Differential Geometry: Circle Patterns (Part 2)

[Discrete Conformal Mappinngs via Circle Patterns. Kharevych, Springborn and Schröder]
Recall:
Given a smooth function $F: \mathbb{R}^n \rightarrow \mathbb{R}$, the function $F$ has an extremum at $x_0$ only if $\nabla F(x_0) = 0$.
Furthermore, if the Hessian of $F$ is either strictly positive definite, or strictly negative definite, $x_0$ is the only extremum of $F$. 
Optimization:

Given a function $f: \mathbb{R} \rightarrow \mathbb{R}$, we can solve for $x_0$ such that $f(x_0) = a$, by optimizing the energy $E(x)$:

$$E(x) = F(x) - ax$$

where $F$ is the integral of $f$:

$$F(x) = \int_{-\infty}^{x} f(s)ds$$
Recall:
Given two triangles $t_1$ and $t_2$ sharing an edge $e$, we denote by $\theta_e$ the (exterior) intersection angle of the circumcircles.
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Given two triangles $t_1$, and $t_2$ sharing an edge $e$, we denote by $\theta_e$ the (exterior) intersection angle of the circumcircles.
And we refer to the triangles defined by the edge $e$ and the circumcenters as the *kite* of $e$. 
Recall:
The angle of intersection $\theta_e$ can be expressed in terms of the angles at the vertices opposite $e$:

$$\theta_e = \pi - \alpha_1 - \alpha_2$$
Circle Patterns

Recall:
If we know the angle $\theta_e$ and we know the radii of the circumcircles, we can figure out the other half-angles of the kite:

$$
\phi^1_e = \tan^{-1}\left(\frac{\sin \theta_e}{r_1 / r_2 - \cos \theta_e}\right)
$$

and the length of $e$:

$$
|e| = 2r_1 \sin(\phi^1_e)
$$
Recall:
In the case that $e \in t$ is a boundary edge, we can still define an angle $\theta_e$ by associating a phantom triangle with vertex infinitely far away as the neighbor of $t$ across $e$.

Then the angle $\theta_e$ can be expressed in terms of the opposite angle(s):

$$\theta_e = \pi - \alpha - 0$$
Recall:
In the case that $e \in t$ is a boundary edge, we can still define an angle $\theta_e$ by associating a phantom triangle with vertex infinitely far away as the neighbor of $t$ across $e$.
And the kite half-angle becomes:
$$\varphi_e^t = \pi - \theta_e$$
Pattern Layout

Recall:
If we know the radius of each circumcircle in a valid circle pattern, we can lay-out the planar triangulation.

Knowing the radii means knowing edge-lengths. So we can lay-out the triangulation by placing the first edge down along the $x$-axis and then successively placing down the vertices across from the edge.
Circle Patterns

Goal:
Given an abstract planar triangulation and values $\theta_e$ on the edges we want to find a circle pattern (assignment of radii) with the same combinatorics and angles of intersection $\theta_e$. 
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Given an abstract planar triangulation and values $\theta_e$ on the edges we want to find a circle pattern (assignment of radii) with the same combinatorics and angles of intersection $\theta_e$.

- Does such a circle pattern exist?
- If it does, is it unique?
- And, how do we find it?
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We will start with the last two questions and work our way backwards.
Circle Patterns

Approach:
Assuming that the circle pattern exists, we will show that it is unique and how to find it.
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**Necessary Condition:** We will define an energy on the set of circle patterns which has to be minimized if the assignment of radii is planar.
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Assuming that the circle pattern exists, we will show that it is unique and how to find it.

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**Sufficient Condition:** Then we will show that this energy has a positive definite Hessian, so there is only one minimum.
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Assuming that the circle pattern exists, we will show that it is unique and how to find it.

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Sufficient Condition: Then we will show that this energy has a positive definite Hessian, so there is only one minimum.

This kills two birds with one stone:
• It shows that the pattern is unique.
• And it tells us that we can get at it using gradient descent.
Suppose that we have an assignment of radii to the triangles, $r(t)$ for $t \in T$, satisfying the angle constraints.
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Building the Energy

Suppose that we have an assignment of radii to the triangles, $r(t)$ for $t \in T$, satisfying the angle constraints.

Then we can compute the kite half-angles.

