Differential Geometry:

Circle Patterns (Part 2)

[*Discrete Conformal Mappinngs via Circle Patterns*. Kharevych, Springborn and Schröder]

Preliminaries

Recall:

Given a smooth function F:$\boldsymbol{R}^n \rightarrow \boldsymbol{R}$, the function F has an extremum at x_0 only if $\nabla F(x_0)=0$.

Furthermore, if the Hessian of F is either strictly positive definite, or strictly negative definite, x_0 is the only extremum of F.

Preliminaries

<u>Optimization</u>:

Given a function f:$\boldsymbol{R}\rightarrow\boldsymbol{R}$, we can solve for x_0 such that $f(x_0)=a$, by optimizing the energy $E(x)$:

$$E(x) = F(x) - ax$$

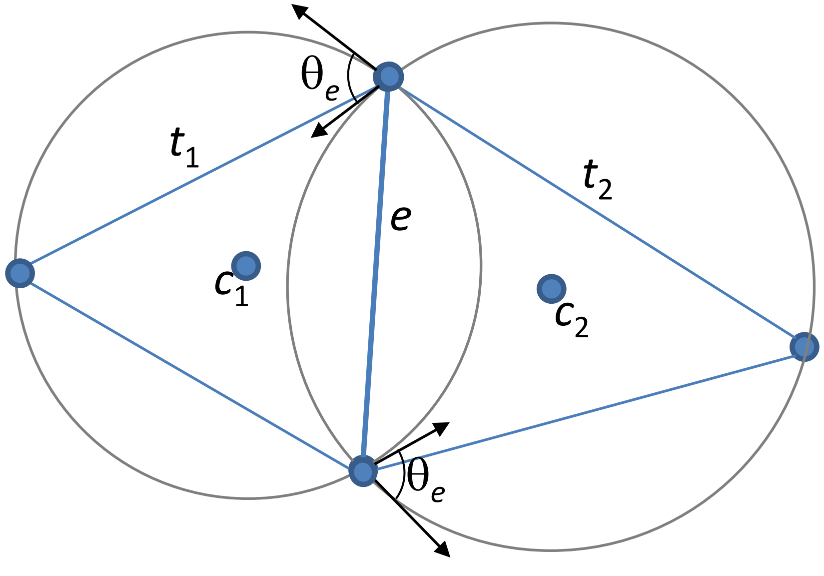
where F is the integral of f:

$$F(x) = \int_{-\infty}^{x} f(s)ds$$

Circle Patterns

Recall:

Given two triangles t_1, and t_2 sharing an edge e, we denote by θ_e the (exterior) intersection angle of the circumcircles.

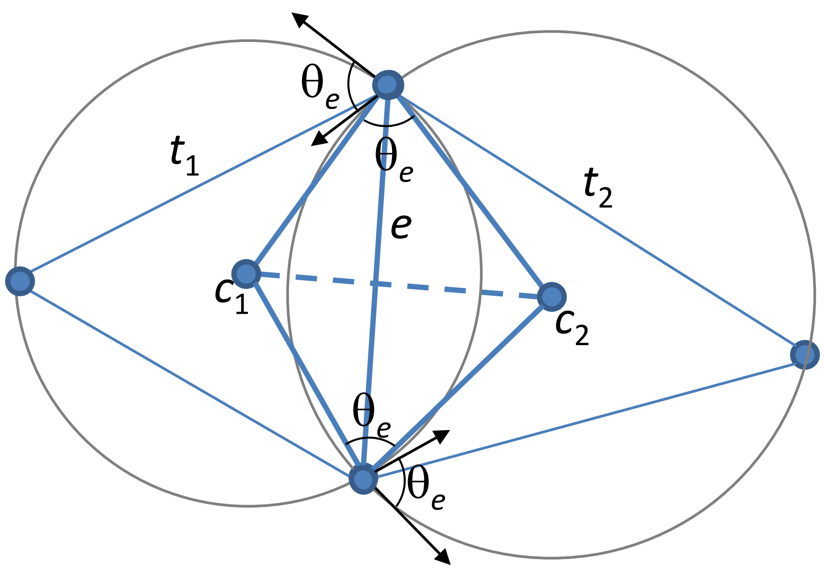


Circle Patterns

Recall:

Given two triangles t_1, and t_2 sharing an edge e, we denote by θ_e the (exterior) intersection angle of the circumcircles.

And we refer to the triangles defined by the edge e and the circumcenters as the *kite* of e.

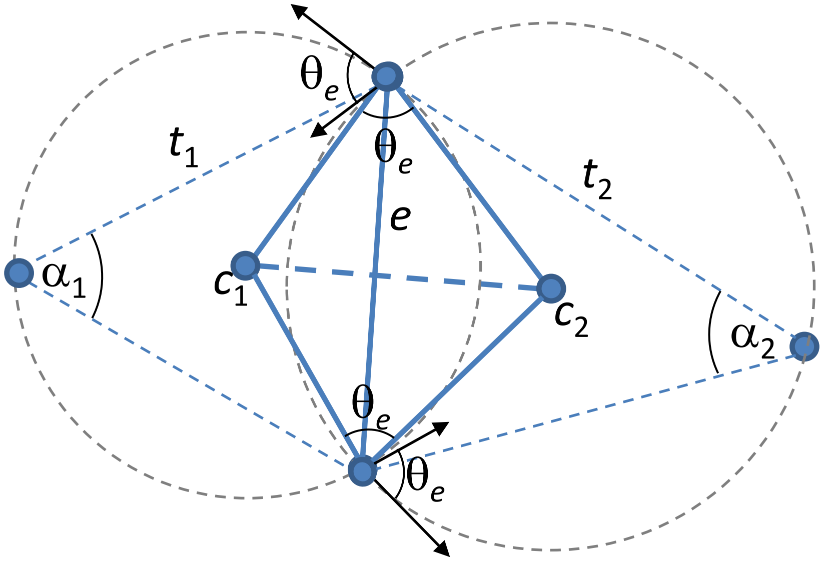


Circle Patterns

Recall:

The angle of intersection θ_e can be expressed in terms of the angles at the vertices opposite e:

$$\theta_e = \pi - \alpha_1 - \alpha_2$$



Circle Patterns

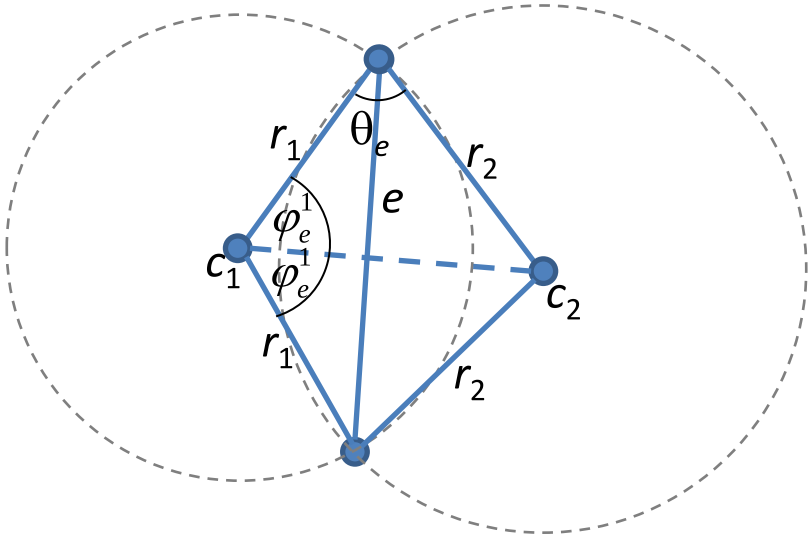
Recall:

If we know the angle θ_e and we know the radii of the circumcircles, we can figure out the other half-angles of the kite:

$$\phi_e^1 = \tan^{-1}\left(\frac{\sin\theta_e}{r_1 / r_2 - \cos\theta_e}\right)$$

and the length of e:

$$|e| = 2r_1 \sin\left(\phi_e^1\right)$$



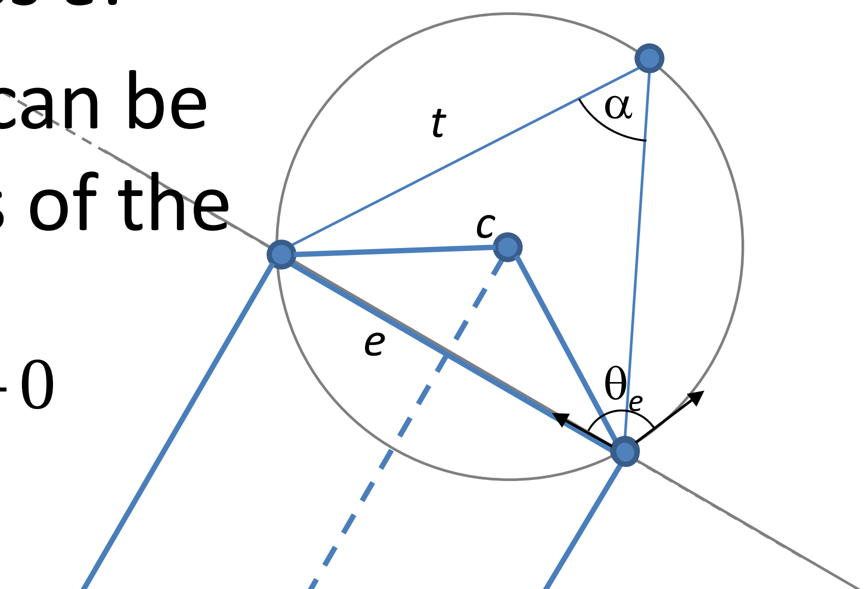
Circle Patterns

Recall:

In the case that $e \in t$ is a boundary edge, we can still define an angle θ_e by associating a phantom triangle with vertex infinitely far away as the neighbor of t across e.

