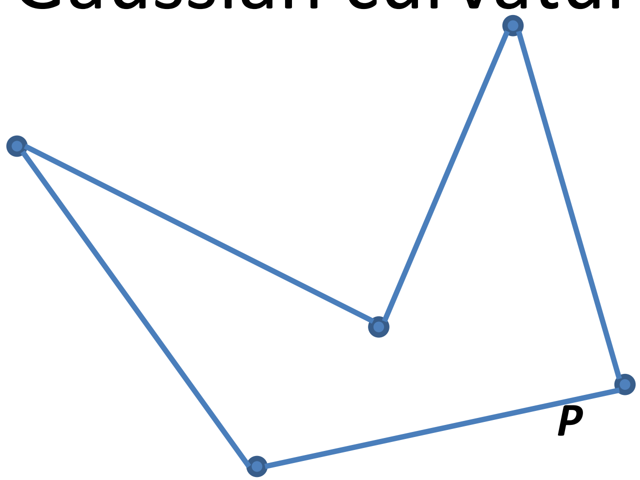
Differential Geometry:
Willmore Flow

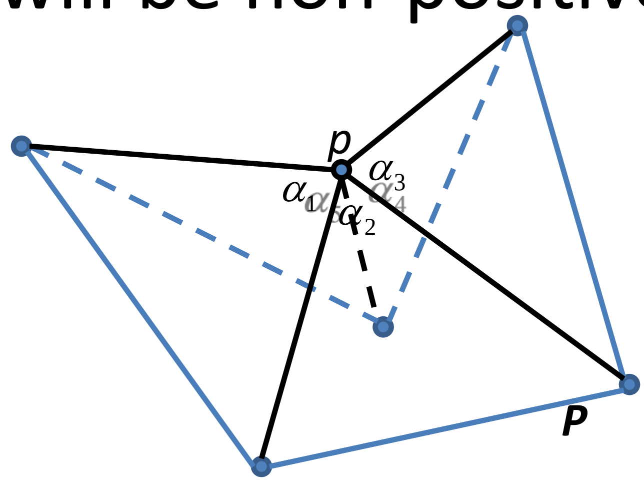
[*Discrete Willmore Flow*. Bobenko and Schröder, 2005]

Gaussian Curvature and Convexity

<u>Claim</u>:

Given a (nice) polygon ***P*** in 3D (not necessarily planar) and a point *p* in the convex hull of ***P***, then if we consider the triangulation derived by connecting *p* to the vertices in ***P*** the discrete Gaussian curvature will be non-positive.





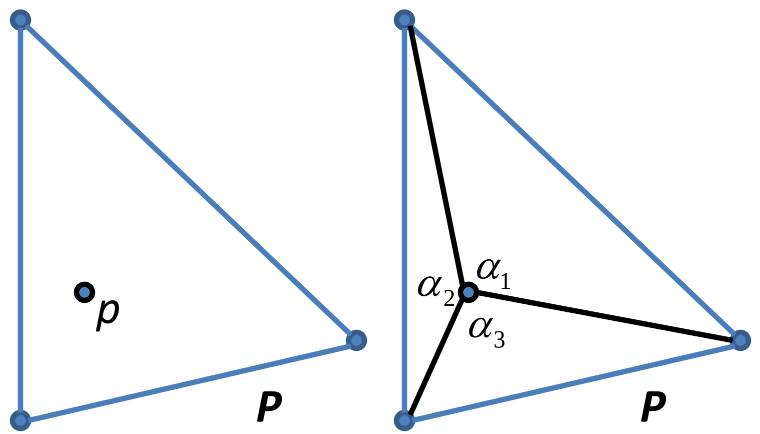
$$2\pi - \sum \alpha_i \leq 0$$

Gaussian Curvature and Convexity

Idea:

Consider a simple case when ***P*** is a triangle and *p* is some point inside the triangle.

Then the sum of the angles about *p* has to be equal to 2π.

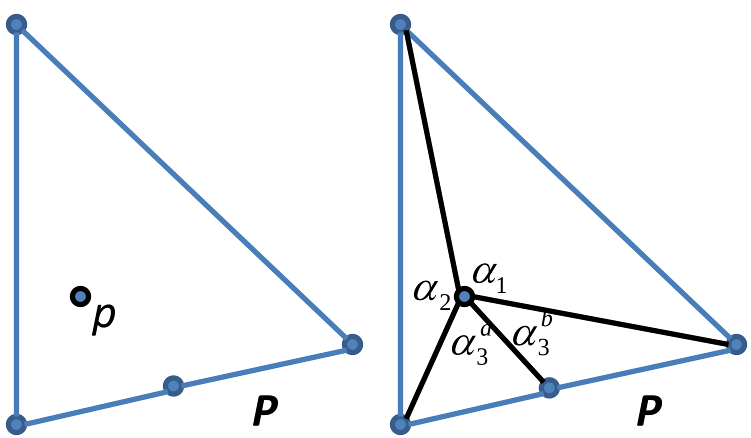


$$2\pi - \sum \alpha_i = 0$$

Gaussian Curvature and Convexity

<u>Idea</u>:

If we split one of the edges adding a new vertex on the edge, then the two new angles have to sum to an angle equal to the angle they replace, so the Gaussian curvature doesn't change.



$$\alpha_i^a + \alpha_i^b = \alpha_i$$

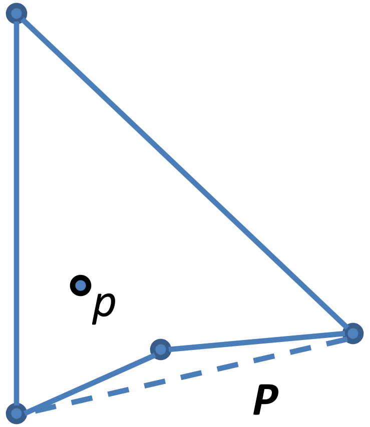
$$\Downarrow$$

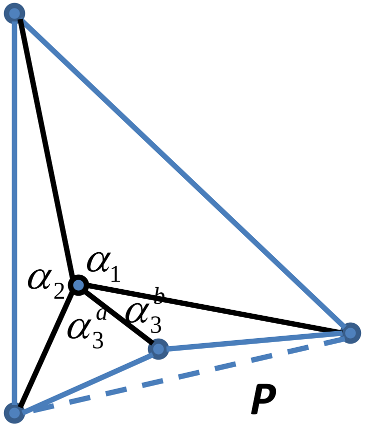
$$2\pi - \sum \alpha_i = 0$$

Gaussian Curvature and Convexity

Idea:

However, if we split one of the edges adding a new vertex off the edge, (not nec. in the plane of the triangle), then the two new angles have to sum to an angle larger than the angle they replace, and the Gaussian curvature decreases.





$$\alpha_i^a + \alpha_i^b \geq \alpha_i$$

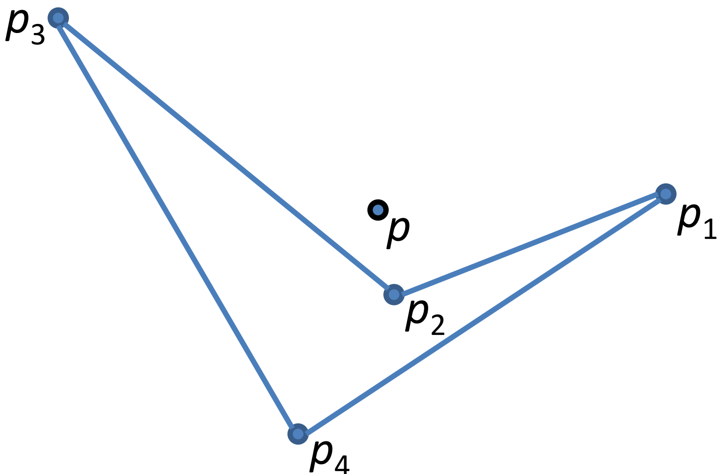
$$\Downarrow$$

$$2\pi - \sum \alpha_i \leq 0$$

Gaussian Curvature and Convexity

Proof (n=4):

We start by proving the case for four vertices.



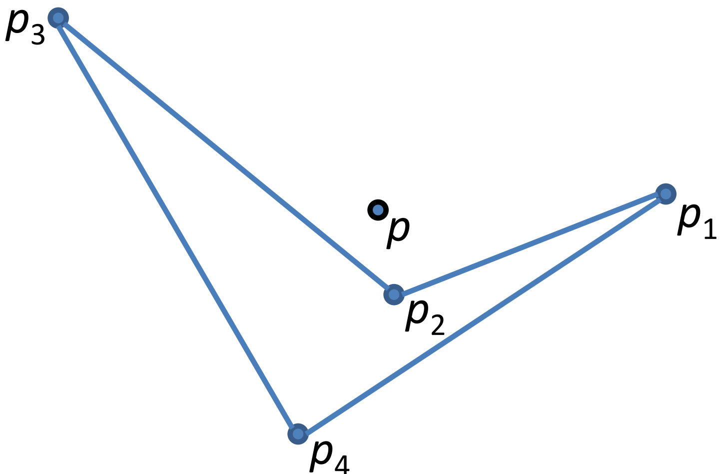
Gaussian Curvature and Convexity

Proof (n=4):

We start by proving the case for four vertices.