**Necessary Condition:**
Since we assume that the circle pattern is valid, the angles must sum to $2\pi$:

$$2 \sum_{e \in t} \phi_e^t = 2\pi$$
Building the Energy

Goal:
Our goal will be to define an energy $E(\{r_t\})$ that is minimized precisely when the assignment of radii, $\{r_t\}$, has the property that the sum of the kite half-angles is $2\pi$. 
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Our goal will be to define an energy $E(\{r_t\})$ that is minimized precisely when the assignment of radii, $\{r_t\}$, has the property that the sum of the kite half-angles is $2\pi$.
Although this is only a necessary condition, we will show that the energy has a unique minimizer.
Building the Energy

Set-Up:
Instead of working with radii $r_t$, we will work with log-radii $\rho_t = \log(r_t)$. 
Building the Energy

Single Triangle Case:
Assume that we have got all the log-radii right except for the log-radius of the triangle $t$. 

![Diagram of a triangle with vertices labeled i, j, and k, and edges labeled with function symbols like $\phi^i_{jk}$ and $\phi^i_{kj}$]
Building the Energy

Single Triangle Case:
Assume that we have got all the log-radii right except for the log-radius of the triangle $t$.
We need to define an energy function $E_t(\rho_t)$ that is minimized when the sum of half-kite angles in $t$ equals $2\pi$:

$$2\pi = 2\sum_{e \in t} \phi^e_t(\rho_t)$$
Building the Energy

Recall:
For an interior edge $e$ adjacent to $t$ and $t'$:

$$\phi_e^t = \tan^{-1}\left(\frac{\sin \theta_e}{r_t / r_{t'} - \cos \theta_e}\right) = \tan^{-1}\left(\frac{\sin \theta_e}{e^{\rho_t - \rho_{t'} - \cos \theta_e}}\right)$$
Building the Energy

Recall:

For an interior edge $e$ adjacent to $t$ and $t'$:

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$$

And, for a boundary edge $e$ adjacent to $t$:

$$
2\phi_e^t = 2\pi - 2\theta_e
$$
Building the Energy

Thus, we can write out the constraint on the kite half-angles as:

\[ 0 = 2\pi - 2 \sum_{e \in t, e \in E^k} \tan^{-1}\left(\frac{\sin \theta_e}{e^\rho - \rho} - \cos \theta_e\right) - 2 \sum_{e \in t, e \in \partial E} \pi - \theta_e \]
Building the Energy

Thus, we can write out the constraint on the kite half-angles as:

\[ 0 = 2\pi - 2 \sum_{e \in t, e \in \partial E} \tan^{-1}\left( \frac{\sin \theta_e}{e^{\rho_i - \rho_i'} - \cos \theta_e} \right) - 2 \sum_{e \in t, e \in \partial E} \pi - \theta_e \]

Thus, we can obtain an energy for the triangle \( t \) by first computing the integral:

\[ F_e(x) = \int_{-\infty}^{x} \tan^{-1}\left( \frac{\sin \theta_e}{e^s - \cos \theta_e} \right) ds \]
Building the Energy

Thus, we can write out the constraint on the kite half-angles as:

\[ 0 = 2\pi - 2 \sum_{e \in t, e \in E^o} \tan^{-1}\left( \frac{\sin \theta_e}{e^{\rho_t - \rho_t'} - \cos \theta_e} \right) - 2 \sum_{e \in t, e \in \partial E} \pi - \theta_e \]

Thus, we can obtain an energy for the triangle \( t \) by first computing the integral:

\[ F_e(x) = \int_{-\infty}^{x} \tan^{-1}\left( \frac{\sin \theta_e}{e^{s} - \cos \theta_e} \right) ds \]

And the energy function becomes:

\[ E_t(\rho_t) = 2\pi \rho_t - 2 \sum_{e \in t, e \in E^o} F_e(\rho_t - \rho_t') - 2 \sum_{e \in t, e \in \partial E} (\pi - \theta_e) \rho_t \]
Building the Energy

If we differentiate this energy, we get:

\[
E'_t(\rho_t) = 2\pi - 2 \sum_{e \in t, e \in E^0} F'_e(\rho_t - \rho_{t'}) - 2 \sum_{e \in t, e \in \partial E} (\pi - \theta_e) \rho_t
\]
Building the Energy

\[ \phi_e^t = \begin{cases} \tan^{-1}\left( \frac{\sin \theta_e}{e^{\rho_t - \rho_{t'}} - \cos \theta_e} \right) & e \in E^o \\ \pi - \theta_e & e \in \partial E \end{cases} \]

\[ F_e(x) = \int_{-\infty}^{x} \tan^{-1}\left( \frac{\sin \theta_e}{e^s - \cos \theta_e} \right) ds \]

\[ E_t(\rho_t) = 2\pi \rho_t - 2 \sum_{e \in t, e \in E^o} F_e(\rho_t - \rho_{t'}) - 2 \sum_{e \in t, e \in \partial E} (\pi - \theta_e) \rho_t \]