Then the angle θ_e can be expressed in terms of the opposite angle(s):

$$\theta_e = \pi - \alpha - 0$$



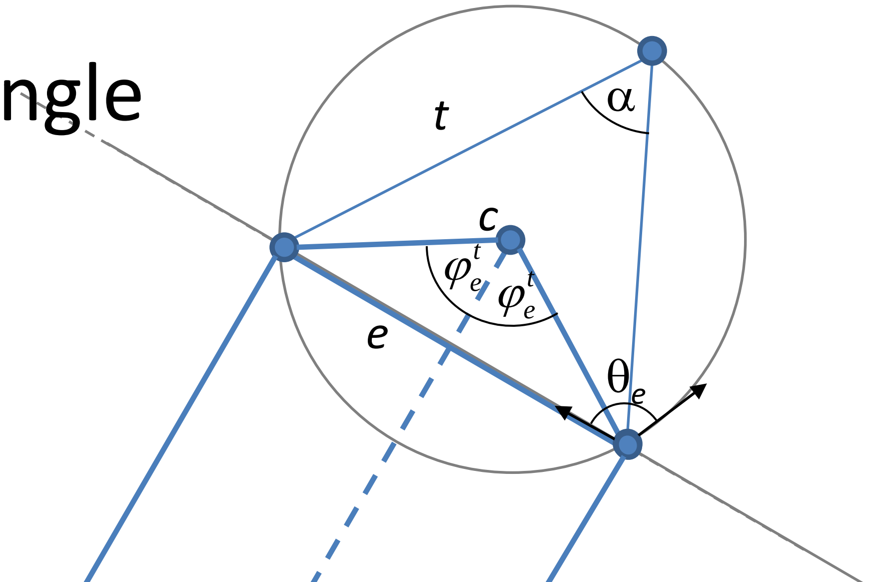
Circle Patterns

Recall:

In the case that $e \in t$ is a boundary edge, we can still define an angle θ_e by associating a phantom triangle with vertex infinitely far away as the neighbor of t across e.

And the kite half-angle becomes:

$$\varphi_e^t = \pi - \theta_e$$



Pattern Layout

Recall:

If we know the radius of each circumcircle in a valid circle pattern, we can lay-out the planar triangulation.

Knowing the radii means knowing edge-lengths.

So we can lay-out the triangulation by placing the first edge down along the *x*-axis and then successively placing down the vertices across from the edge.

Circle Patterns

<u>Goal</u>:

Given an abstract planar triangulation and values θ_e on the edges we want to find a circle pattern (assignment of radii) with the same combinatorics and angles of intersection θ_e.

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- Does such a circle pattern exist?
- If it does, is it unique?
- And, how do we find it?

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We will start with the last two questions and work our way backwards.

Circle Patterns

Approach:

Assuming that the circle pattern exists, we will show that it is unique and how to find it.

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Sufficient Condition: Then we will show that this energy has a positive definite Hessian, so there is only one minimum.

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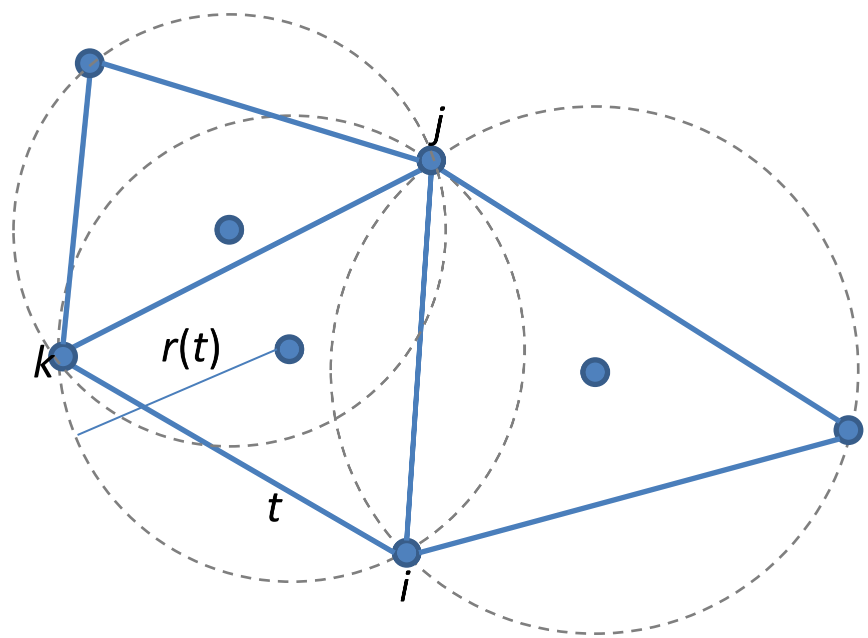
This kills two birds with one stone:

- It shows that the pattern is unique.
- And it tells us that we can get at it using gradient descent.

Building the Energy

Suppose that we have an assignment of radii to the triangles, $r(t)$ for $t \in T$, satisfying the angle constraints.



Building the Energy

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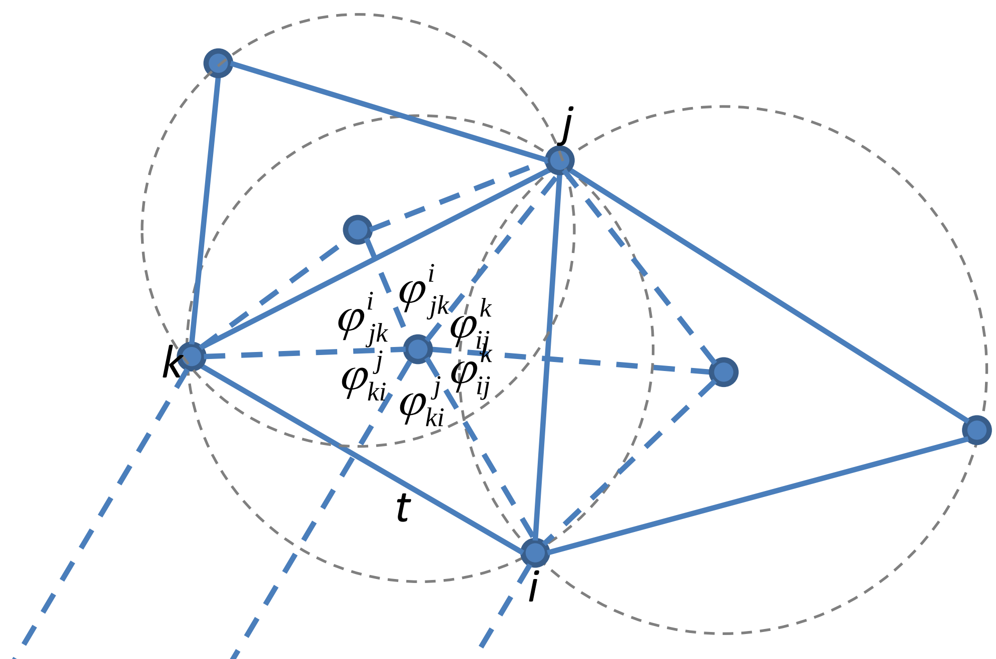
Then we can compute the kite half-angles.

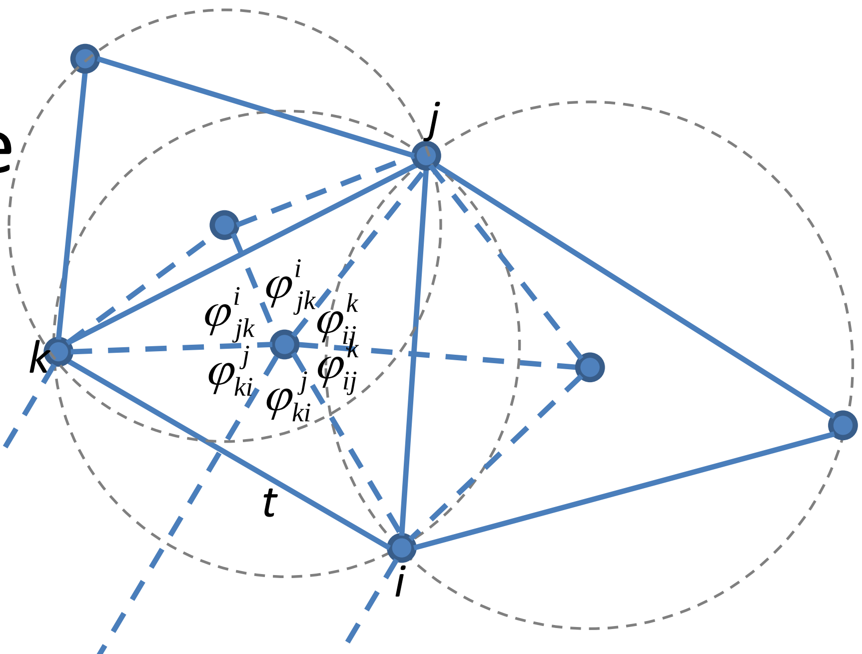


Building the Energy

Suppose that we have an assignment of radii to the triangles, $r(t)$ for $t \in T$, satisfying the angle constraints.
Then we can compute the kite half-angles.