Since p is in the convex hull, it can be expressed as a non-negative combination of the vertices:

$$p = \sum_{i=1}^{4} \beta_i p_i \quad \text{with} \quad \beta_i \geq 0 \text{ and } \sum_{i=1}^{4} \beta_i = 1$$



Gaussian Curvature and Convexity

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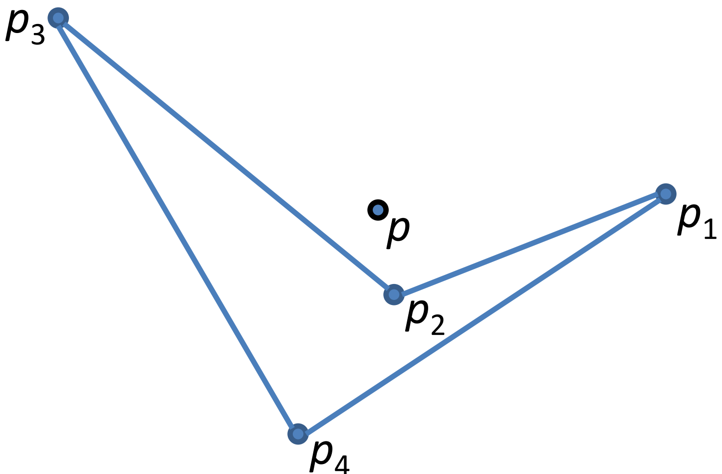
Proof (n=4):

Re-writing, we get:

$$p = \beta_1 p_1 + (1-\beta_1)\left(\frac{\beta_2}{1-\beta_1} p_2 + \frac{\beta_3}{1-\beta_1} p_3 + \frac{\beta_4}{1-\beta_1} p_4 \right)$$



Gaussian Curvature and Convexity

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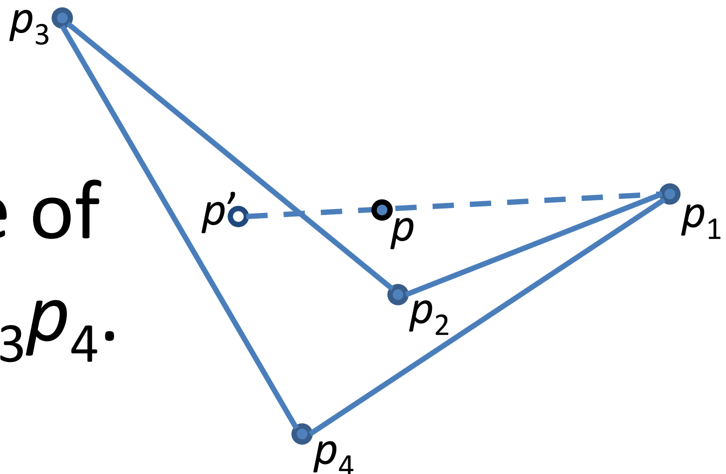
<u>Proof (n=4)</u>:

Since the β_i sum to 1 and since $0 \geq \beta_1 \geq 1$, we can write out the expression for p:

$$p = \beta_1 p_1 + (1-\beta_1)\left(\gamma_2 p_2 + \gamma_3 p_3 + \gamma_4 p_4\right)$$

where the γ_i are non-negative and sum to one.

Thus, p is the weighted average of p_1 and a point p' in triangle $p_2 p_3 p_4$.



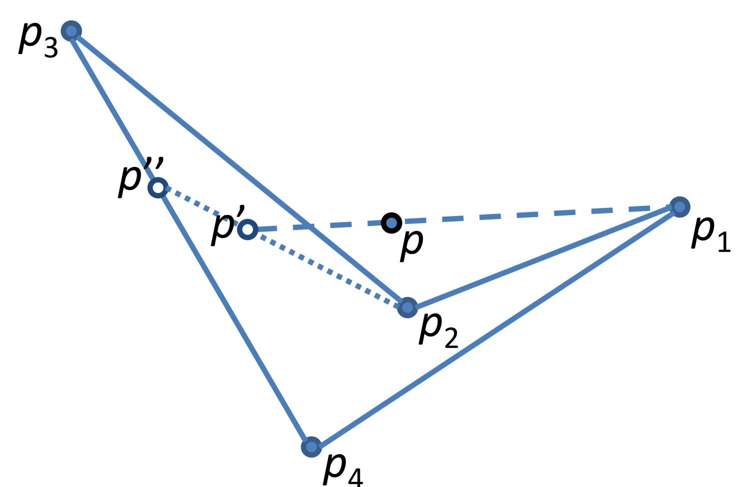
Gaussian Curvature and Convexity

$$p = \beta_1 p_1 + (1-\beta_1)\underbrace{\left(\gamma_2 p_2 + \gamma_3 p_3 + \gamma_4 p_4\right)}_{p'}$$

Proof (n=4):

Similarly, we can write out the point p' as the (non-negatively) weighted average of p_2 and a point on the edge $p_3 p_4$:

$$p' = \gamma_2 p_2 + (1-\gamma_2)\left(\frac{\gamma_3}{1-\gamma_2} p_3 + \frac{\gamma_4}{1-\gamma_2} p_4\right)$$



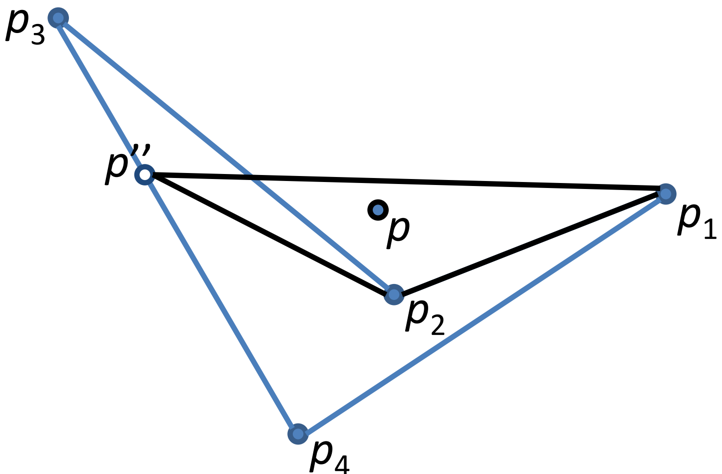
Gaussian Curvature and Convexity

$$p = \beta_1 p_1 + (1-\beta_1)\underbrace{\left(\gamma_2 p_2 + \gamma_3 p_3 + \gamma_4 p_4\right)}_{p'}$$

$$p' = \gamma_2 p_2 + (1-\gamma_2)\underbrace{\left(\delta_3 p_3 + \delta_4 p_4\right)}_{p''}$$

Proof (n=4):

Thus, the point p lives on the triangle passing through p_1, p_2, and a point p'' on the edge p_3p_4.



Gaussian Curvature and Convexity

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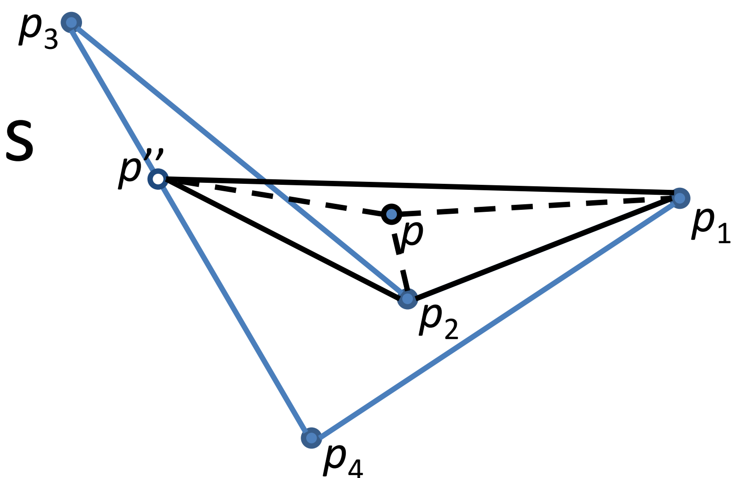
Proof (n=4):

Thus, the point p lives on the triangle passing through p_1, p_2, and a point p'' on the edge $p_3 p_4$.

Since p is inside the triangle, we know that the sum of angles has to be 2π.



Gaussian Curvature and Convexity

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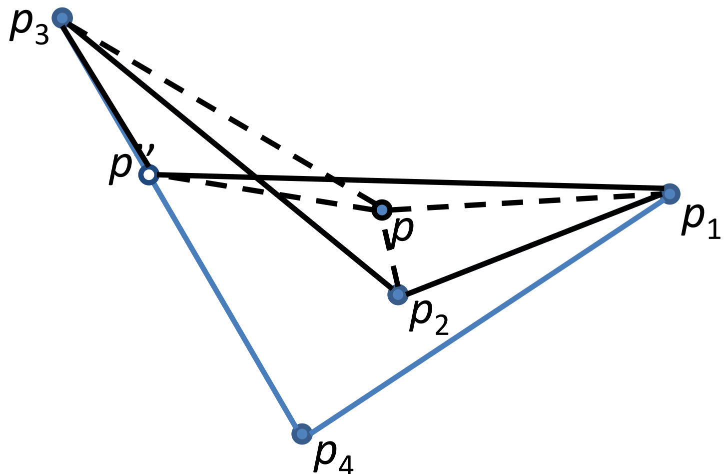
Proof (n=4):

Inserting the point p_3 on the edge between p_2 and p'', we do not decrease the angle sum.