Sanity Check:

If we differentiate this energy, we get:

\[ E_t'(\rho_t) = 2\pi - 2 \sum_{e \in t, e \in E^o} F_e'(\rho_t - \rho_{t'}) - 2 \sum_{e \in t, e \in \partial E} (\pi - \theta_e) \]

\[ = 2\pi - 2 \sum_{e \in t, e \in E^o} \tan^{-1}\left( \frac{\sin \theta_e}{e^{\rho_t - \rho_{t'}} - \cos \theta_e} \right) - 2 \sum_{e \in t, e \in \partial E} (\pi - \theta_e) \]
Building the Energy

\[ \phi_e^t = \begin{cases} \tan^{-1}\left(\frac{\sin \theta_e}{e^{\rho_t - \rho_t} - \cos \theta_e}\right) & e \in E^\circ \\ \pi - \theta_e & e \in \partial E \end{cases} \]

\[ F_e(x) = \int_{-\infty}^{x} \tan^{-1}\left(\frac{\sin \theta_e}{e^s - \cos \theta_e}\right) ds \]

\[ E_t(\rho_t) = 2\pi \rho_t - 2 \sum_{e \in t, e \in E^\circ} F_e(\rho_t - \rho_t) - 2 \sum_{e \in t, e \in \partial E} (\pi - \theta_e) \rho_t \]

**Sanity Check:**

If we differentiate this energy, we get:

\[ E_t'(\rho_t) = 2\pi - 2 \sum_{e \in t, e \in E^\circ} F_e'(\rho_t - \rho_t') - 2 \sum_{e \in t, e \in \partial E} (\pi - \theta_e) - 2 \sum_{e \in t, e \in \partial E} \sum_{e \in E} \phi_e^t - 2 \sum_{e \in t, e \in \partial E} \phi_e^t = 2\pi - 2 \sum_{e \in t} \phi_e^t \]
Building the Energy

We would like to define the total energy over all the triangles as the sum of triangle energies:

\[ E(\{\rho\}) = \sum_t E_t(\rho_t) \]

But this won’t work.
Building the Energy

We would like to define the total energy over all the triangles as the sum of triangle energies:

$$E(\{\rho\}) = \sum_t E_t(\rho_t)$$

But this won’t work.

If $t$ and $t'$ share an edge $e$, then the energy that is contributed by triangle $t'$ also depends on $\rho_t$. 
Building the Energy

Now the component of the energy that is a function of the radius $\rho_t$ is:

$$2\pi \rho_t - 2 \sum_{e \in t, e \in E^o} F_e(\rho_t - \rho_{t'}) - 2 \sum_{e \in t, e \in \partial E} (\pi - \theta_e) \rho_t - 2 \sum_{e \in t, e \in E^o} F_e(\rho_{t'} - \rho_t)$$

Original (Desired) Energy

Energy from Neighbors
Building the Energy

Now the component of the energy that is a function of the radius $\rho_t$ is:

$$2\pi\rho_t - 2 \sum_{e \in t, e \in E^\circ} F_e(\rho_t - \rho_t') - 2 \sum_{e \in t, e \in \partial E} (\pi - \theta_e)\rho_t - 2 \sum_{e \in t, e \in E^\circ} F_e(\rho_t' - \rho_t)$$

Thus, if we differentiate the total energy with respect to $\rho_t$, we get:

$$\frac{\partial E}{\partial \rho_t} = 2\pi - 2 \sum_{e \in t} \phi_e + 2 \sum_{e \in t, e \in E^\circ} \phi_e'$$
Building the Energy

\[ \frac{\partial E}{\partial \rho_t} = 2\pi - 2 \sum_{e \in t} \phi^t_e + 2 \sum_{e \in t, e \in E^o} \phi^{t'}_e \]