<u>Necessary Condition</u>:
Since we assume that the circle pattern is valid, the angles must sum to 2π:

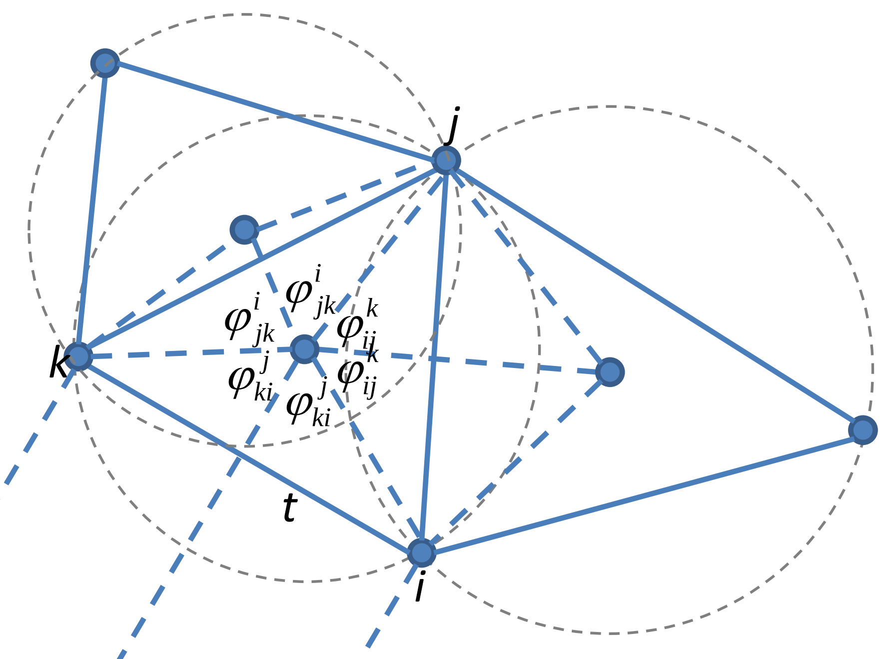
$$2\sum_{e \in t} \phi_e^t = 2\pi$$



Building the Energy

<u>Goal</u>:

Our goal will be to define an energy $E(\{r_t\})$ that is minimized precisely when the assignment of radii, $\{r_t\}$, has the property that the sum of the kite half-angles is 2π.

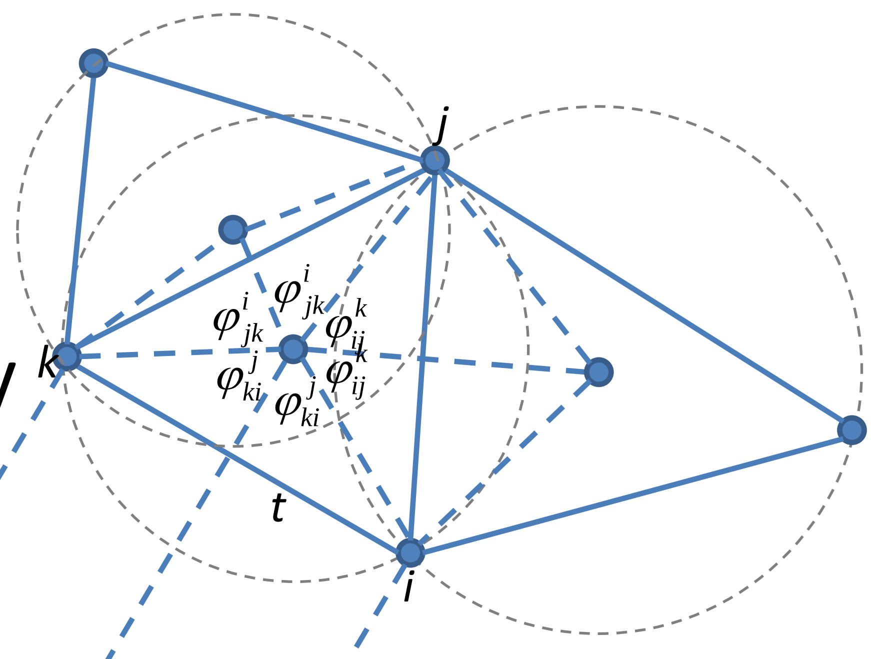


Building the Energy

Goal:

Our goal will be to define an energy $E(\{r_t\})$ that is minimized precisely when the assignment of radii, $\{r_t\}$, has the property that the sum of the kite half-angles is 2π.

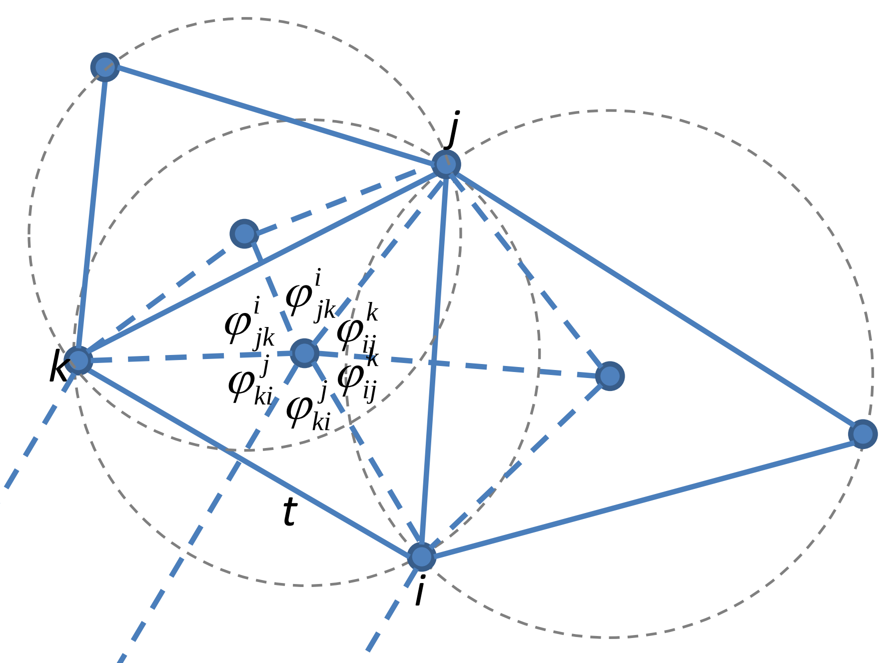
Although this is only a necessary condition, we will show that the energy has a unique minimizer.



Building the Energy

<u>Set-Up</u>:

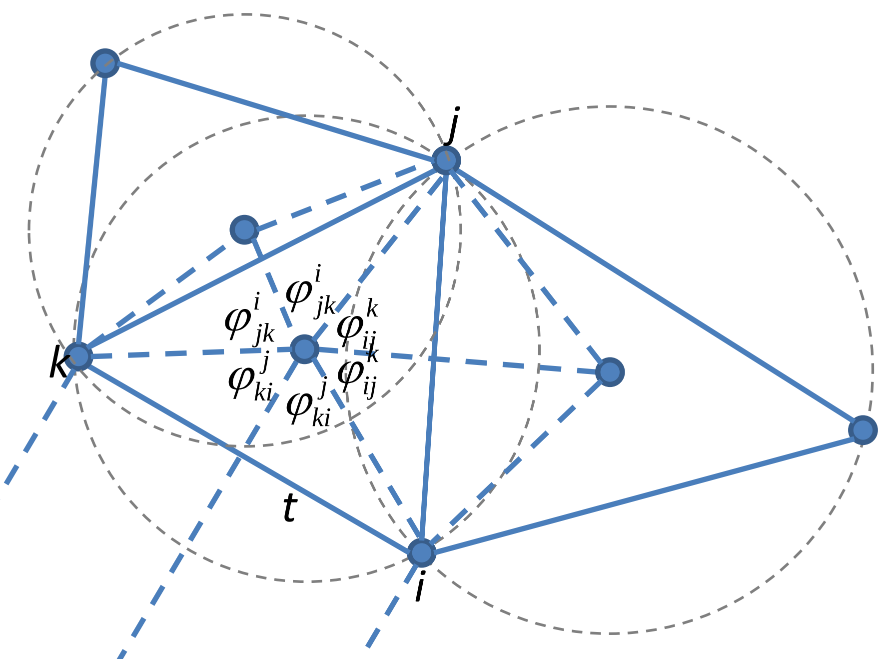
Instead of working with radii r_t, we will work with log-radii $\rho_t = \log(r_t)$.



Building the Energy

<u>Single Triangle Case</u>:

Assume that we have got all the log-radii right except for the log-radius of the triangle *t*.



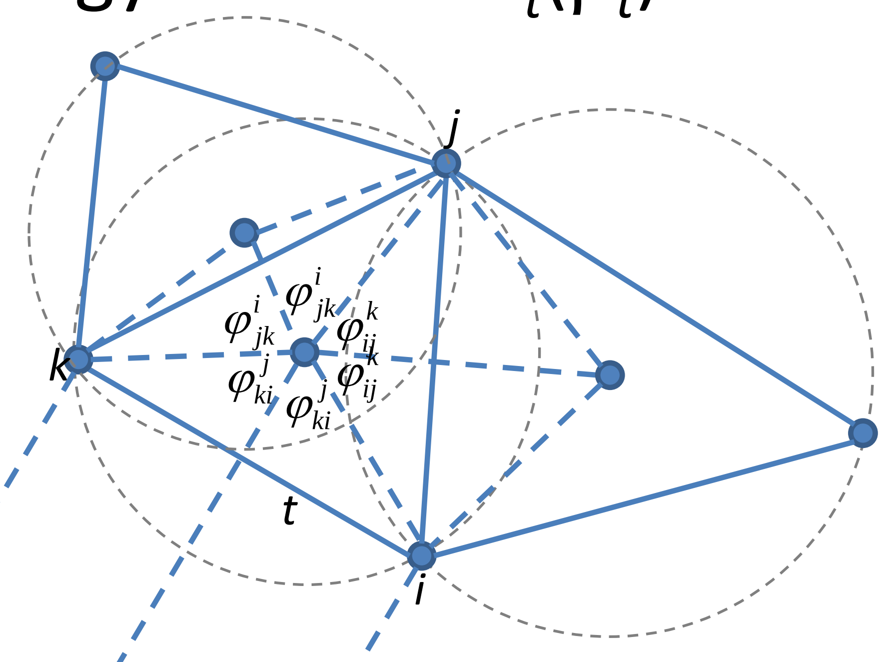
Building the Energy

Single Triangle Case:

Assume that we have got all the log-radii right except for the log-radius of the triangle *t*.