Gaussian Curvature and Convexity

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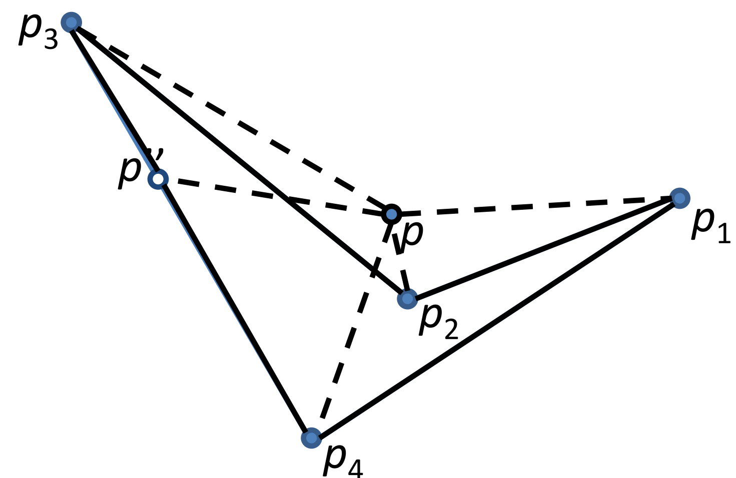
$$p' = \gamma_2 p_2 + (1-\gamma_2)\underbrace{\left(\delta_3 p_3 + \delta_4 p_4\right)}_{p''}$$

<u>Proof (n=4)</u>:

And again, inserting the point p_4 on the edge between p_1 and p'', we do not decrease the angle sum.



Gaussian Curvature and Convexity

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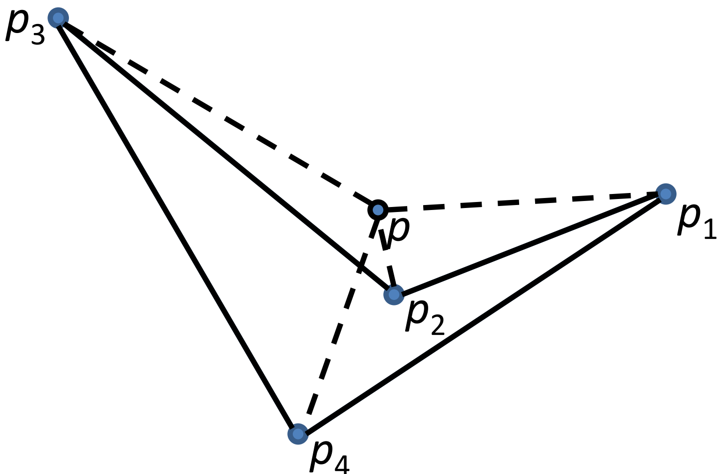
Proof (n=4):

Finally, since the point p'' lies on the edge between p_3 and p_4, removing it does not change the angle sum.



Gaussian Curvature and Convexity

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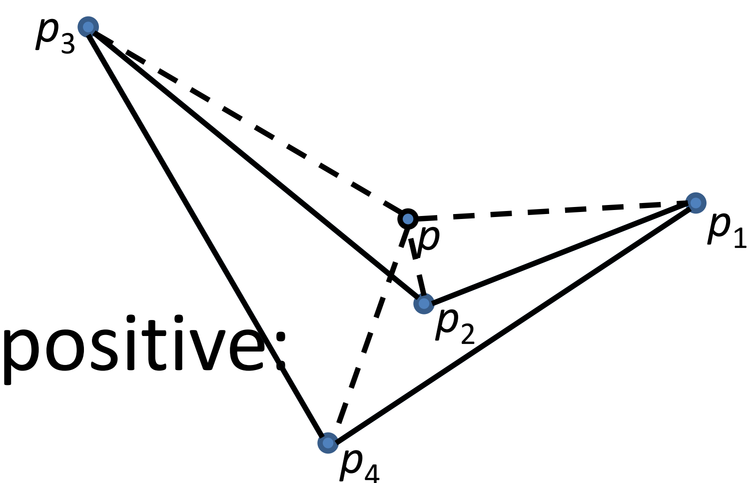
$$p' = \gamma_2 p_2 + (1-\gamma_2)\underbrace{\left(\delta_3 p_3 + \delta_4 p_4\right)}_{p''}$$

Proof (n=4):

Finally, since the point p'' lies on the edge between p_3 and p_4, removing it does not change the angle sum.

Thus, the total sum of angles around p is at least 2π and the Gaussian curvature cannot be positive:

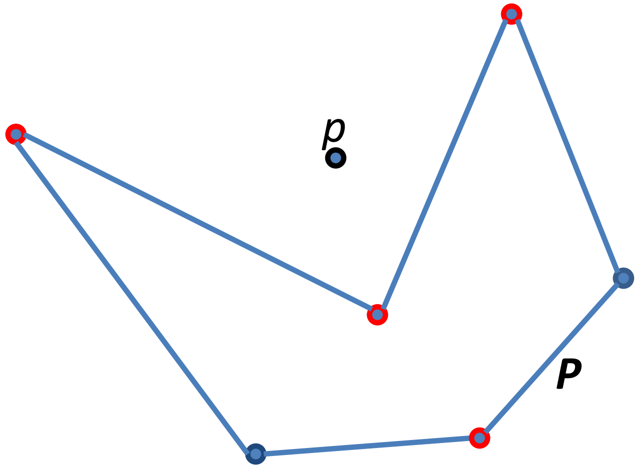
$$2\pi - \sum \alpha_i \leq 0$$



Gaussian Curvature and Convexity

Proof:

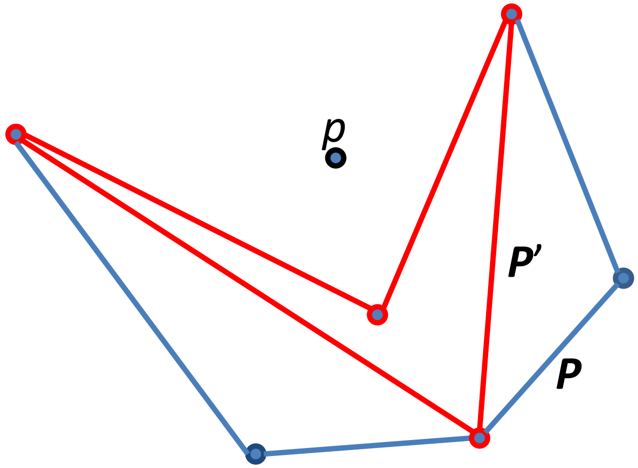
To prove the claim, for a general polygon ***P***, we use the fact that if *p* is in the convex hull of ***P*** then it is the (non-negative) weighted average of four of the vertices.



Gaussian Curvature and Convexity

Proof:

Connecting these vertices in the order in which they appear on ***P***, we get a polygon ***P'*** with four vertices.

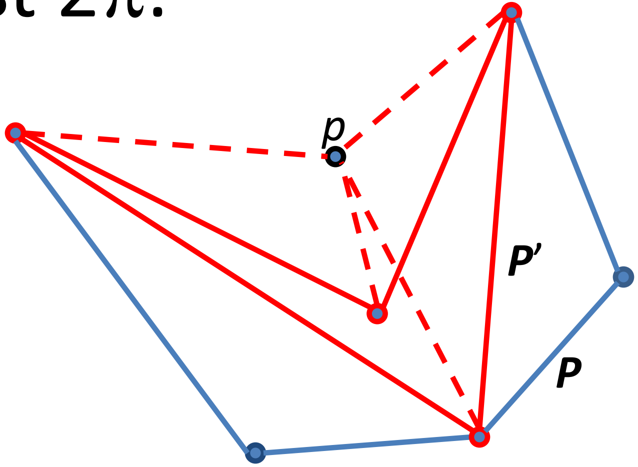


Gaussian Curvature and Convexity

Proof:

Connecting these vertices in the order in which they appear on ***P***, we get a polygon ***P'*** with four vertices.

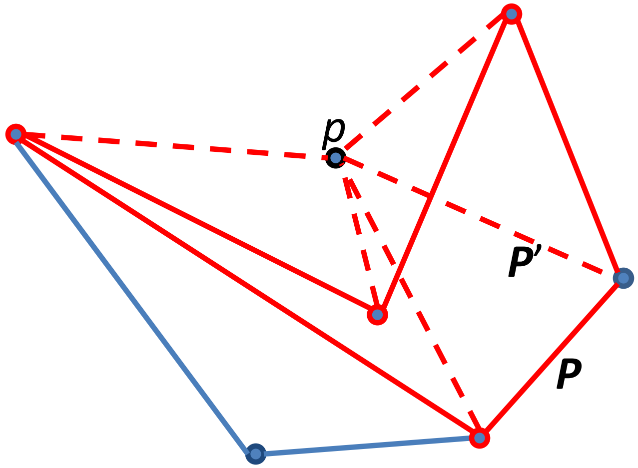
We know that connecting these vertices to *p*, the angle sum has to be at least 2π.



