Differential Geometry:

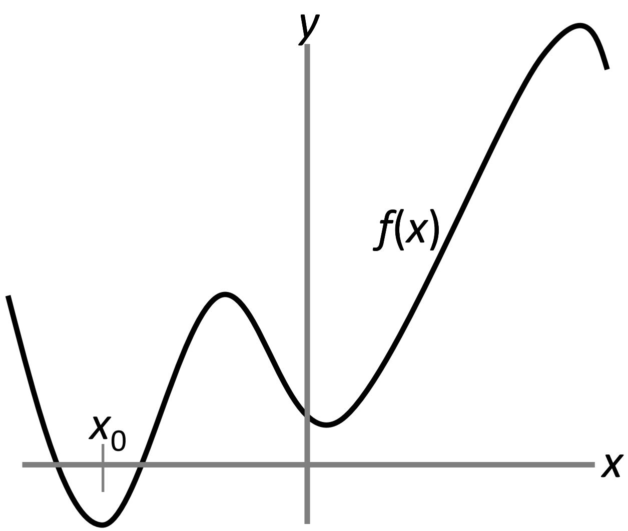
Circle Patterns (Part 1)

[*Discrete Conformal Mappinngs via Circle Patterns*. Kharevych, Springborn and Schröder]

Preliminaries

Recall:

Given a smooth function f:$\boldsymbol{R}\rightarrow\boldsymbol{R}$, the function f has an extremum at x_0 only if $f'(x_0)=0$.



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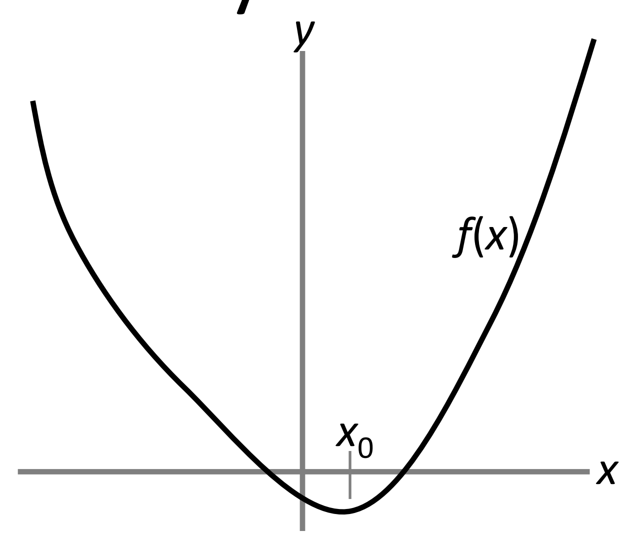
Given a smooth function f:$\boldsymbol{R}\rightarrow\boldsymbol{R}$, the function f has an extremum at x_0 only if $f'(x_0)=0$.

Furthermore, by the mean-value-theorem, if f'' is never zero then the value x_0 is the only extremum of f.



Preliminaries

Recall:

Given a smooth function F:$\boldsymbol{R}^n \rightarrow \boldsymbol{R}$, the *gradient* of F is the vector valued function ∇F:$\boldsymbol{R}^n \rightarrow \boldsymbol{R}^n$ obtained by computing the partial derivatives:

$$\nabla F = \left(\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n} \right)$$

Preliminaries

Recall:

Given a smooth function $F:\mathbf{R}^n \to \mathbf{R}$, the *Hessian* of F is the (symmetric) matrix-valued function $H(F):\mathbf{R}^n \to \mathbf{R}^{n\times n}$ obtained by computing the mixed second derivatives of F:

$$H(F) = \begin{pmatrix} \frac{\partial^2 F}{\partial x_1 \partial x_1} & \frac{\partial^2 F}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 F}{\partial x_n \partial x_1} \\ \frac{\partial^2 F}{\partial x_1 \partial x_2} & \frac{\partial^2 F}{\partial x_2 \partial x_2} & \cdots & \frac{\partial^2 F}{\partial x_n \partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 F}{\partial x_1 \partial x_n} & \frac{\partial^2 F}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 F}{\partial x_n \partial x_n} \end{pmatrix}$$

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Recall:

Given a smooth function $F:\mathbf{R}^n\rightarrow\mathbf{R}$, if we choose a position $x_0\in\mathbf{R}^n$ and direction $v\in\mathbf{R}^n$, we can define the 1D function $f(t)=F(x_0+tv)$.

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The derivative of $f(t)$ at t=0 is the dot-product of the direction v with the gradient of F:

$$f'(0) = \left\langle \nabla F(x_0), v \right\rangle$$

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The second-derivative of $f(t)$ at $t=0$ is the square norm of v with respect to the Hessian of F:

$$f''(0) = v^t \left[H(F)(x_0) \right] v$$

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Given a smooth function $F:\mathbf{R}^n \rightarrow \mathbf{R}$, the function F has an extremum at x_0 only if $\nabla F(x_0)=0$.

Furthermore, if the Hessian of F is either strictly positive definite, or strictly negative definite, x_0 is the only extremum of F.

Preliminaries

Optimization:

Given a function $f:\boldsymbol{R}\rightarrow\boldsymbol{R}$, suppose that we would like to solve for x_0 such that $f(x_0)=a$.

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$$E'(x) = f(x) - a$$

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Then energy minimization/maximization at x_0 implies that the derivative is zero, and hence that $f(x_0)=a$.

Preliminaries

<u>Optimization</u>:

Given a function f:$\boldsymbol{R}\rightarrow\boldsymbol{R}$, suppose that we would like to solve for x_0 such that $f(x_0)=a$.