We need to define an energy function $E_t(\rho_t)$ that is minimized when the sum of half-kite angles in *t* equals 2π:

$$2\pi = 2\sum_{e\in t} \phi_e^t(\rho_t)$$

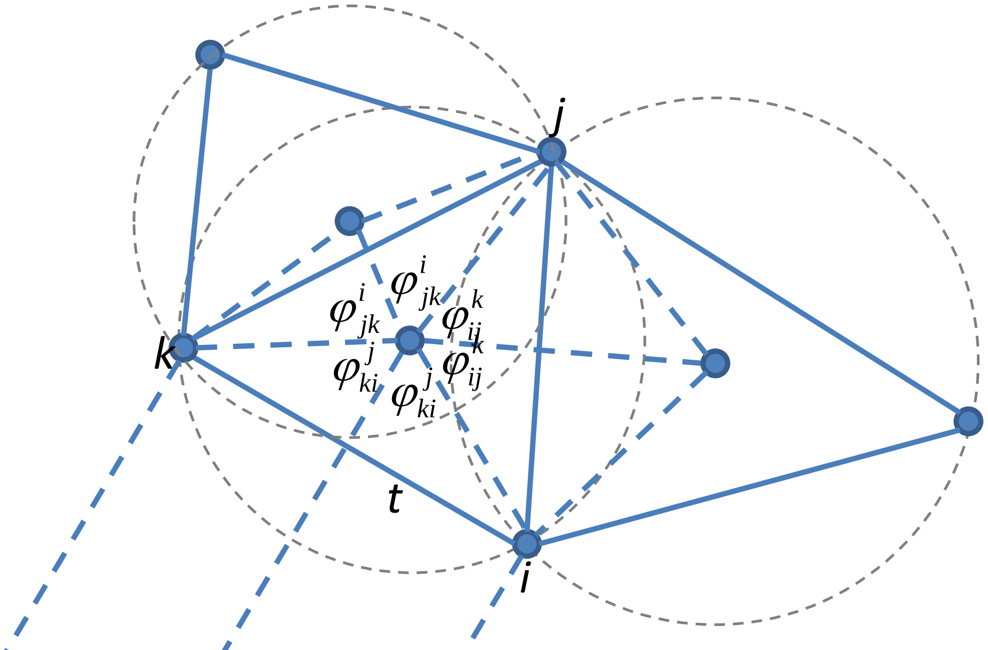


Building the Energy

<u>Recall</u>:

For an interior edge *e* adjacent to *t* and *t*':

$$\phi_e^t = \tan^{-1}\left(\frac{\sin\theta_e}{r_t / r_{t'} - \cos\theta_e}\right) = \tan^{-1}\left(\frac{\sin\theta_e}{e^{\rho_t - \rho_{t'}} - \cos\theta_e}\right)$$



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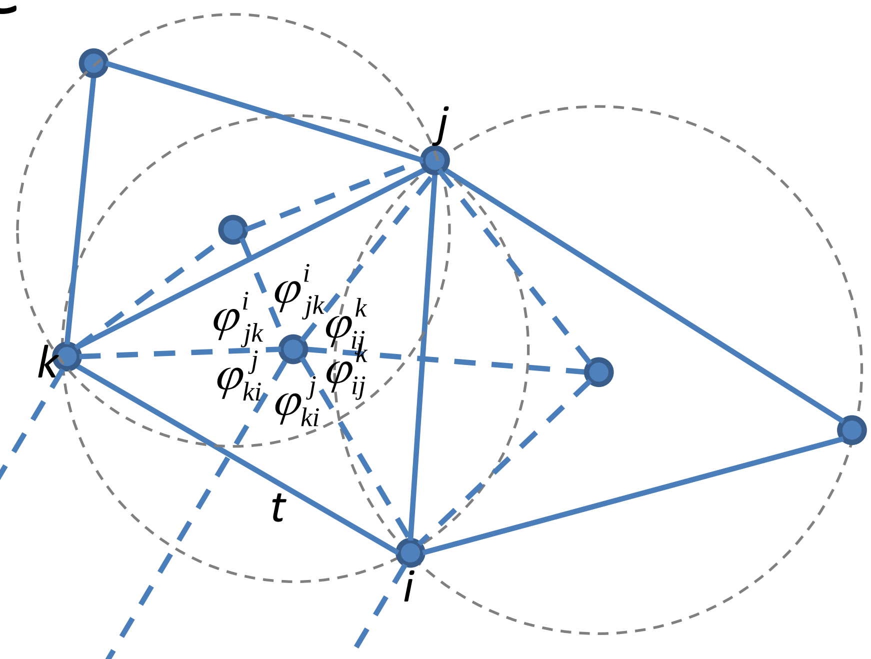
And, for a boundary edge *e* adjacent to *t*:

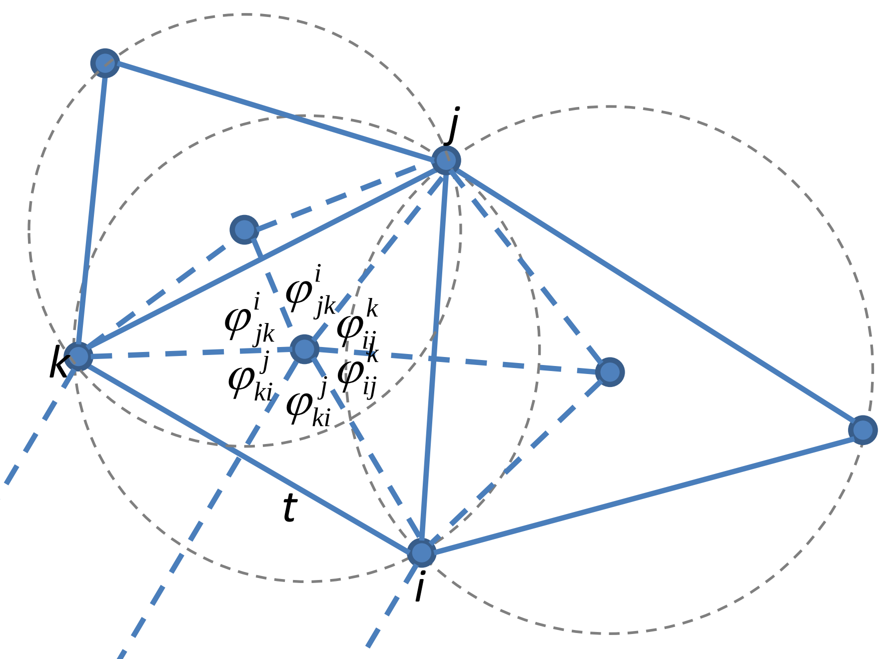
$$2\phi_e^t = 2\pi - 2\theta_e$$



Building the Energy

Thus, we can write out the constraint on the kite half-angles as:

$$0 = 2\pi - 2\sum_{e\in t, e\in E^\circ} \tan^{-1}\left(\frac{\sin\theta_e}{e^{\rho_t - \rho_{t'}} - \cos\theta_e}\right) - 2\sum_{e\in t, e\in \partial E} \pi - \theta_e$$



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Thus, we can obtain an energy for the triangle *t* by first computing the integral:

$$F_e(x) = \int_{-\infty}^{x} \tan^{-1}\left(\frac{\sin\theta_e}{e^{s} - \cos\theta_e}\right) ds$$

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And the energy function becomes:

$$E_t(\rho_t) = 2\pi\rho_t - 2\sum_{e\in t, e\in E^{\circ}} F_e(\rho_t - \rho_{t'}) - 2\sum_{e\in t, e\in \partial E} (\pi - \theta_e)\rho_t$$

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$$\phi_e^t = \begin{cases} \tan^{-1}\left(\dfrac{\sin\theta_e}{e^{\rho_t-\rho_{t'}} - \cos\theta_e}\right) & e \in E^\circ \\ \pi - \theta_e & e \in \partial E \end{cases}$$

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<u>Sanity Check</u>:

If we differentiate this energy, we get:

$$E_t'(\rho_t) = 2\pi - 2\sum_{e\in t, e\in E^\circ} F_e'(\rho_t - \rho_{t'}) - 2\sum_{e\in t, e\in\partial E} (\pi - \theta_e)$$

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Building the Energy

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$$= 2\pi - 2\sum_{e\in t, e\in E^\circ}\phi_e^t - 2\sum_{e\in t, e\in\partial E}\phi_e^t = 2\pi - 2\sum_{e\in t}\phi_e^t$$

Building the Energy

We would like to define the total energy over all the triangles as the sum of triangle energies:

$$E(\{\rho\}) = \sum_t E_t(\rho_t)$$

But this won't work.

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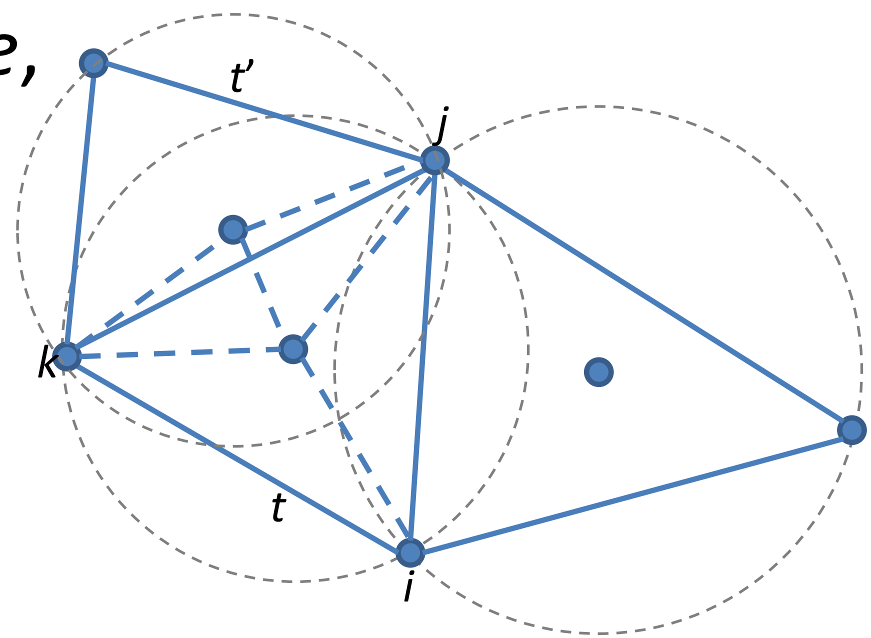
But this won't work.