Gaussian Curvature and Convexity

Proof:

But now we can proceed as before, splitting the edges of ***P'*** and adding back in the vertices of ***P***…



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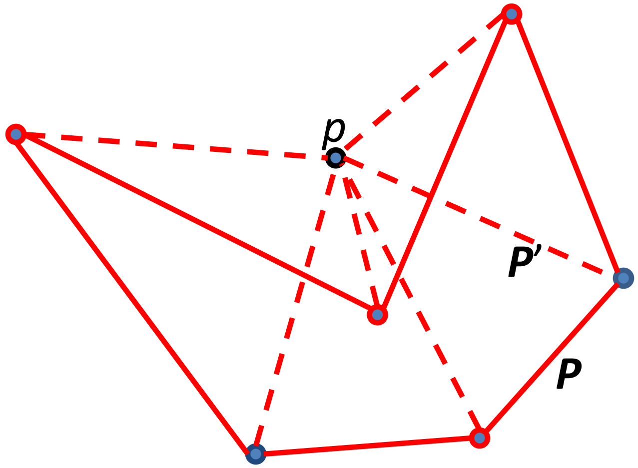
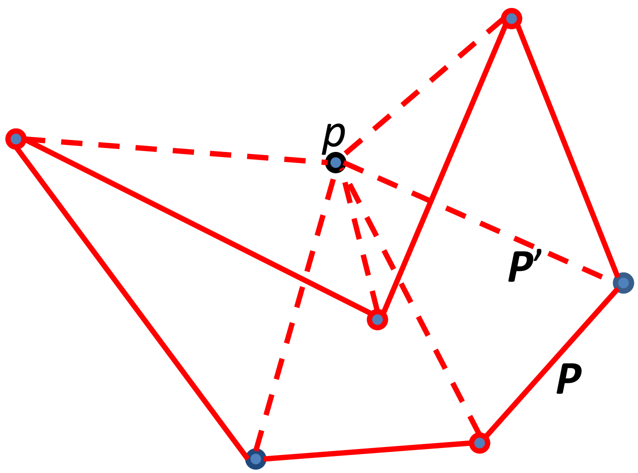


Gaussian Curvature and Convexity

Proof:

But now we can proceed as before, splitting the edges of ***P'*** and adding back in the vertices of ***P***…

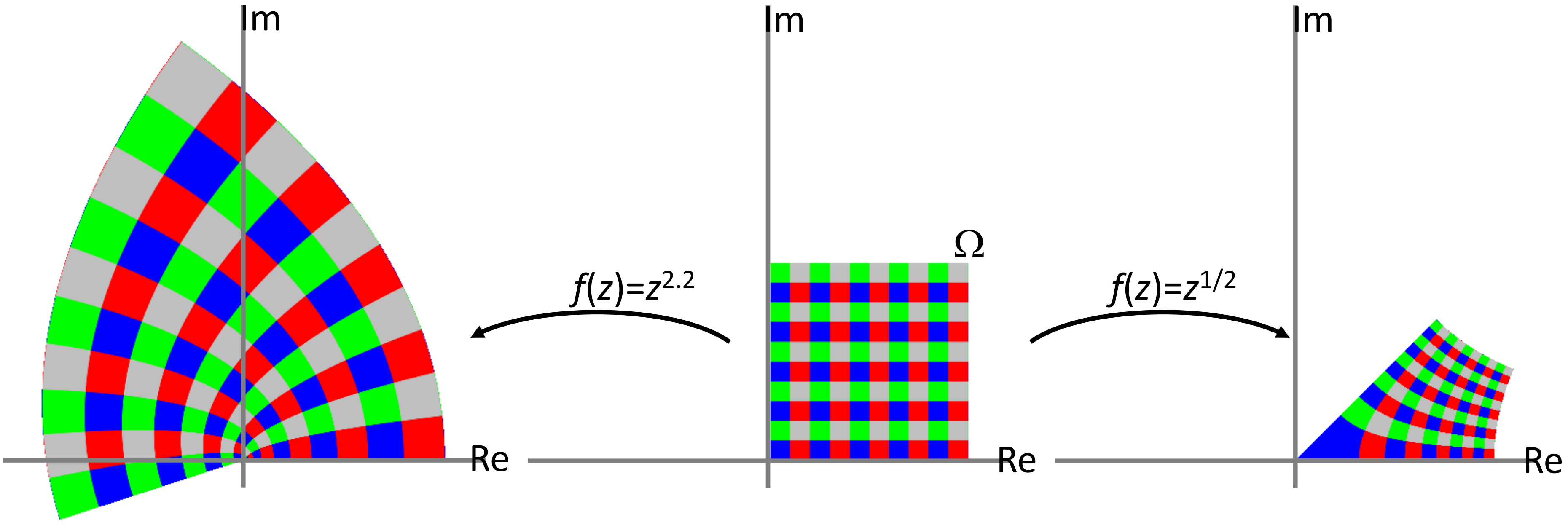
Since the introduction of new vertices cannot decrease the angle sum, the Gaussian curvature at *p* cannot be positive.



Recall

<u>Conformal Maps</u>:

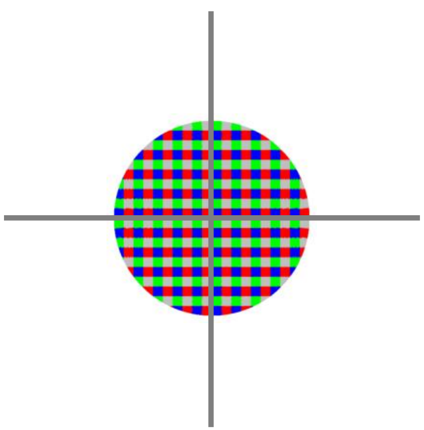
In looking at maps $f:\Omega \subset \mathbf{R}^2 \rightarrow \mathbf{R}^2$, we had looked at *conformal maps* which are maps that preserve angles, though not necessarily scale.

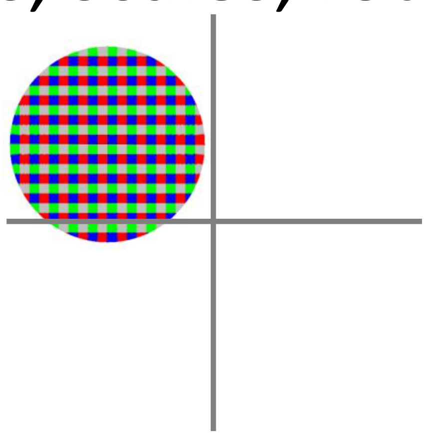


Recall

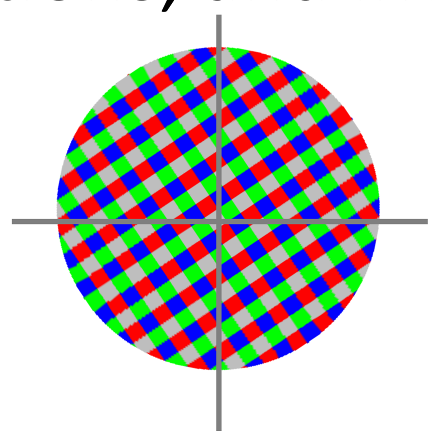
<u>Conformal Maps</u>:

The only conformal maps from the entirety of the extended plane back into the extended plane, f:$\boldsymbol{R}^2\cup\{\infty\}\rightarrow\boldsymbol{R}^2\cup\{\infty\}$, are the *Möbius transformations*, expressable as combinations of translations, scales, rotations, and inversions.

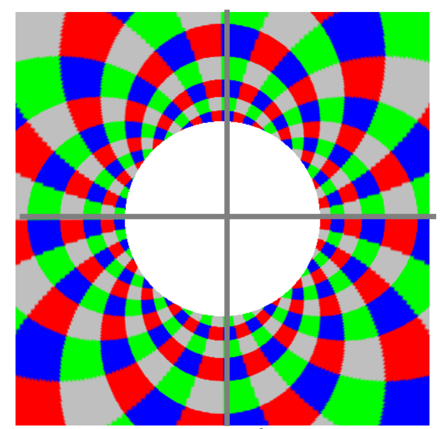




Translation



Scale + Rotation



Inversion

Recall

Conformal Maps:

These maps take all circles to circles (with a line being considered a generalization of a circle going through infinity).

In particular, if a point on a circle/line gets mapped to ∞, that circle/line must get mapped to a line.

Recall

Conformal Maps:

In higher dimensions, we also define conformal maps as the maps that preserve angles, but it turns out that the *Möbius transformations* are the only conformal maps (Liouville's Theorem).

Recall

Conformal Maps:

In higher dimensions, we also define conformal maps as the maps that preserve angles, but it turns out that the *Möbius transformations* are the only conformal maps (Liouville's Theorem).

These maps takes (generalized) circles to circles and (generalized) spheres to spheres.

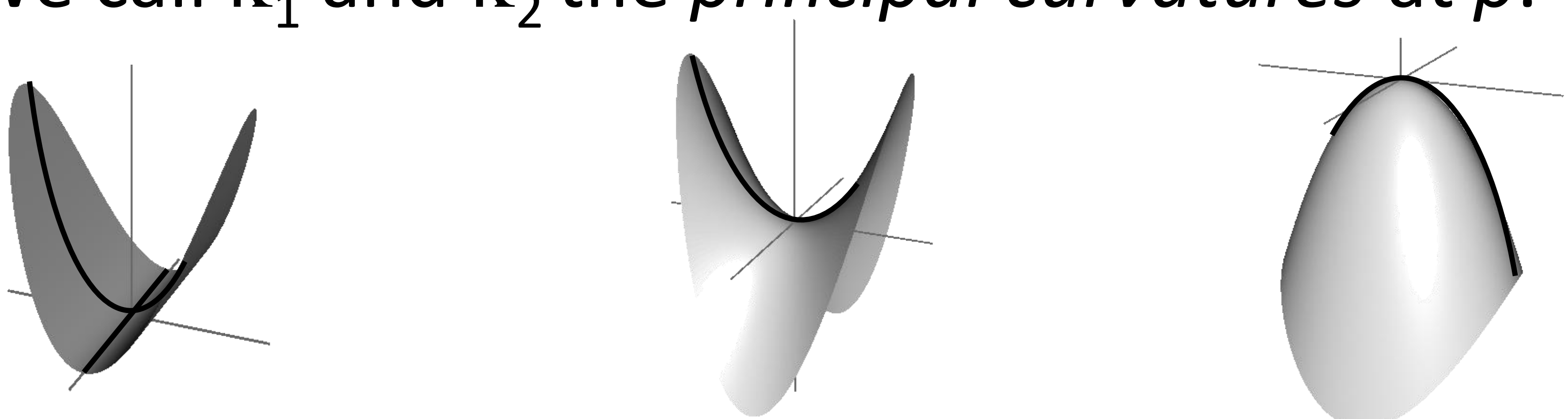
Willmore Flow

Recall:

If we express a surface at a point p as the graph of a function defined over the tangent plane, then, up to rotation, the function is of the form:

$$f(x,y) \approx \frac{\kappa_1}{2}x^2 + \frac{\kappa_2}{2}y^2$$

We call κ_1 and κ_2 the *principal curvatures* at p.