Desired Constraint
Extra Stuff

To address, this we will tweak the energy a bit.
In particular, we will use the fact that for interior edges, the sum of the half-kite angles satisfies:

\[ 2\phi^t_e + 2\phi^{t'}_e = 2\pi - 2\theta_e \]
Thus, if we modify the triangle energy:

\[
E_t(\rho_t) = 2\pi \rho_t - 2 \sum_{e \in t, e \in E^o} F_e(\rho_t - \rho_{t'}) - 2 \sum_{e \in t, e \in \partial E} (\pi - \theta_e) \rho_t + \sum_{e \in t, e \in E^o} F_e(\rho_t - \rho_{t'})
\]

Original (Desired) Energy

Energy Adjustment
Thus, if we modify the triangle energy:

\[
E_t(\rho_t) = 2\pi\rho_t - 2 \sum_{e\in t, e\in E^o} F_e(\rho_t - \rho_{t'}) - 2 \sum_{e\in t, e\in \partial E} (\pi - \theta_e)\rho_t + \sum_{e\in t, e\in E^o} F_e(\rho_t - \rho_{t'})
\]

the energy gradient of the total energy with respect to the log-radius at \( t \) becomes:

\[
\frac{\partial E}{\partial \rho_t} = 2\pi - 2\sum_{e\in t} \phi_e^t + \sum_{e\in t, e\in E^o} \phi_e^t + 2\sum_{e\in t, e\in E^o} \phi_e^{t'} - \sum_{e\in t, e\in E^o} \phi_e^{t'}
\]
Building the Energy (Tweak 1)

Thus, if we modify the triangle energy:

\[
E_t(\rho_t) = 2\pi \rho_t - 2 \sum_{e \in t, e \in E}\frac{F_e(\rho_t - \rho_{t'})}{e \in t, e \in \partial E} - 2 \sum_{e \in t, e \in E^\circ}(\pi - \theta_e)\rho_t + \sum_{e \in t, e \in E^\circ}F_e(\rho_t - \rho_{t'})
\]

the energy gradient of the total energy with respect to the log-radius at \(t\) becomes:

\[
\frac{\partial E}{\partial \rho_t} = 2\pi - 2\sum_{e \in t} \phi_e^t + \sum_{e \in t, e \in E^\circ} \phi_e^t + 2 \sum_{e \in t, e \in E^\circ} \phi_e^{t'} - \sum_{e \in t, e \in E^\circ} \phi_e^{t''}
\]

\[
= 2\pi - 2\sum_{e \in t} \phi_e^t + \sum_{e \in t, e \in E^\circ} \left(\phi_e^t + \phi_e^{t''}\right)
\]
Building the Energy (Tweak 1)

Thus, if we modify the triangle energy:

\[ E_t(\rho_t) = 2\pi \rho_t - 2 \sum_{e \in t, e \in E^o} F_e(\rho_t - \rho_{t'}) - 2 \sum_{e \in t, e \in \partial E} (\pi - \theta_e) \rho_t + \sum_{e \in t, e \in E^o} F_e(\rho_t - \rho_{t'}) \]

the energy gradient of the total energy with respect to the log-radius at \( t \) becomes:

\[
\frac{\partial E}{\partial \rho_t} = 2\pi - 2 \sum_{e \in t} \phi_e^t + \sum_{e \in t, e \in E^o} \phi_e^t + 2 \sum_{e \in t, e \in E^o} \phi_e^{t'} - \sum_{e \in t, e \in E^o} \phi_e^{t'}
\]

\[
= 2\pi - 2 \sum_{e \in t} \phi_e^t + \sum_{e \in t, e \in E^o} (\phi_e^t + \phi_e^{t'})
\]

\[
= 2\pi - 2 \sum_{e \in t} \phi_e^t + \sum_{e \in t, e \in E^o} (\pi - \theta_e)
\]
Building the Energy (Tweak 1)

Thus, if we modify the triangle energy:

\[ E_t(\rho_t) = 2\pi\rho_t - 2 \sum_{e \in t, e \in E^o} F_e(\rho_t - \rho_{t'}) - 2 \sum_{e \in t, e \in \partial E} (\pi - \theta_e)\rho_t + \sum_{e \in t, e \in E^o} F_e(\rho_t - \rho_{t'}) \]

the energy gradient of the total energy with respect to the log-radius at \( t \) becomes:

\[
\frac{\partial E}{\partial \rho_t} = 2\pi - 2\sum_{e \in t} \phi^t_e + \sum_{e \in t, e \in E^o} (\pi - \theta_e)
\]