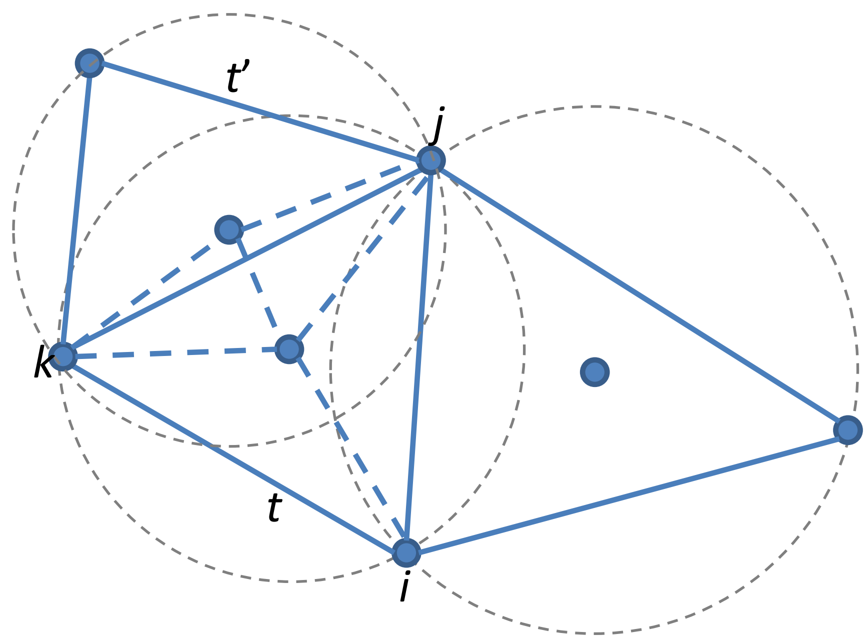
If *t* and *t'* share an edge *e*, then the energy that is contributed by triangle *t'* also depends on ρ_t.



Building the Energy

Now the component of the energy that is a function of the radius ρ_t is:

$$\underbrace{2\pi\rho_t - 2\sum_{e\in t, e\in E^\circ} F_e(\rho_t - \rho_{t'}) - 2\sum_{e\in t, e\in \partial E}(\pi-\theta_e)\rho_t}_{\text{Original (Desired) Energy}} \underbrace{- 2\sum_{e\in t, e\in E^\circ} F_e(\rho_{t'} - \rho_t)}_{\text{Energy from Neighbors}}$$



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Thus, if we differentiate the total energy with respect to ρ_t, we get:

$$\frac{\partial E}{\partial \rho_t} = 2\pi \underbrace{- 2\sum_{e\in t}\phi_e^t}_{\substack{\text{Desired}\\\text{Constraint}}} \underbrace{+ 2\sum_{e\in t, e\in E^\circ}\phi_e^{t'}}_{\substack{\text{Extra}\\\text{Stuff}}}$$

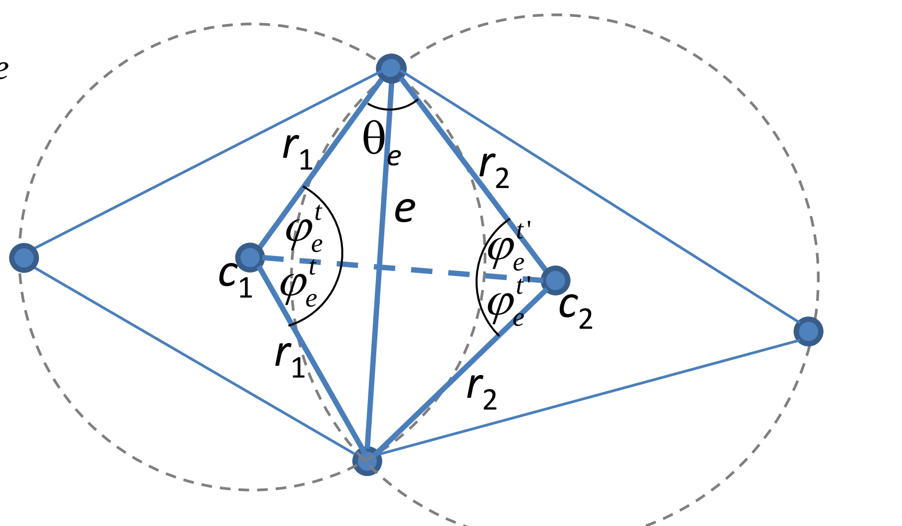
Building the Energy

$$\frac{\partial E}{\partial \rho_t} = 2\pi - 2\underbrace{\sum_{e \in t} \phi_e^t}_{\text{Desired Constraint}} + 2\underbrace{\sum_{e \in t, e \in E^\circ} \phi_e^{t'}}_{\text{Extra Stuff}}$$

To address, this we will tweak the energy a bit.

In particular, we will use the fact that for interior edges, the sum of the half-kite angles satisfies:

$$2\phi_e^t + 2\phi_e^{t'} = 2\pi - 2\theta_e$$



Building the Energy (Tweak 1)

Thus, if we modify the triangle energy:

$$E_t(\rho_t) = \underbrace{2\pi\rho_t - 2\sum_{e\in t, e\in E^\circ} F_e(\rho_t - \rho_{t'}) - 2\sum_{e\in t, e\in \partial E}(\pi - \theta_e)\rho_t}_{\text{Original (Desired) Energy}} + \underbrace{\sum_{e\in t, e\in E^\circ} F_e(\rho_t - \rho_{t'})}_{\text{Energy Adjustment}}$$

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the energy gradient of the total energy with respect to the log-radius at *t* becomes:

$$\frac{\partial E}{\partial \rho_t} = 2\pi - 2\sum_{e\in t}\phi_e^t + \sum_{e\in t, e\in E^\circ}\phi_e^t + 2\sum_{e\in t, e\in E^\circ}\phi_e^{t'} - \sum_{e\in t, e\in E^\circ}\phi_e^{t'}$$

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$$= 2\pi - 2\sum_{e\in t}\phi_e^t + \sum_{e\in t, e\in E^\circ}\left(\phi_e^t + \phi_e^{t'}\right)$$

Building the Energy (Tweak 1)

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the energy gradient of the total energy with respect to the log-radius at *t* becomes:

$$\begin{aligned}\frac{\partial E}{\partial \rho_t} &= 2\pi - 2\sum_{e\in t}\phi_e^t + \sum_{e\in t, e\in E^\circ}\phi_e^t + 2\sum_{e\in t, e\in E^\circ}\phi_e^{t'} - \sum_{e\in t, e\in E^\circ}\phi_e^{t'} \\ &= 2\pi - 2\sum_{e\in t}\phi_e^t + \sum_{e\in t, e\in E^\circ}\left(\phi_e^t + \phi_e^{t'}\right) \\ &= 2\pi - 2\sum_{e\in t}\phi_e^t + \sum_{e\in t, e\in E^\circ}\left(\pi - \theta_e\right)\end{aligned}$$

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the energy gradient of the total energy with respect to the log-radius at *t* becomes:

$$\frac{\partial E}{\partial \rho_t} = 2\pi - 2\sum_{e\in t}\phi_e^t + \sum_{e\in t, e\in E^\circ}(\pi - \theta_e)$$

We still don't have the constraint we want, but now we are off by a constant factor that is independent of the log-radii.

Building the Energy (Tweak 1)

<u>So Far</u>:

If we set the per-triangle energy to be:

$$E_t(\{\rho_t\}) = \underbrace{2\pi\rho_t - 2\sum_{e\in t, e\in E^\circ} F_e(\rho_t - \rho_{t'}) - 2\sum_{e\in t, e\in \partial E}(\pi - \theta_e)\rho_t}_{\text{Original (Desired) Energy}} + \underbrace{\sum_{e\in t, e\in E^\circ} F_e(\rho_t - \rho_{t'})}_{\text{Energy Adjustment}}$$

and we set the total energy to be the sum of the per-triangle energies:

$$E(\{\rho\}) = \sum_{t\in T} E_t(\{\rho\})$$

then the gradient becomes:

$$\frac{\partial E}{\partial \rho_t} = \underbrace{2\pi - 2\sum_{e\in t}\phi_e^t}_{\substack{\text{Desired}\\ \text{Constraint}}} + \underbrace{\sum_{e\in t, e\in E^\circ}(\pi - \theta_e)}_{\substack{\text{Extra}\\ \text{Stuff}}}$$

Building the Energy (Tweak 2)

To get rid of the last bit, we adjust the total energy term a bit more:

$$E(\{\rho\}) = \underbrace{\sum_{t \in T} E_t(\{\rho\})}_{\text{Per-Triangle Contribution}} - \underbrace{\sum_{e \in E^{\circ}} (\pi - \theta_e)(\rho_t + \rho_{t'})}_{\text{Adjustment Term}}$$

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Now, the total energy has the desired property:

$$\frac{\partial E}{\partial \rho_t} = 2\pi - 2\sum_{e\in t} \phi_e^t$$

so the energy is minimized only if we satisfy the planarity constraint on the kite half-angles.