Willmore Flow

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We call κ_1 and κ_2 the *principal curvatures* at *p*.

We define *Gaussian* (resp. *mean*) *curvature* as the product (resp. sum) of principal curvatures:

$$K = \kappa_1 \cdot \kappa_2 \qquad\qquad H = \kappa_1 + \kappa_2$$

Willmore Flow

Goal:

We would like to evolve the surface so that it minimizes the total curvature:

$$E(S) = \int_{p \in S} \kappa_1^2(p) + \kappa_2^2(p)\, dp$$

Willmore Flow

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Equivalently, we could express this as:

$$E(S) = \int_S (\kappa_1 - \kappa_2)^2 + 2\kappa_1\kappa_2\, dp$$

Willmore Flow

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$$= \int_S (\kappa_1 - \kappa_2)^2\,dp + 4\pi\chi_S$$

by the Gauss-Bonnet theorem, with χ_S the Euler characteristic of S (i.e. χ_S=2-2g).

Willmore Flow

<u>Goal</u>:

We would like to evolve the surface so that it minimizes the total curvature:

$$E(S) = \int_{p \in S} \kappa_1^2(p) + \kappa_2^2(p)\, dp = \int_S (\kappa_1 - \kappa_2)^2\, dp + 4\pi\chi_S$$

Since the surface will not change genus throughout the evolution, minimizing the total curvature is equivalent to minimizing the curvature difference:

$$E(S) = \int_S (\kappa_1 - \kappa_2)^2\, dp$$

Willmore Flow

$$E(S) = \int_S (\kappa_1 - \kappa_2)^2 \, dp$$

Challenge:

We would like to extend the definition of the energy to discrete surfaces so that we can evolve discrete surfaces to reduce their total curvature.

The problem is that we don't know how to compute the principal curvatures in a discrete setting.

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Pass 1:

Rearranging the energy, we get:

$$E(S)=\int_S \left(\kappa_1-\kappa_2\right)^2\, dp$$

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<u>Pass 1</u>:

Rearranging the energy, we get:

$$E(S) = \int_S (\kappa_1 - \kappa_2)^2 \, dp$$

$$= \int_S \kappa_1^2 - 2\kappa_1\kappa_2 + \kappa_2^2 \, dp$$

Willmore Flow

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Rearranging the energy, we get:

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$$= \int_S \kappa_1^2 - 2\kappa_1\kappa_2 + \kappa_2^2 \, dp$$

$$= \int_S \kappa_1^2 + 2\kappa_1\kappa_2 + \kappa_2^2 - 4\kappa_1\kappa_2 \, dp$$

Willmore Flow

Pass 1:

Rearranging the energy, we get:

$$
\begin{aligned}
E(S) &= \int_S (\kappa_1 - \kappa_2)^2 \, dp \\
&= \int_S \kappa_1^2 - 2\kappa_1\kappa_2 + \kappa_2^2 \, dp \\
&= \int_S \kappa_1^2 + 2\kappa_1\kappa_2 + \kappa_2^2 - 4\kappa_1\kappa_2 \, dp \\
&= \int_S H^2 - 4K \, dp
\end{aligned}
$$

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$$E(S) = \int_S (\kappa_1 - \kappa_2)^2 \, dp = \int_S H^2 - 4K \, dp$$

Pass 1:

While this seems promising, since we know how to compute Gaussian and mean curvatures in the discrete settings, the problem is that our discrete curvatures are not guaranteed so satisfy H^2-4$K \geq 0$.

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$$E(S) = \int_S (\kappa_1 - \kappa_2)^2 \, dp = \int_S H^2 - 4K \, dp$$

Pass 2:

Rather than building up the integrand by components, we will try to construct it as a single, discrete entity.

Willmore Flow

$$E(S)=\int_S(\kappa_1-\kappa_2)^2\,dp=\int_S H^2-4K\,dp$$

Pass 2:

Rather than building up the integrand by components, we will try to construct it as a single, discrete entity.

Continuous Properties:

- The Willmore energy at *p* is non-negative.
- It is zero only if the shape is locally spherical (i.e. if the point *p* is *umbilical*).
- The integrand is Möbius-invariant.

Willmore Flow

$$E(S) = \int_S (\kappa_1 - \kappa_2)^2 \, dp = \int_S H^2 - 4K \, dp$$

Pass 2:

Rather than building up the integrand by components, we will try to construct it as a single, discrete entity.

Discrete Properties:

- The Willmore energy at *v* should be non-negative.
- It is zero only if the vertex and it's one-ring lie on a sphere (and are convex).
- The Willmore energy at *v* is a Möbius-invariant.

Willmore Flow

$$E(S) = \int_S (\kappa_1 - \kappa_2)^2 \, dp = \int_S H^2 - 4K \, dp$$

Pass 2:

Rather than building up the integrand by components, we will try to construct it as a single, discrete entity.

Discrete Properties:

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–

–

Definition:

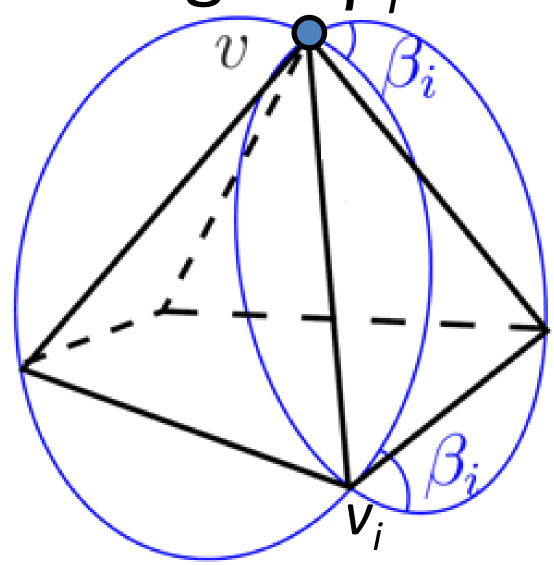
The vertex v and its one-ring are *convex* if, for each triangle (v, v_i, v_{i+1}) the vertices are all on one side of the triangle.

Willmore Flow

$$E(S)=\int_S (\kappa_1-\kappa_2)^2\,dp=\int_S H^2-4K\,dp$$

<u>Approach</u>:

Use the triangles to define circumcircles in 3D, and consider the exterior intersection angles β_i of the circumcircles.



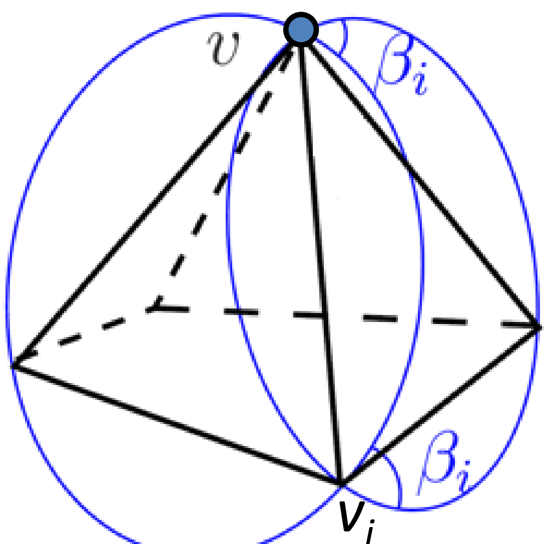
Willmore Flow

$$E(S) = \int_S (\kappa_1 - \kappa_2)^2 \, dp = \int_S H^2 - 4K \, dp$$

<u>Claim</u>:

The deficit of the sum of exterior intersection angles:

$$E(v) = \sum_{v_i \in N(v)} \beta_i - 2\pi$$



has exactly the properties we want:

- $E(v) \geq 0$ for every vertex v.
- $E(v) = 0$ iff. v and its one-ring lie on a common sphere and are convex.
- The Willmore energy at v is a Möbius-invariant.

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Proof:

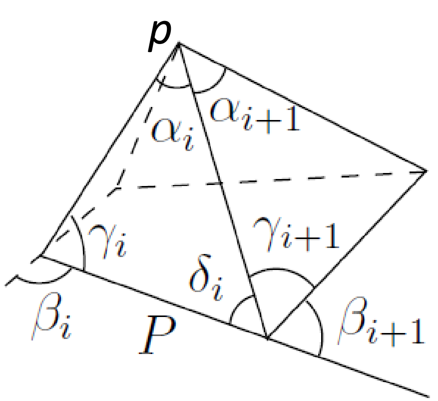
Auxiliary Lemma:

Let ***P*** be a polygon with external angles β_i, (not nec. planar). Choose a point *p* and connect the vertices of ***P*** to *p* in order to get a triangulation.

