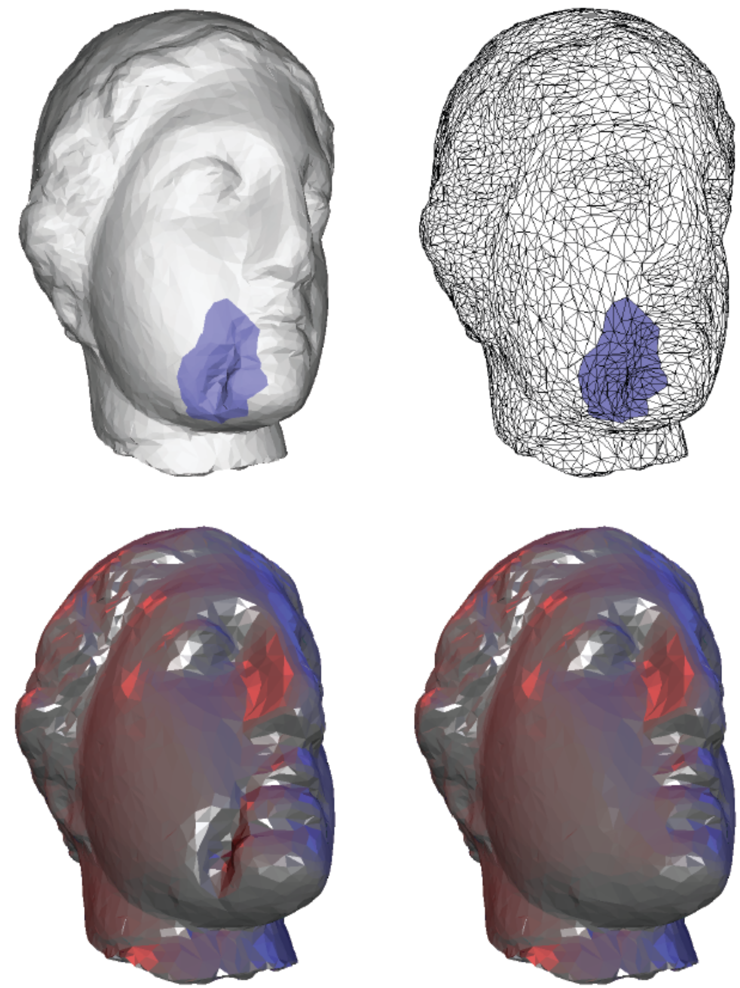
Willmore Energy

Since the system is fourth order, the flow is constrained by specifying both the positions and the normals of points on the boundary.



Willmore Energy

If we constrain the points and normals on the boundary of a patch inside a surface, we can use Willmore flow to smoothly evolve the patch while ensuring that the normals of the evolved patch match up with the original surface.



Willmore Energy

$$\frac{\partial \beta}{\partial p_1} = -\frac{1}{\sin\beta}\left[\left(\frac{\cos\beta}{|A|^2} - \frac{\langle b,c\rangle}{|A||D|}\right)A + \left(\frac{\langle a,c\rangle}{|B||D|} - \frac{\langle c,d\rangle}{|A||B|}\right)B - \left(\frac{\langle a,b\rangle}{|C||D|} - \frac{\langle b,d\rangle}{|A||C|}\right)C - \left(\frac{\cos\beta}{|D|^2} - \frac{\langle b,c\rangle}{|A||D|}\right)D\right]$$

Some Subtlety:

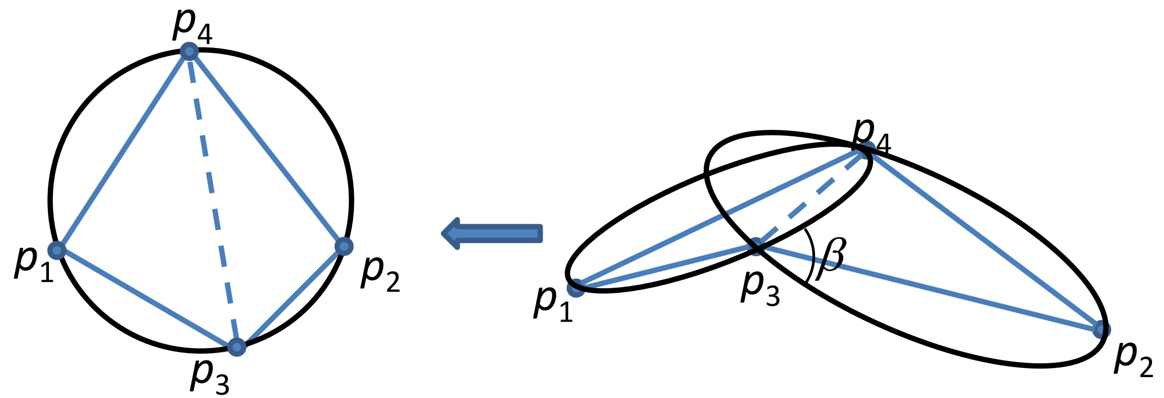
In the case that $\beta=0$, the expression for the energy gradient falls apart.

Willmore Energy

$$\frac{\partial \beta}{\partial p_1} = -\frac{1}{\sin\beta}\left[\left(\frac{\cos\beta}{|A|^2} - \frac{\langle b,c\rangle}{|A||D|}\right)A + \left(\frac{\langle a,c\rangle}{|B||D|} - \frac{\langle c,d\rangle}{|A||B|}\right)B - \left(\frac{\langle a,b\rangle}{|C||D|} - \frac{\langle b,d\rangle}{|A||C|}\right)C - \left(\frac{\cos\beta}{|D|^2} - \frac{\langle b,c\rangle}{|A||D|}\right)D\right]$$

Some Subtlety:

If we assume that the angle β is zero, then this must imply that the four points all reside on a common circle.



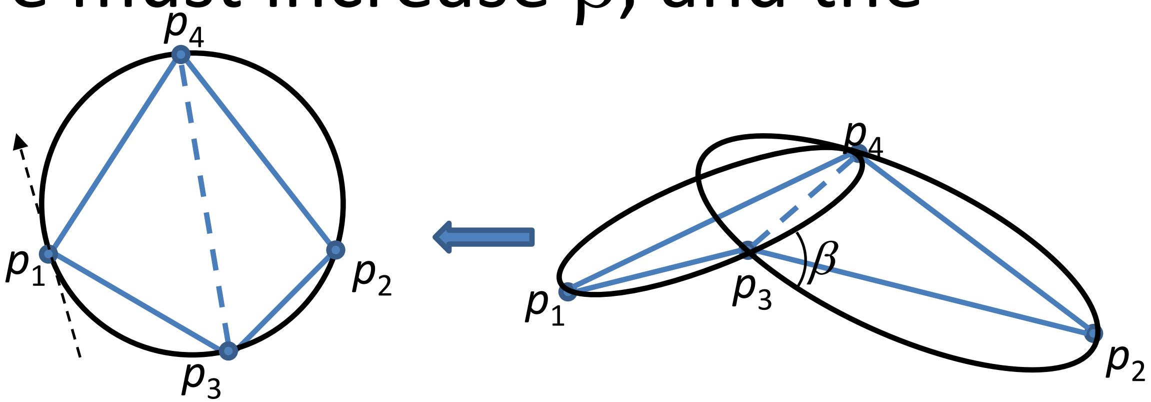
Willmore Energy

$$\frac{\partial\beta}{\partial p_1} = -\frac{1}{\sin\beta}\left[\left(\frac{\cos\beta}{|A|^2} - \frac{\langle b,c\rangle}{|A||D|}\right)A + \left(\frac{\langle a,c\rangle}{|B||D|} - \frac{\langle c,d\rangle}{|A||B|}\right)B - \left(\frac{\langle a,b\rangle}{|C||D|} - \frac{\langle b,d\rangle}{|A||C|}\right)C - \left(\frac{\cos\beta}{|D|^2} - \frac{\langle b,c\rangle}{|A||D|}\right)D\right]$$

Some Subtlety:

Thus, in moving p_1 (infinitesimally) we will only effect the angle β if we move p_1 in a direction perpendicular to the tangent of the circle at p_1.