The Energy Hessian

$$\frac{\partial E}{\partial \rho_t} = 2\pi - 2\sum_{e \in t, e \in E^{\circ}} \tan^{-1}\left(\frac{\sin\theta_e}{e^{\rho_t - \rho_{t'}} - \cos\theta_e}\right) - 2\sum_{e \in t, e \in \partial E} (\pi - \theta_e)$$

If we go through the messiness of differentiating the functions:

$$f_e(x) = \tan^{-1}\left(\frac{\sin\theta_e}{e^x - \cos\theta_e}\right)$$

we get:

$$f'_e(x) = \frac{-\sin\theta_e}{2[\cosh x - \cos\theta_e]}$$

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And the Hessian of the energy becomes:

$$\frac{\partial^2 E}{\partial \rho_t \partial \rho_t} = -2\sum_{e \in t, e \in E^\circ} f_e'(\rho_t - \rho_{t'}) \qquad \frac{\partial^2 E}{\partial \rho_t \partial \rho_{t'}} = 2\sum_{e \in t, e \in E^\circ} f_e'(\rho_t - \rho_{t'})$$

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$$v^t H(E)(\{\rho\})v = \sum_{e\in E^\circ} 2\frac{\partial^2 E}{\partial \rho_t \partial \rho_{t'}} v_t v_{t'} + \frac{\partial^2 E}{\partial \rho_t \partial \rho_t} v_t v_t + \frac{\partial^2 E}{\partial \rho_{t'} \partial \rho_{t'}} v_{t'} v_{t'}$$

The Energy Hessian

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$$= \sum_{e\in E^\circ} 4 f_e'(\rho_t - \rho_{t'}) v_t v_{t'} - 2 f_e'(\rho_t - \rho_{t'}) v_t v_t - 2 f_e'(\rho_t - \rho_{t'}) v_{t'} v_{t'}$$

The Energy Hessian

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The Energy Hessian

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$$= 2\sum_{e\in E^\circ} -\left(v_t - v_{t'}\right)^2 f_e'(\rho_t - \rho_{t'})$$

$$= \sum_{e\in E^\circ} \frac{\sin\theta_e}{\cosh(\rho_t - \rho_{t'}) - \cos\theta_e}\left(v_t - v_{t'}\right)^2$$

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So, if the original triangulation is locally Delaunay ($0<\theta_e<\pi$), the Hessian is never negative, and is only 0 when $v=\{c,\ldots,c\}$, i.e. we get no change in the gradient only if we offset all the log-radii by the same amount.

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That is, if we apply a uniform scale to the circle pattern.

Circle Patterns

Recap:

We know that if a circle pattern is valid then the sum of the kite-half angles has to add up to 2π.

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We know that if a circle pattern is valid then the sum of the kite-half angles has to add up to 2π.

We have designed an energy whose gradient vanishes whenever this condition is satisfied.

We have shown that the gradient of the energy can only vanish once.

Circle Patterns

<u>Recap</u>:

Thus, at most one circle pattern can satisfy the planarity condition and have the prescribed angles θ_e, and we can find it through gradient descent along the energy.

Circle Patterns

Thought Question:

In order to prove uniqueness, we used the existence of the function:

$$F_e(x) = \int_{-\infty}^{x} \tan^{-1}\left(\frac{\sin\theta_e}{e^s - \cos\theta_e}\right) ds$$

Do we ever need to know its value?

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In order to prove uniqueness, we used the existence of the function:

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Do we ever need to know its value?

If we did, we would find that:

$$F_e(x) = \operatorname{Im}\operatorname{Li}_2\left(e^{x+i\theta_e}\right)$$

where Li_2 is the dilogarithm function, defined for complex *z* as:

$$\operatorname{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}$$

Circle Patterns

<u>Question</u>:

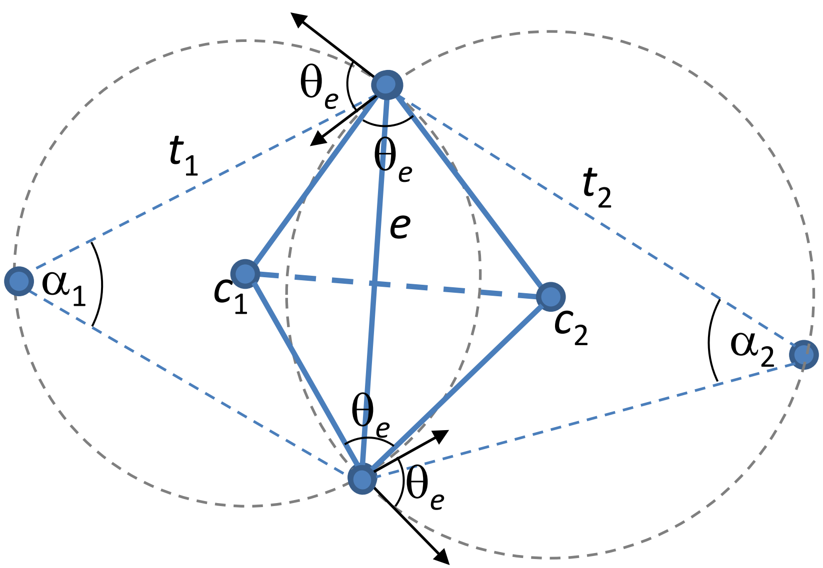
Given an abstract triangulation T (of a disk) and an assignment of angles θ_e to each edge, is there a circle pattern that has the combinatorics of T and defines the prescribed angles θ_e?

Circle Patterns

Necessary Properties of the Angles θ_e:

1. **Delaunay**: Since we are assuming that the triangulation is locally Delaunay, we know that on interior edges we have $\alpha_1+\alpha_2<\pi$, so:

$$0 < \theta_e < \pi$$

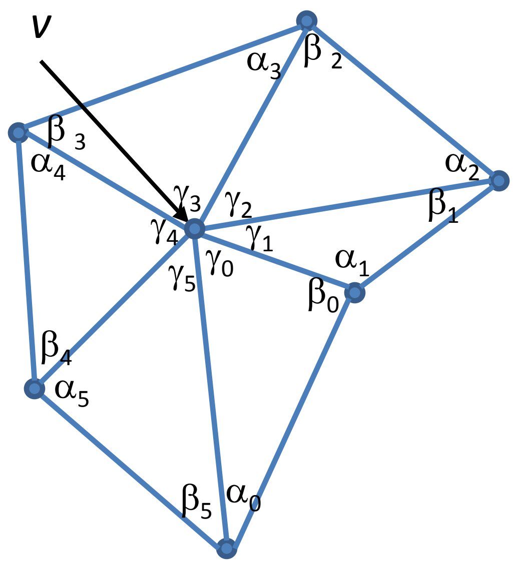


Circle Patterns

Necessary Properties of the Angles θ_e:

2. **Planarity**: For an interior vertex *v*, the sum of angles θ_e over all edges *e* adjacent to *v* is 2π:

$$\sum_{e \ni v} \theta_e = 2\pi \qquad \forall\, v \in V^\circ$$



$$\sum_{e \ni v} \theta_e = \sum_{i=0}^{n} \pi - \alpha_i - \beta_{i+1}$$

$$= \sum_{i=0}^{n} \pi - \alpha_i - \beta_i$$

$$= \sum_{i=0}^{n} \gamma_i = 2\pi$$

Circle Patterns

<u>Question</u>:

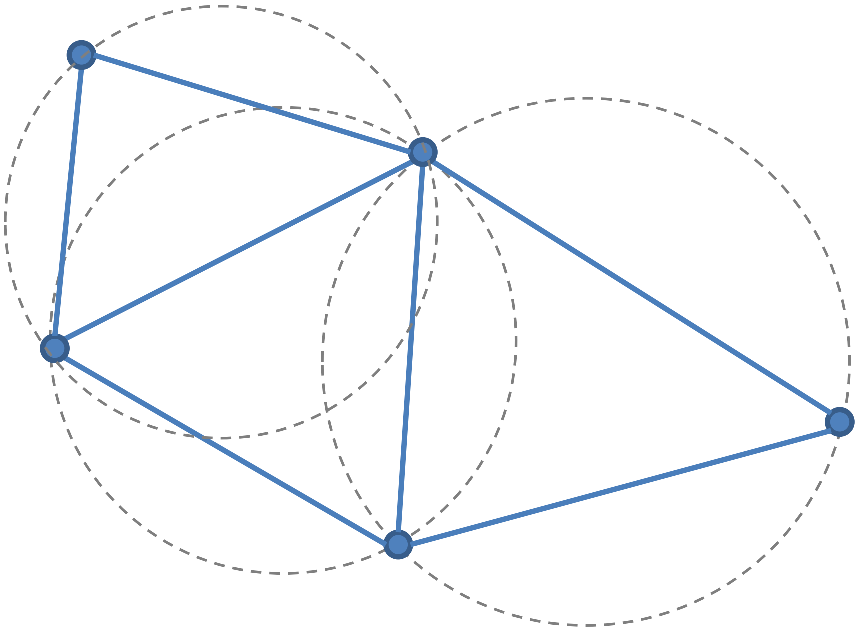
Given an abstract triangulation T (of a disk) and an assignment of angles θ_e to each edge, is there a circle pattern that has the combinatorics of T and defines the prescribed angles θ_e?

What if we assume that the constraint angles are reasonable (i.e. satisfy the Delaunay and Planarity condition)?