Then the angles α_i at the tip of the pyramid satisfy:

$$\sum_i \beta_i \geq \sum_i \alpha_i$$

with equality iff. ***P*** is planar and convex, and *p* is inside of ***P***.



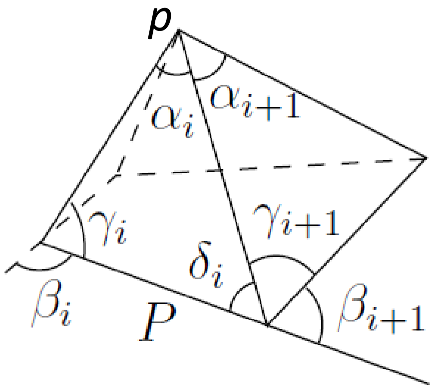
Willmore Flow

Proof:

Auxiliary Lemma (Proof):

If we consider the angles in the triangulation, we can observe that:

$$\pi = \alpha_i + \delta_i + \gamma_i \qquad \pi \le \beta_i + \delta_i + \gamma_{i+1}$$



Willmore Flow

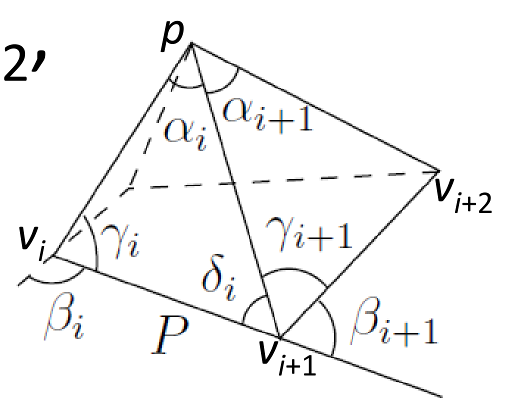
Proof:

Auxiliary Lemma (Proof):

If we consider the angles in the triangulation, we can observe that:

$$\pi = \alpha_i + \delta_i + \gamma_i \qquad \pi \le \beta_i + \delta_i + \gamma_{i+1}$$

Furthermore, we have equality in the second equation iff. vertices p, v_i, v_{i+1}, and v_{i+2}, all reside in the same plane, and p is on the convex side of the angle $\angle v_i v_{i+1} v_{i+2}$.



Willmore Flow

Proof:

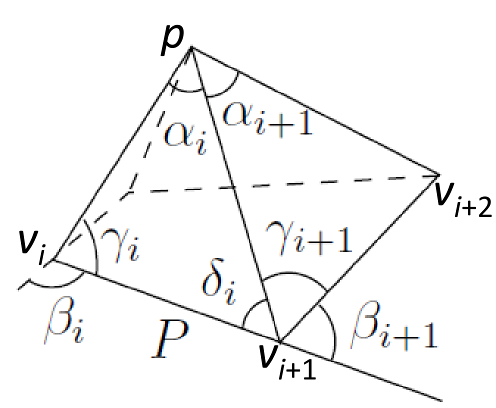
Auxiliary Lemma (Proof):

If we consider the angles in the triangulation, we can observe that:

$$\pi = \alpha_i + \delta_i + \gamma_i \qquad \pi \leq \beta_i + \delta_i + \gamma_{i+1}$$

Summing over all vertices in polygon ***P***, we get:

$$\sum_i \beta_i + \delta_i + \gamma_{i+1} \geq \sum_i \alpha_i + \delta_i + \gamma_i$$



Willmore Flow

<u>Proof</u>:

<u>Auxiliary Lemma (Proof)</u>:

If we consider the angles in the triangulation, we can observe that:

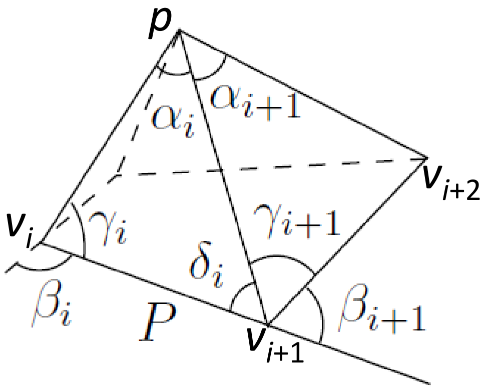
$$\pi = \alpha_i + \delta_i + \gamma_i \qquad \pi \leq \beta_i + \delta_i + \gamma_{i+1}$$

Summing over all vertices in polygon *P*, we get:

$$\sum_i \beta_i + \delta_i + \gamma_{i+1} \geq \sum_i \alpha_i + \delta_i + \gamma_i$$

$$\Downarrow$$

$$\sum_i \beta_i + \delta_i + \gamma_i \geq \sum_i \alpha_i + \delta_i + \gamma_i$$



Willmore Flow

Proof:

Auxiliary Lemma (Proof):

If we consider the angles in the triangulation, we can observe that:

$$\pi = \alpha_i + \delta_i + \gamma_i \qquad \pi \leq \beta_i + \delta_i + \gamma_{i+1}$$

Summing over all vertices in polygon *P*, we get:

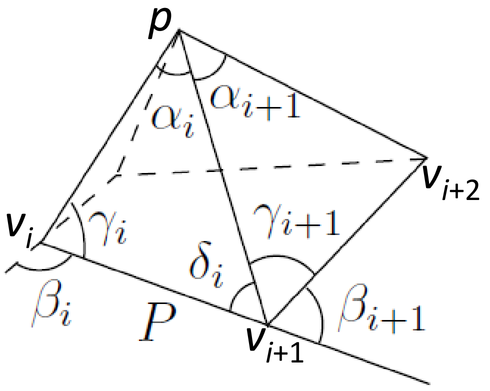
$$\sum_i \beta_i + \delta_i + \gamma_{i+1} \geq \sum_i \alpha_i + \delta_i + \gamma_i$$

$$\Downarrow$$

$$\sum_i \beta_i + \delta_i + \gamma_i \geq \sum_i \alpha_i + \delta_i + \gamma_i$$

$$\Downarrow$$

$$\sum_i \beta_i \geq \sum_i \alpha_i$$



Willmore Flow

Proof:

Auxiliary Lemma (Proof):

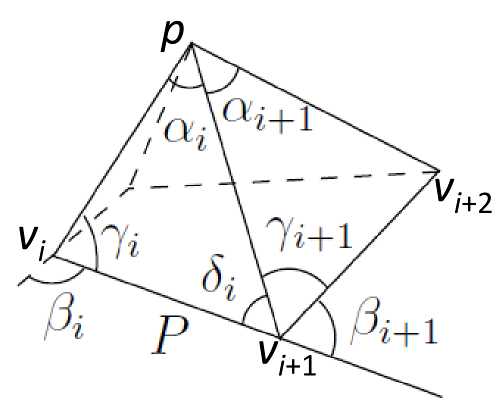
If we consider the angles in the triangulation, we can observe that:

$$\pi = \alpha_i + \delta_i + \gamma_i \qquad \pi \le \beta_i + \delta_i + \gamma_{i+1}$$

Summing over all vertices in polygon ***P***, we get:

$$\sum_i \beta_i \ge \sum_i \alpha_i$$

with equality iff. ***P*** is planar and convex, and the *p* is inside ***P***.



Willmore Flow

<u>Proof</u>:

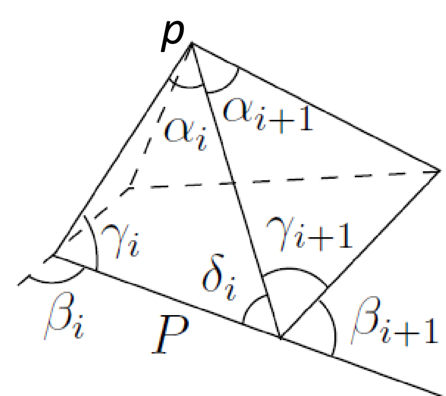
<u>Auxiliary Lemma (Corollary)</u>:

Note that since:

$$\sum_i \beta_i \geq \sum_i \alpha_i$$

for any *p*, if we place *p* inside the convex hull of ***P***, then we know that the Gauss curvature at *p* must be negative, so:

$$\sum_i \beta_i \geq \sum_i \alpha_i \geq 2\pi$$

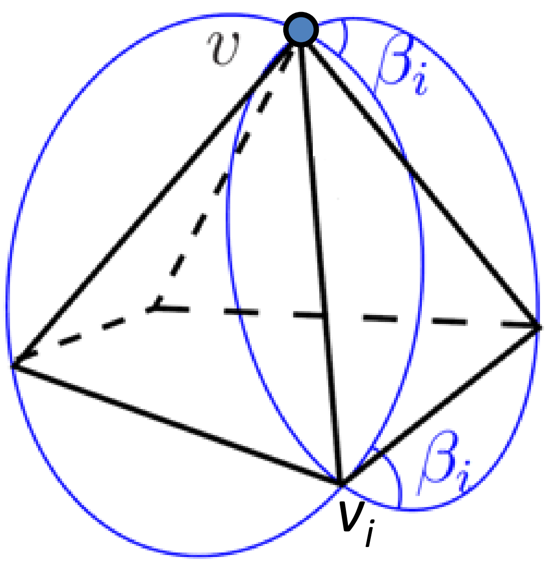


Willmore Flow

$$E(v) = \sum_{v_i \in N(v)} \beta_i - 2\pi$$

<u>Claim</u>:

- $E(v) \geq 0$ for every vertex v.
- $E(v) = 0$ iff. v and its one-ring lie on a common sphere and are convex.
- The Willmore energy at v is a Möbius-invariant.