We still don’t have the constraint we want, but now we are off by a constant factor that is independent of the log-radii.
Building the Energy (Tweak 1)

So Far:

If we set the per-triangle energy to be:

$$E_t(\{\rho_t\}) = 2\pi\rho_t - 2 \sum_{e \in t, e \in E^\circ} F_e(\rho_t - \rho_{t'}) - 2 \sum_{e \in t, e \in \partial E} (\pi - \theta_e)\rho_t + \sum_{e \in t, e \in E^\circ} F_e(\rho_t - \rho_{t'})$$

and we set the total energy to be the sum of the per-triangle energies:

$$E(\{\rho\}) = \sum_{t \in T} E_t(\{\rho\})$$

then the gradient becomes:

$$\frac{\partial E}{\partial \rho_t} = 2\pi - 2 \sum_{e \in t} \phi^t_e + \sum_{e \in t, e \in E^\circ} (\pi - \theta_e)$$

Original (Desired) Energy

Energy Adjustment

Desired Constraint

Extra Stuff
Building the Energy (Tweak 2)

To get rid of the last bit, we adjust the total energy term a bit more:

\[
E(\{\rho\}) = \sum_{t \in T} E_t(\{\rho\}) - \sum_{e \in E^o} (\pi - \theta_e)(\rho_t + \rho_t')
\]

- Per-Triangle Contribution
- Adjustment Term
Building the Energy (Tweak 2)

To get rid of the last bit, we adjust the total energy term a bit more:

\[ E(\{\rho\}) = \sum_{t \in T} E_t(\{\rho\}) - \sum_{e \in E^o} (\pi - \theta_e)(\rho_t + \rho_{t'}) \]

Now, the total energy has the desired property:

\[ \frac{\partial E}{\partial \rho_t} = 2\pi - 2\sum_{e \in t} \phi^t_e \]

so the energy is minimized only if we satisfy the planarity constraint on the kite half-angles.
The Energy Hessian

\[ \frac{\partial E}{\partial \rho_t} = 2\pi - 2 \sum_{e \in t, e \in E^o} \tan^{-1}\left( \frac{\sin \theta_e}{e^{\rho_i - \rho_r} - \cos \theta_e} \right) - 2 \sum_{e \in t, e \in \partial E} (\pi - \theta_e) \]

If we go through the messiness of differentiating the functions:

\[ f_e(x) = \tan^{-1}\left( \frac{\sin \theta_e}{e^x - \cos \theta_e} \right) \]

we get:

\[ f'_e(x) = \frac{-\sin \theta_e}{2[\cosh x - \cos \theta_e]} \]
The Energy Hessian

\[
\frac{\partial E}{\partial \rho_t} = 2\pi - 2 \sum_{e \in t, e \in E^o} \tan^{-1}\left(\frac{\sin \theta_e}{e^{\rho_t - \rho_i} - \cos \theta_e}\right) - 2 \sum_{e \in t, e \in \partial E} (\pi - \theta_e)
\]

\[
f'_e(x) = \frac{-\sin \theta_e}{2[\cosh x - \cos \theta_e]}
\]

And the Hessian of the energy becomes:

\[
\frac{\partial^2 E}{\partial \rho_t \partial \rho_t} = -2 \sum_{e \in t, e \in E^o} f'_e(\rho_t - \rho_i) \quad \frac{\partial^2 E}{\partial \rho_t \partial \rho_i} = 2 \sum_{e \in t, e \in E^o} f'_e(\rho_t - \rho_i)
\]
The Energy Hessian

\[
\frac{\partial E}{\partial \rho_t} = 2\pi - 2 \sum_{e \in t, e \in E^o} \tan^{-1}\left(\frac{\sin \theta_e}{e^{\rho_t - \rho_{t'}} - \cos \theta_e}\right) - 2 \sum_{e \in t, e \in \partial E} (\pi - \theta_e)
\]

\[
f'_e(x) = \frac{-\sin \theta_e}{2[\cosh x - \cos \theta_e]}
\]

\[
\frac{\partial^2 E}{\partial \rho_t \partial \rho_t} = -2 \sum_{e \in t, e \in E^o} f'_e(\rho_t - \rho_{t'})
\]

\[
\frac{\partial^2 E}{\partial \rho_t \partial \rho_{t'}} = 2 \sum_{e \in t, e \in E^o} f'_e(\rho_t - \rho_{t'})
\]