Circle Patterns

<u>Towards an Answer</u>:

If we can find a circle pattern, then that would give us a triangulation in the plane:

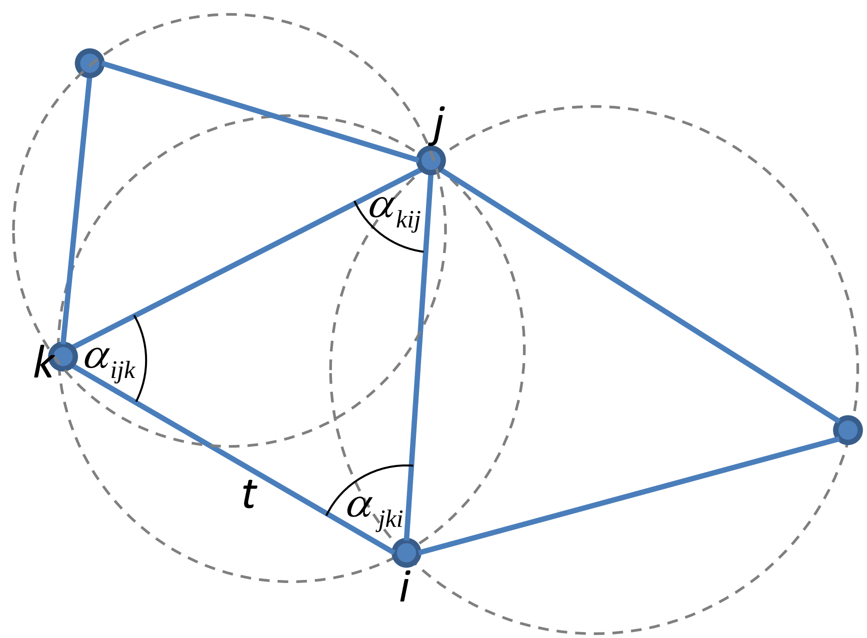


Circle Patterns

Towards an Answer:

If we can find a circle pattern, then that would give us a triangulation in the plane:
And each triangle $t=(i,j,k)$ would have interior angles that satisfy:

- $\alpha_{ijk}>0$, $\alpha_{ijk}+\alpha_{kij}+\alpha_{jki}=\pi$



Circle Patterns

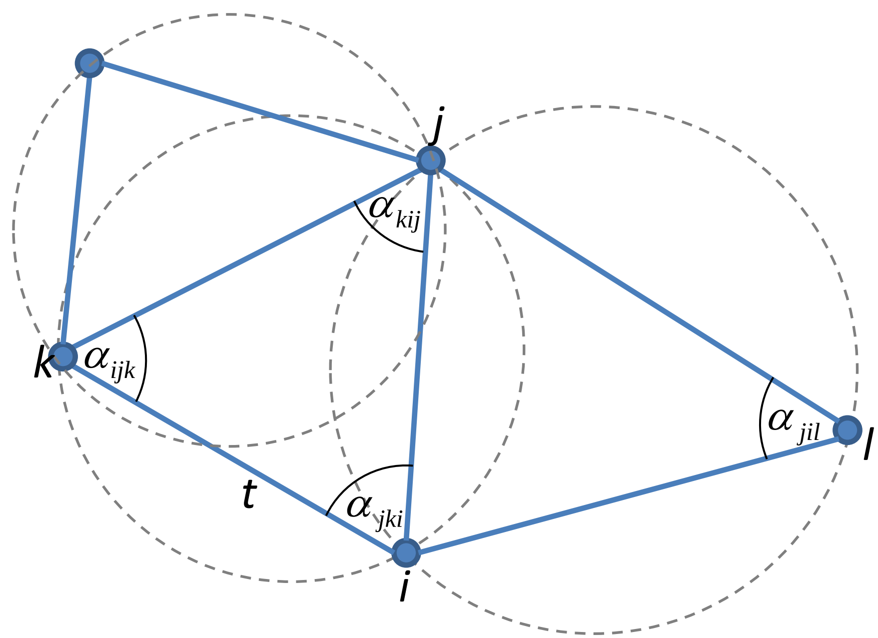
Towards an Answer:

If we can find a circle pattern, then that would give us a triangulation in the plane:
And each triangle $t=(i,j,k)$ would have interior angles that satisfy:

- $\alpha_{ijk}>0$, $\alpha_{ijk}+\alpha_{kij}+\alpha_{jki}=\pi$
- For interior edges:

  $\theta_{ij}=\pi-\alpha_{ijk}-\alpha_{jik}$
- For boundary edges:

  $\theta_{ik}=\pi-\alpha_{kij}$



Circle Patterns

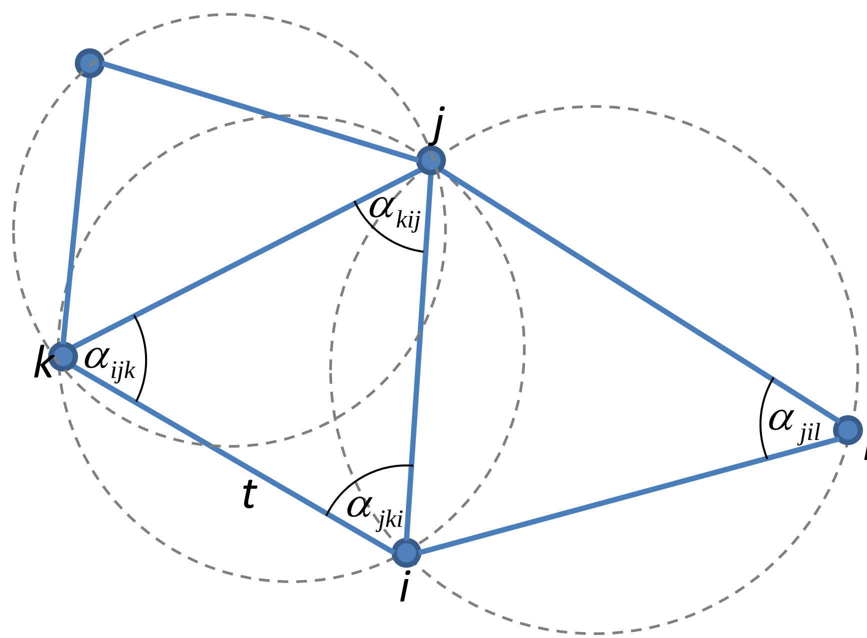
Definition:

Given a triangle mesh T and a set of angle constraints θ_e, we say that an assignment of angles α_{ijk} to the angles of the triangles in T is a *coherent angle system* if:

- $\alpha_{ijk}>0$, $\alpha_{ijk}+\alpha_{kij}+\alpha_{jki}=\pi$
- For interior edges:

  $\theta_{ij}=\pi- \alpha_{ijk}-\alpha_{jik}$
- For boundary edges:

  $\theta_{ik}=\pi- \alpha_{kij}$



Existence of Circle Patterns

Theorem [Bobenko and Springborn 2004]:

The circle pattern has a (unique up to scale) solution iff. a coherent angle system exists.

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This doesn’t imply that the angles in the angle system will be the angles of the pattern.

Existence of Circle Patterns

Theorem [Bobenko and Springborn 2004]:

The circle pattern has a (unique up to scale) solution iff. a coherent angle system exists.

Note:

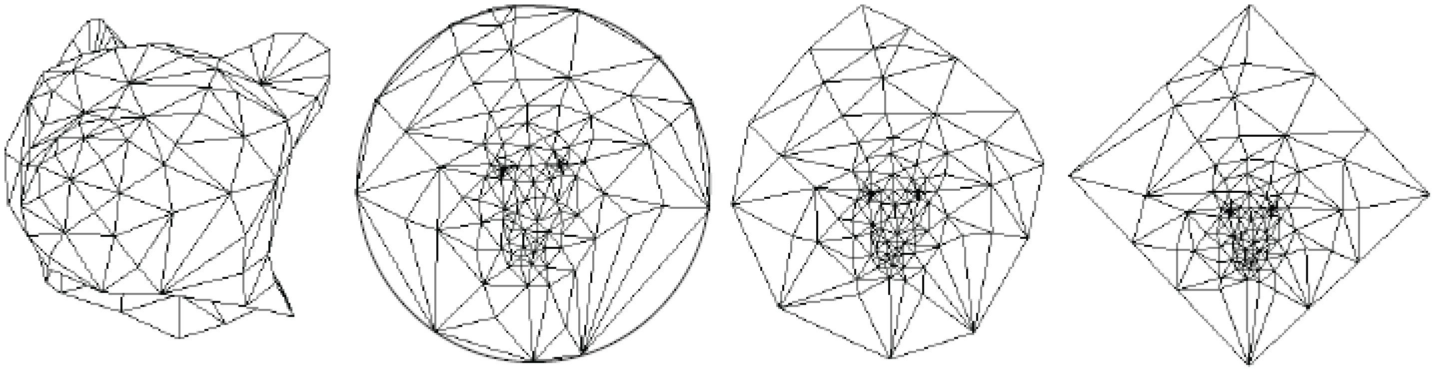
This doesn’t imply that the angles in the angle system will be the angles of the pattern.

A coherent angle system has $3|T|$ degrees of freedom, and $|e|+|T|\approx 2.5|T|$ constraints, so we expect many CAS’s for a set of edge weights θ_e.

Extending to Meshes

<u>Challenge</u>:

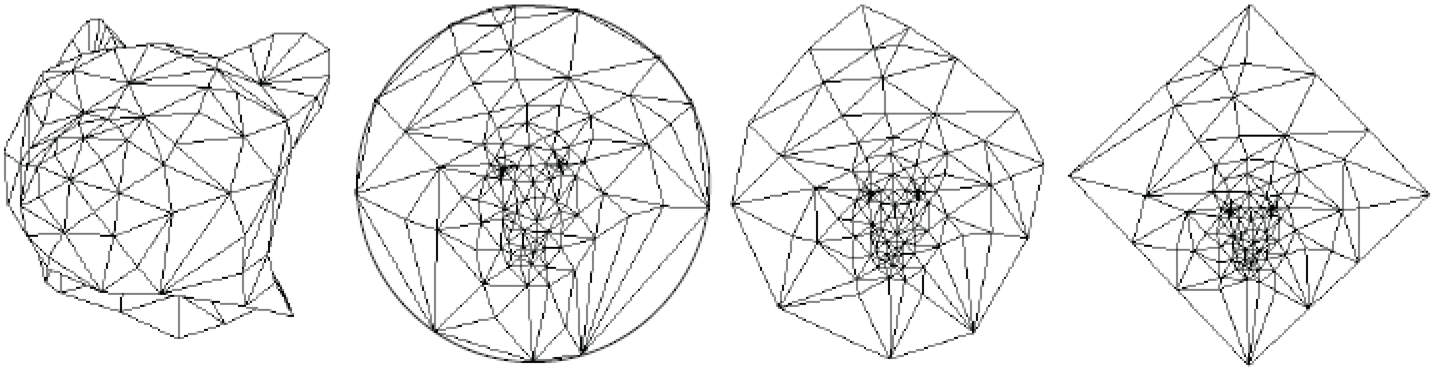
Given an arbitrary triangle mesh, the interior angles $\{\alpha_i\}$ do not necessarily define a coherent angle system.