And in that case, we must increase β, and the rate of change will not depend on the perpendicular direction.

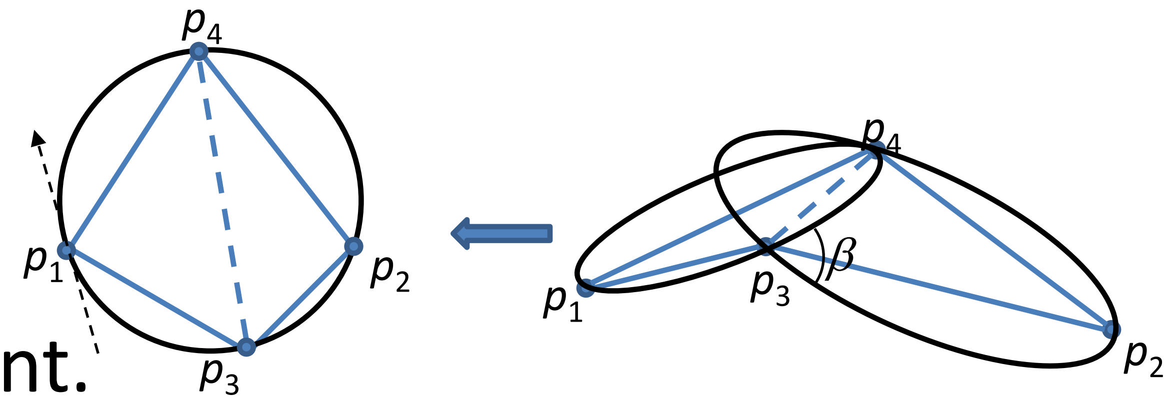


Willmore Energy

$$\frac{\partial\beta}{\partial p_1}=-\frac{1}{\sin\beta}\left[\left(\frac{\cos\beta}{|A|^2}-\frac{\langle b,c\rangle}{|A||D|}\right)A+\left(\frac{\langle a,c\rangle}{|B||D|}-\frac{\langle c,d\rangle}{|A||B|}\right)B-\left(\frac{\langle a,b\rangle}{|C||D|}-\frac{\langle b,d\rangle}{|A||C|}\right)C-\left(\frac{\cos\beta}{|D|^2}-\frac{\langle b,c\rangle}{|A||D|}\right)D\right]$$

<u>Some Subtlety</u>:

So, if the contribution to the gradient of the Willmore energy w.r.t. p_1 from all other triangles is G, the final gradient will use the (negative) part of G that is parallel to the tangent and may use a part in the perpendicular direction if it can offset the detriment.



Willmore Energy

$$\frac{\partial \beta}{\partial p_1} = -\frac{1}{\sin\beta}\left[\left(\frac{\cos\beta}{|A|^2} - \frac{\langle b,c\rangle}{|A||D|}\right)A + \left(\frac{\langle a,c\rangle}{|B||D|} - \frac{\langle c,d\rangle}{|A||B|}\right)B - \left(\frac{\langle a,b\rangle}{|C||D|} - \frac{\langle b,d\rangle}{|A||C|}\right)C - \left(\frac{\cos\beta}{|D|^2} - \frac{\langle b,c\rangle}{|A||D|}\right)D\right]$$

Some Subtlety:

Things get messier still if the point p_1 is adjacent to multiple edges with vanishing angles of circum-circle intersection.