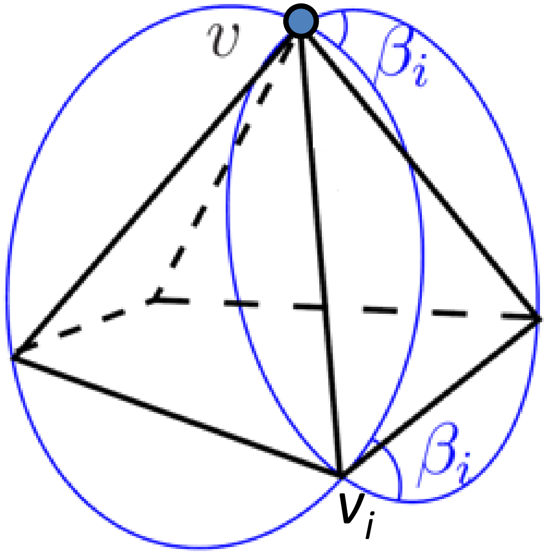
Willmore Flow

$$E(v) = \sum_{v_i \in N(v)} \beta_i - 2\pi$$

Claim:

– The Willmore energy at *v* is a Möbius-invariant.

The Möbius-invariance follows from the fact that we have defined the energy in terms of angles between circles.



Willmore Flow

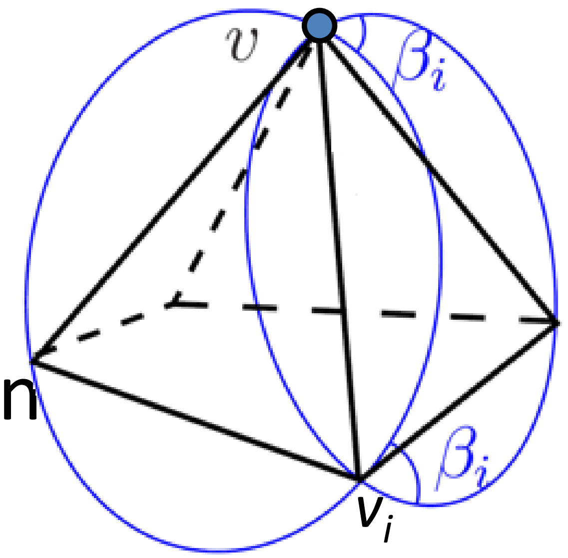
$$E(v) = \sum_{v_i \in N(v)} \beta_i - 2\pi$$

<u>Claim</u>:

– The Willmore energy at *v* is a Möbius-invariant.

The Möbius-invariance follows from the fact that we have defined the energy in terms of angles between circles.

It also means that nothing changes if we apply a Möbius transformation to the geometry.

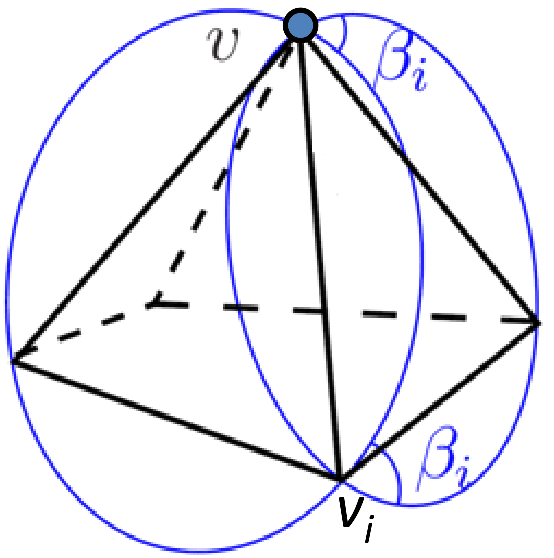


Willmore Flow

$$E(v) = \sum_{v_i \in N(v)} \beta_i - 2\pi$$

<u>Claim</u>:

Apply some Möbius transformation, *M*, that sends the vertex *v* to ∞.



Willmore Flow

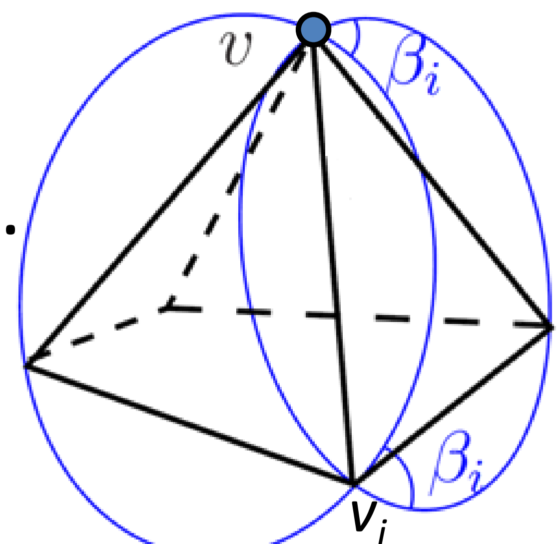
$$E(v) = \sum_{v_i \in N(v)} \beta_i - 2\pi$$

<u>Claim</u>:

Apply some Möbius transformation, *M*, that sends the vertex *v* to ∞.

The circum-circles become lines.

The lines through *v* and v_i stay lines.



Willmore Flow

$$E(v) = \sum_{v_i \in N(v)} \beta_i - 2\pi$$

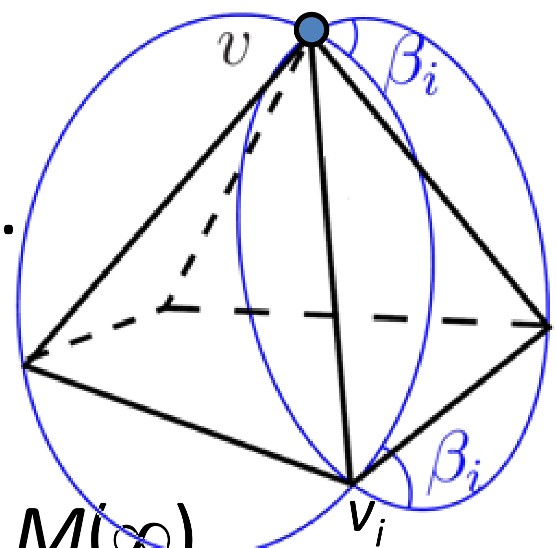
<u>Claim</u>:

Apply some Möbius transformation, *M*, that sends the vertex *v* to ∞.

The circum-circles become lines.

The lines through *v* and v_i stay lines.

The original lines through *v* and v_i intersected at ∞, so the new lines through *M*(*v*) and *M*(v_i) intersect at *M*(∞).

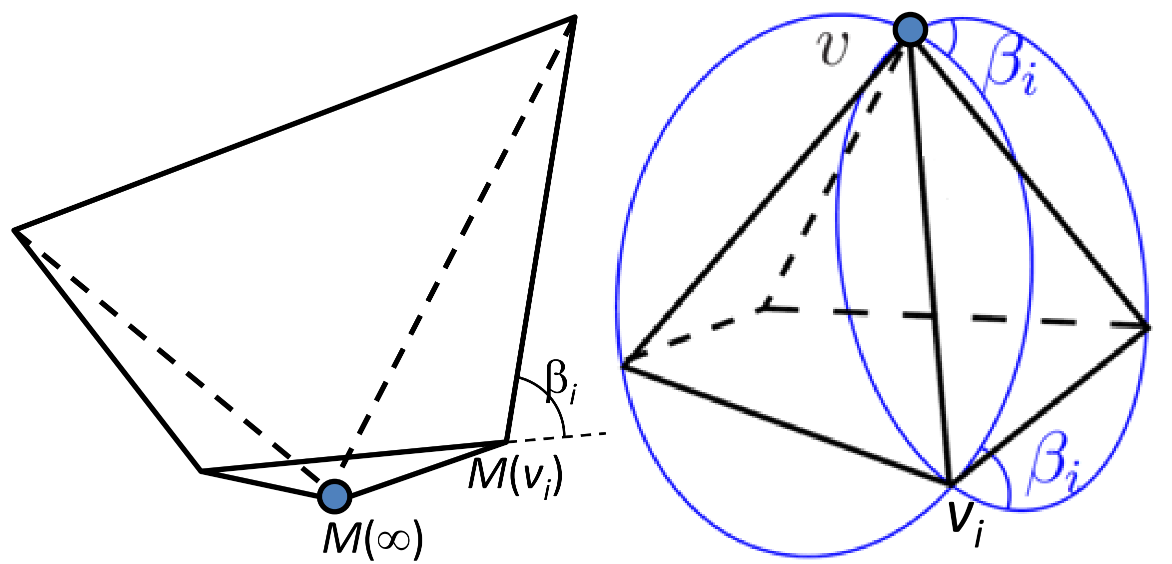


Willmore Flow

$$E(v) = \sum_{v_i \in N(v)} \beta_i - 2\pi$$

Claim:

Apply some Möbius transformation, *M*, that sends the vertex *v* to ∞.