So for a vector \(v=\{v_t\}\) we get:

\[
v' H(E)(\{\rho\})v = \sum_{e \in E^o} 2 \frac{\partial^2 E}{\partial \rho_t \partial \rho_{t'}} v_t v_{t'} + \frac{\partial^2 E}{\partial \rho_t \partial \rho_t} v_t v_t + \frac{\partial^2 E}{\partial \rho_{t'} \partial \rho_{t'}} v_{t'} v_{t'}
\]
The Energy Hessian

\[
\frac{\partial E}{\partial \rho_t} = 2\pi - 2 \sum_{e \in t, e \in E^\circ} \tan^{-1}\left(\frac{\sin \theta_e}{e^{\rho_t - \rho_t'} - \cos \theta_e}\right) - 2 \sum_{e \in t, e \in \partial E} (\pi - \theta_e)
\]

\[
f'_e(x) = \frac{-\sin \theta_e}{2[\cosh x - \cos \theta_e]}
\]

\[
\frac{\partial^2 E}{\partial \rho_t \partial \rho_t'} = -2 \sum_{e \in t, e \in E^\circ} f'_e(\rho_t - \rho_t') \quad \frac{\partial^2 E}{\partial \rho_t \partial \rho_t'} = 2 \sum_{e \in t, e \in E^\circ} f'_e(\rho_t - \rho_t')
\]

So for a vector \(v = \{v_t\}\) we get:

\[
v' H(E)(\{\rho\})v = \sum_{e \in E^\circ} 2 \frac{\partial^2 E}{\partial \rho_t \partial \rho_t'} v_t v_{t'} + \frac{\partial^2 E}{\partial \rho_t \partial \rho_t'} v_t v_t + \frac{\partial^2 E}{\partial \rho_t \partial \rho_t'} v_{t'} v_{t'}
\]

\[
= \sum_{e \in E^\circ} 4 f'_e(\rho_t - \rho_t') v_t v_{t'} - 2 f'_e(\rho_t - \rho_t') v_t v_t - 2 f'_e(\rho_t - \rho_t') v_{t'} v_{t'}
\]
The Energy Hessian

\[
\frac{\partial E}{\partial \rho_t} = 2\pi - 2 \sum_{e\in t, e\in E^*} \tan^{-1}\left(\frac{\sin \theta_e}{e^{\rho_t - \rho_t'} - \cos \theta_e}\right) - 2 \sum_{e\in t, e\in \partial E} (\pi - \theta_e)
\]

\[
f'_e(x) = \frac{-\sin \theta_e}{2[\cosh x - \cos \theta_e]}
\]

\[
\frac{\partial^2 E}{\partial \rho_t \partial \rho_t'} = -2 \sum_{e\in t, e\in E^*} f'_e(\rho_t - \rho_t')
\]

\[
\frac{\partial^2 E}{\partial \rho_t \partial \rho_t'} = 2 \sum_{e\in t, e\in E^*} f'_e(\rho_t - \rho_t')
\]

So for a vector \(v=\{v_t\}\) we get:

\[
v' H(E)(\{\rho\})v = \sum_{e\in E^*} 2 \frac{\partial^2 E}{\partial \rho_t \partial \rho_t'} v_t v_{t'} + \frac{\partial^2 E}{\partial \rho_t \partial \rho_t'} v_t v_t + \frac{\partial^2 E}{\partial \rho_t \partial \rho_t'} v_{t'} v_{t'}
\]

\[
= \sum_{e\in E^*} 4 f'_e(\rho_t - \rho_t') v_t v_{t'} - 2 f'_e(\rho_t - \rho_t') v_t v_t - 2 f'_e(\rho_t - \rho_t') v_{t'} v_{t'}
\]

\[
= 2 \sum_{e\in E^*} (v_t - v_{t'})^2 f'_e(\rho_t - \rho_t')
\]
The Energy Hessian

\[
\frac{\partial E}{\partial \rho_t} = 2\pi - 2 \sum_{e \in t, e \in E^o} \tan^{-1} \left( \frac{\sin \theta_e}{e^{\rho_t - \rho_t'} - \cos \theta_e} \right) - 2 \sum_{e \in t, e \in \partial E} (\pi - \theta_e)
\]

\[
f_e'(x) = \frac{-\sin \theta_e}{2[\cosh x - \cos \theta_e]}
\]

\[
\frac{\partial^2 E}{\partial \rho_t \partial \rho_t'} = -2 \sum_{e \in t, e \in E^o} f'_e(\rho_t - \rho_t')
\]