Extending to Meshes

<u>Solution</u>:

Find a coherent angle system whose angles $\{\beta_i\}$ are as close as possible to the angles $\{\alpha_i\}$ in the original mesh.



Extending to Meshes

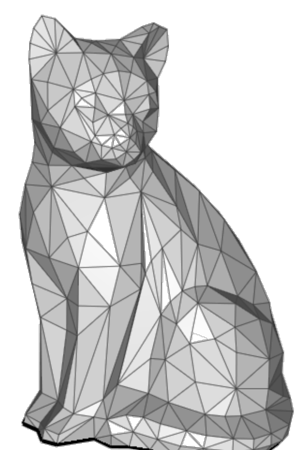
Solution:

This problem can be formulated as a quadratic minimization of:

$$Q(\{\beta\}) = \sum_{t \in T} \sum_{e \in t} \left| \alpha_e^t - \beta_e^t \right|^2$$

subject to:

- Positivity: For all angles $\beta_e^t > 0$
- Delaunay: For all interior edges $\beta_e^t + \beta_e^{t'} < \pi$
- Triangle Sum: For all triangles $\sum_{e \in t} \beta_e^t = 2\pi$
- Vertex Sum: For all interior verts. $\sum_{e \ni v} \beta_e^t = 2\pi$



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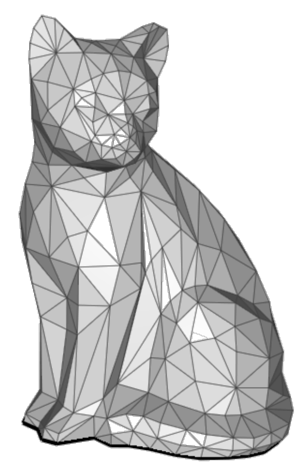
On boundary vertices, we can either place hard constraints prescribing curvature:

$$\sum_{e \ni v} \beta_e^t = \pi - \kappa_v$$

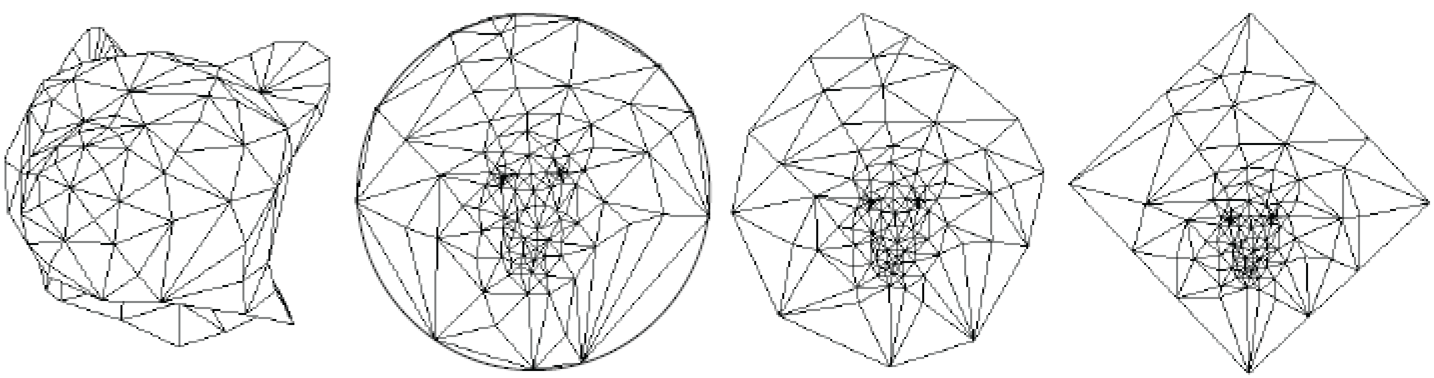
or place "natural" constraints:

$$\sum_{e \ni v} \beta_e^t < 2\pi$$



Extending to Meshes

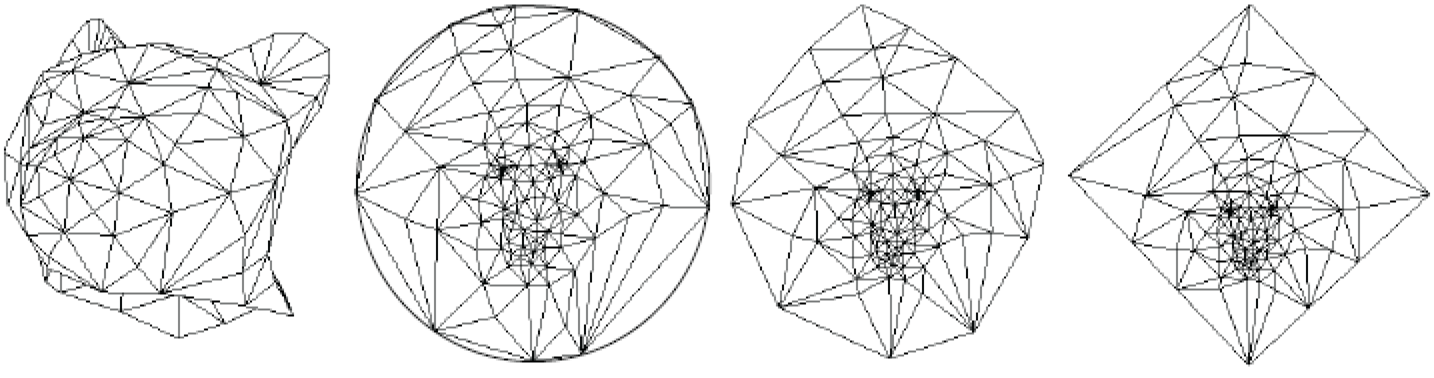
If the minimization problem is satisfiable, we get a valid angle system approximating the original angles and encoding the boundary constraints.



Extending to Meshes

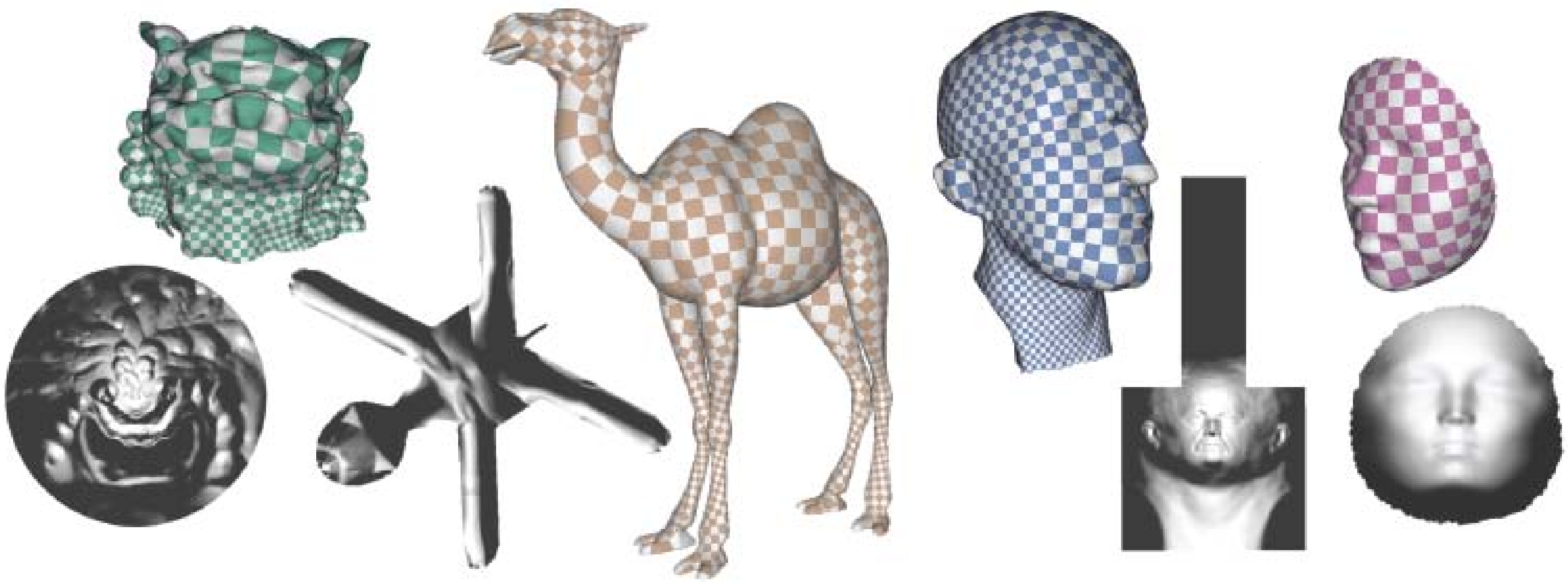
If the minimization problem is satisfiable, we get a valid angle system approximating the original angles and encoding the boundary constraints.

We can use these to define constraint angle θ_e and use gradient descent on the energy to get the circle pattern.



Extending to Meshes

This will define a mapping from the mesh into the plane which is very close to conformal.



Extending to Meshes

Note that the conformal map is defined with respect to angles θ_e that are defined from the fit angles $\{\beta\}$ and not from the mesh angles, so the "conformality" of the map will be tied to the closeness of the fit angles to the original angles.