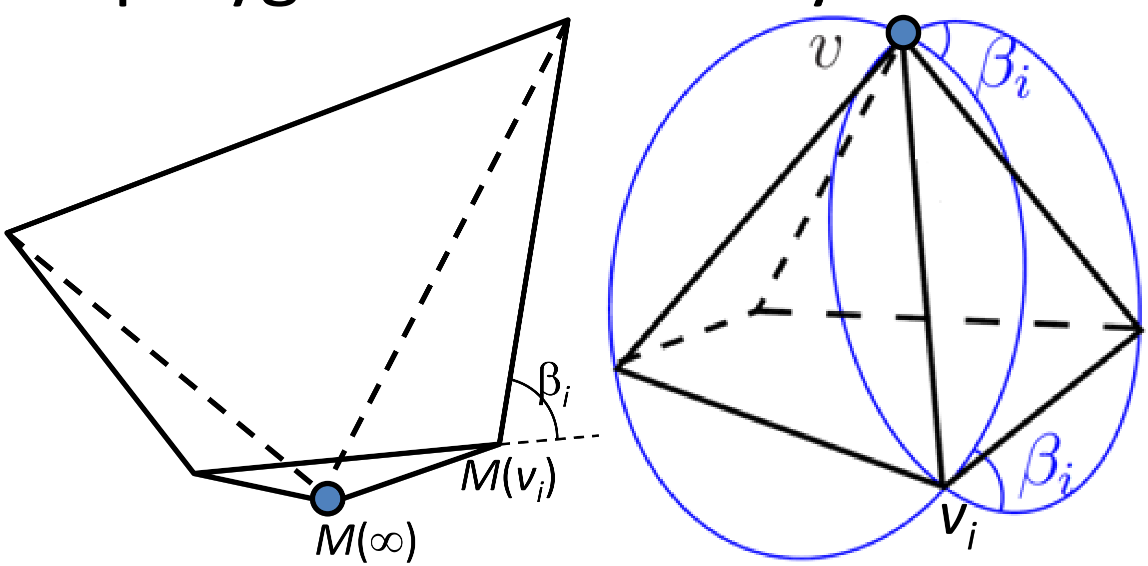


Willmore Flow

$$E(v) = \sum_{v_i \in N(v)} \beta_i - 2\pi$$

<u>Claim</u>:

But now we have the situation from the auxiliary lemma with polygon ***P*** defined by the vertices $M(v_i)$.

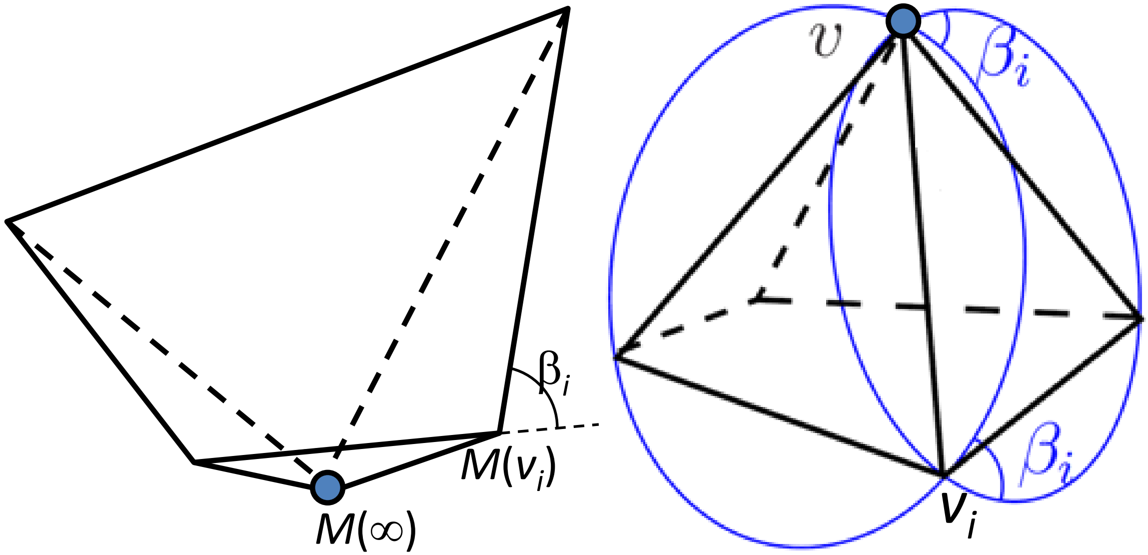


Willmore Flow

$$E(v) = \sum_{v_i \in N(v)} \beta_i - 2\pi$$

<u>Claim</u>:

Thus, we must have: $\sum_{v_i \in N(v)} \beta_i - 2\pi \geq 0$



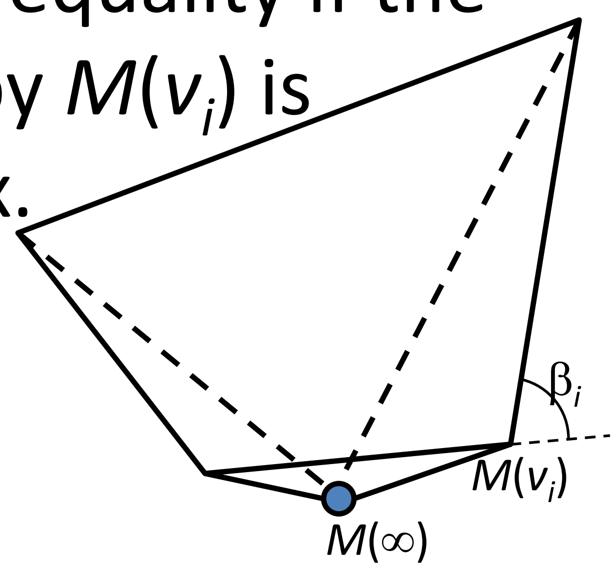
Willmore Flow

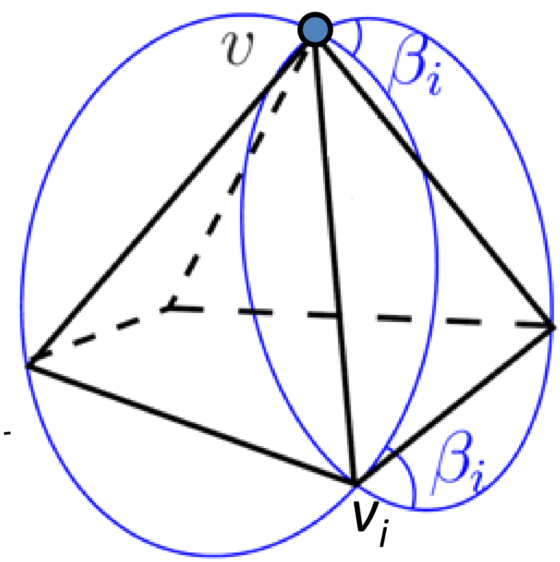
$$E(v) = \sum_{v_i \in N(v)} \beta_i - 2\pi$$

<u>Claim</u>:

Thus, we must have: $\sum_{v_i \in N(v)} \beta_i - 2\pi \geq 0$

We can only have equality if the polygon defined by $M(v_i)$ is planar and convex.





Willmore Flow

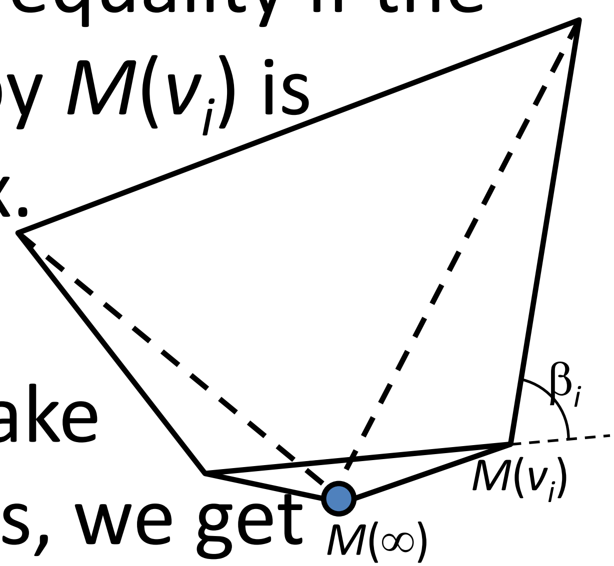
$$E(v) = \sum_{v_i \in N(v)} \beta_i - 2\pi$$

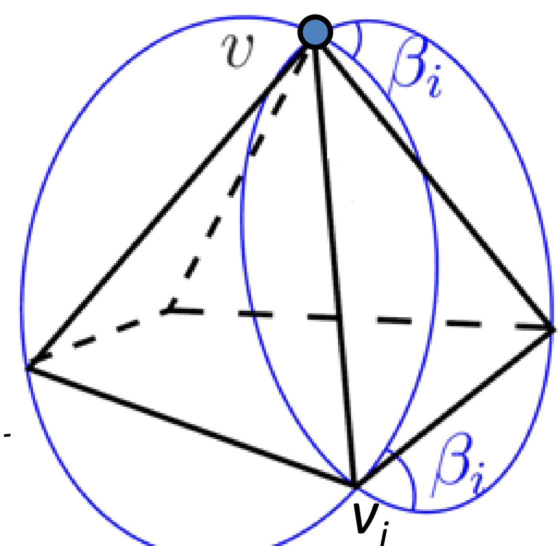
<u>Claim</u>:

Thus, we must have: $\sum_{v_i \in N(v)} \beta_i - 2\pi \geq 0$

We can only have equality if the polygon defined by $M(v_i)$ is planar and convex.

But since Möbius transformations take spheres to spheres, we get planarity only if the v_i live on a common sphere.





Willmore Flow

$$E(v) = \sum_{v_i \in N(v)} \beta_i - 2\pi$$

<u>Claim</u>:

Thus, we must have: $\sum_{v_i \in N(v)} \beta_i - 2\pi \geq 0$

We can only have equality if the polygon defined by $M(v_i)$ is planar and convex.

And we only get convexity if v and its one-ring are convex.