So for a vector \( v = \{v_t\} \) we get:

\[
v' H(E)(\{\rho\})v = \sum_{e \in E^o} 2 \frac{\partial^2 E}{\partial \rho_t \partial \rho_t'} v_t v_{t'} + \frac{\partial^2 E}{\partial \rho_t \partial \rho_t} v_t v_t + \frac{\partial^2 E}{\partial \rho_t' \partial \rho_t'} v_{t'} v_{t'}
\]

\[
= \sum_{e \in E^o} 4 f'_e(\rho_t - \rho_t') v_t v_{t'} - 2 f'_e(\rho_t - \rho_t') v_t v_t - 2 f'_e(\rho_t - \rho_t') v_{t'} v_{t'}
\]

\[
= 2 \sum_{e \in E^o} - (v_t - v_{t'})^2 f'_e(\rho_t - \rho_t')
\]

\[
= \sum_{e \in E^o} \frac{\sin \theta_e}{\cosh(\rho_t - \rho_t') - \cos \theta_e} (v_t - v_{t'})^2
\]
The Energy Hessian

$$v^i H(E)(\{\rho\})v = \sum_{e \in E} \frac{\sin \theta_e}{\cosh(\rho_t - \rho_{t'}) - \cos \theta_e} (v_t - v_{t'})^2$$

So, if the original triangulation is locally Delaunay ($0 < \theta_e < \pi$), the Hessian is never negative, and is only 0 when $v = \{c, \ldots, c\}$, i.e. we get no change in the gradient only if we offset all the log-radii by the same amount.
The Energy Hessian

\[ v^T H(E)(\{\rho\})v = \sum_{e \in E} \frac{\sin \theta_e}{\cosh(\rho_t - \rho_{t'}) - \cos \theta_e} (v_t - v_{t'})^2 \]

So, if the original triangulation is locally Delaunay \((0 < \theta_e < \pi)\), the Hessian is never negative, and is only 0 when \(v = \{c, ..., c\}\), i.e. we get no change in the gradient only if we offset all the log-radii by the same amount.

That is, if we apply a uniform scale to the circle pattern.
Circle Patterns

Recap:
We know that if a circle pattern is valid then the sum of the kite-half angles has to add up to $2\pi$. 
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Recap:
We know that if a circle pattern is valid then the sum of the kite-half angles has to add up to $2\pi$. We have designed an energy whose gradient vanishes whenever this condition is satisfied. We have shown that the gradient of the energy can only vanish once.
Recap:
Thus, at most one circle pattern can satisfy the planarity condition and have the prescribed angles $\theta_e$, and we can find it through gradient descent along the energy.
Thought Question:
In order to prove uniqueness, we used the existence of the function:

\[ F_e(x) = \int_{-\infty}^{x} \tan^{-1} \left( \frac{\sin \theta_e}{e^s - \cos \theta_e} \right) ds \]

Do we ever need to know its value?
Circle Patterns

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Do we ever need to know its value?

If we did, we would find that:

\[ F_e(x) = \text{Im} \text{Li}_2(e^{x+i\theta_e}) \]

where \( \text{Li}_2 \) is the dilogarithm function, defined for complex \( z \) as:

\[ \text{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} \]
Circle Patterns

Question:
Given an abstract triangulation $T$ (of a disk) and an assignment of angles $\theta_e$ to each edge, is there a circle pattern that has the combinatorics of $T$ and defines the prescribed angles $\theta_e$?
Circle Patterns

Necessary Properties of the Angles $\theta_e$:

1. **Delaunay**: Since we are assuming that the triangulation is locally Delaunay, we know that on interior edges we have $\alpha_1 + \alpha_2 < \pi$, so:

$$0 < \theta_e < \pi$$
Circle Patterns

Necessary Properties of the Angles $\theta_e$:

2. **Planarity**: For an interior vertex $v$, the sum of angles $\theta_e$ over all edges $e$ adjacent to $v$ is $2\pi$:

$$\sum_{e \ni v} \theta_e = 2\pi \quad \forall \ v \in V^\circ$$

$$\sum_{e \ni v} \theta_e = \sum_{i=0}^{n} \pi - \alpha_i - \beta_{i+1}$$

$$\sum_{i=0}^{n} \pi - \alpha_i - \beta_{i} = \sum_{i=0}^{n} \gamma_i = 2\pi$$
Circle Patterns

Question:
Given an abstract triangulation $T$ (of a disk) and an assignment of angles $\theta_e$ to each edge, is there a circle pattern that has the combinatorics of $T$ and defines the prescribed angles $\theta_e$?
What if we assume that the constraint angles are reasonable (i.e. satisfy the Delaunay and Planarity condition)?
Circle Patterns

Towards an Answer:
If we can find a circle pattern, then that would give us a triangulation in the plane:
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And each triangle $t=(i,j,k)$ would have interior angles that satisfy:

- $\alpha_{ijk} > 0$, $\alpha_{ijk} + \alpha_{kij} + \alpha_{jki} = \pi$
Circle Patterns

Towards an Answer:
If we can find a circle pattern, then that would give us a triangulation in the plane:
And each triangle $t=(i,j,k)$ would have interior angles that satisfy:

- $\alpha_{ijk} > 0$, $\alpha_{ijk} + \alpha_{kij} + \alpha_{jki} = \pi$
- For interior edges: $\theta_{ij} = \pi - \alpha_{ijk} - \alpha_{jik}$
- For boundary edges: $\theta_{ik} = \pi - \alpha_{kij}$
Circle Patterns

Definition:
Given a triangle mesh $T$ and a set of angle constraints $\theta_e$, we say that an assignment of angles $\alpha_{ijk}$ to the angles of the triangles in $T$ is a coherent angle system if:

- $\alpha_{ijk} > 0$, $\alpha_{ijk} + \alpha_{kij} + \alpha_{jki} = \pi$
- For interior edges:
  $\theta_{ij} = \pi - \alpha_{ijk} - \alpha_{jki}$
- For boundary edges:
  $\theta_{ik} = \pi - \alpha_{kij}$
Existence of Circle Patterns

Theorem [Bobenko and Springborn 2004]:
The circle pattern has a (unique up to scale) solution iff. a coherent angle system exists.
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Note:
This doesn’t imply that the angles in the angle system will be the angles of the pattern.
Existence of Circle Patterns

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Note:
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A coherent angle system has $3|T|$ degrees of freedom, and $|e| + |T| \approx 2.5|T|$ constraints, so we expect many CAS’s for a set of edge weights $\theta_e$. 
Extending to Meshes

Challenge:
Given an arbitrary triangle mesh, the interior angles $\{ \alpha_i \}$ do not necessarily define a coherent angle system.
Extending to Meshes

Solution:
Find a coherent angle system whose angles $\{\beta_i\}$ are as close as possible to the angles $\{\alpha_i\}$ in the original mesh.
Extending to Meshes

Solution:
This problem can be formulated as a quadratic minimization of:

\[ Q(\{\beta\}) = \sum_{t \in T} \sum_{e \in t} |\alpha_e^t - \beta_e^t|^2 \]

subject to:

- **Positivity**: For all angles \( \beta_e^t > 0 \)
- **Delaunay**: For all interior edges \( \beta_e^t + \beta_e^{t'} < \pi \)
- **Triangle Sum**: For all triangles \( \sum_{e \in t} \beta_e^t = 2\pi \)
- **Vertex Sum**: For all interior verts. \( \sum_{e \in v} \beta_e^t = 2\pi \)
Extending to Meshes

Solution:
This problem can be formulated as a quadratic minimization of:

\[ Q(\{\beta\}) = \sum_{t \in T} \sum_{e \in t} |\alpha^t_e - \beta^t_e|^2 \]

On boundary vertices, we can either place hard constraints prescribing curvature:

\[ \sum_{e \in \partial v} \beta^t_e = \pi - \kappa_v \]

or place “natural” constraints:

\[ \sum_{e \in \partial v} \beta^t_e < 2\pi \]
Extending to Meshes

If the minimization problem is satisfiable, we get a valid angle system approximating the original angles and encoding the boundary constraints.
Extending to Meshes

If the minimization problem is satisfiable, we get a valid angle system approximating the original angles and encoding the boundary constraints. We can use these to define constraint angle $\theta_e$ and use gradient descent on the energy to get the circle pattern.
Extending to Meshes

This will define a mapping from the mesh into the plane which is very close to conformal.
Extending to Meshes

Note that the conformal map is defined with respect to angles $\theta_e$ that are defined from the fit angles $\{\beta\}$ and not from the mesh angles, so the “conformality” of the map will be tied to the closeness of the fit angles to the original angles.