To get at such an energy, we can simply set:

$$E(x) = F(x) - ax$$

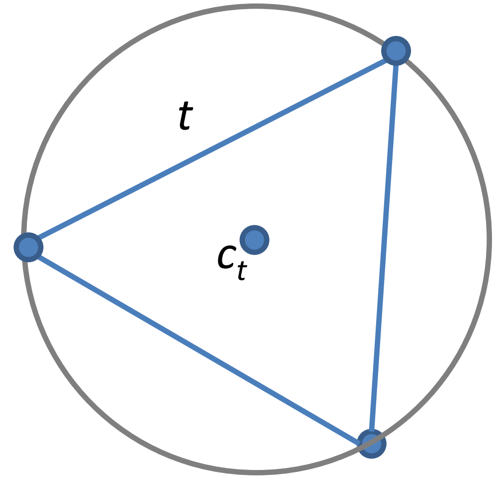
where F is the integral of f:

$$F(x) = \int_{-\infty}^{x} f(s)ds$$

Preliminaries

<u>Circumcircles and Kites</u>:

Given a triangle *t*, the *circumcircle* of the triangle is the circle, centered at c_t, intersecting the triangle's vertices.

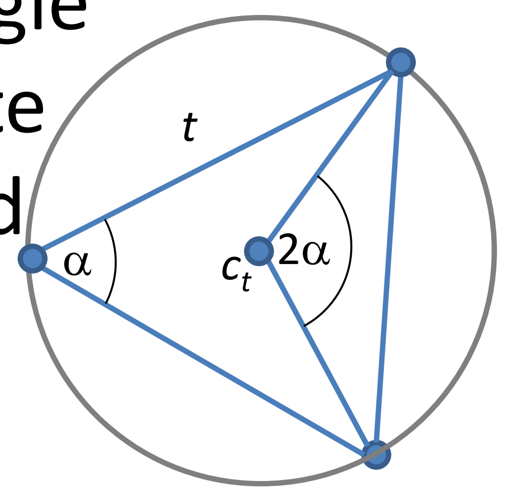


Preliminaries

<u>Circumcircles and Kites</u>:

Given a triangle t, the *circumcircle* of the triangle is the circle, centered at c_t, intersecting the triangle's vertices.

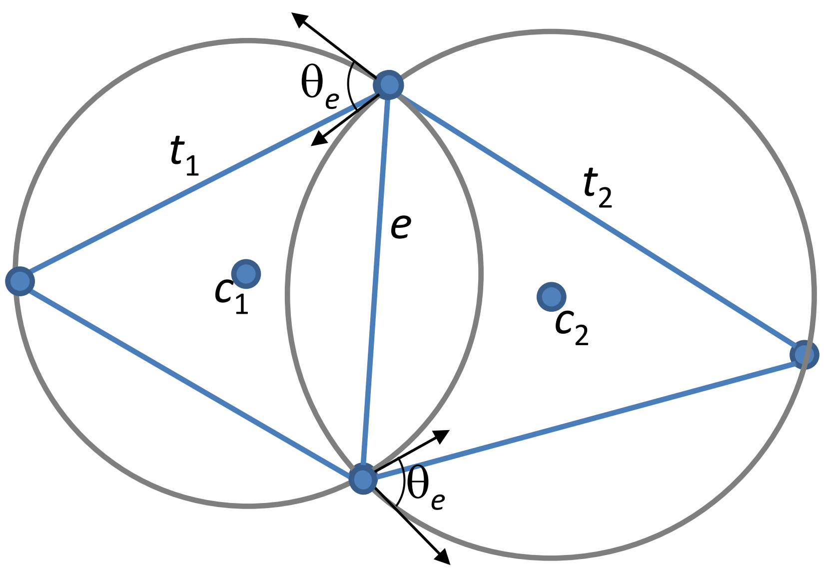
If e is an edge of the triangle and α is the angle opposite e, then the made by c_t and the edge e is 2α.



Preliminaries

<u>Circumcircles and Kites</u>:

Given two triangles t_1, and t_2 sharing an edge e, we denote by θ_e the (exterior) intersection angle of the circumcircles.



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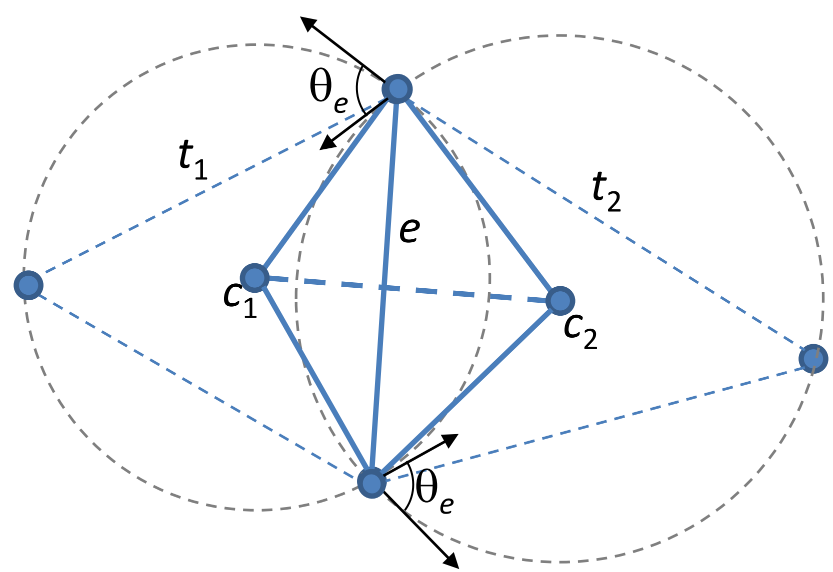
<u>Circumcircles and Kites</u>:

Given two triangles t_1, and t_2 sharing an edge *e*, we denote by θ_e the (exterior) intersection angle of the circumcircles.

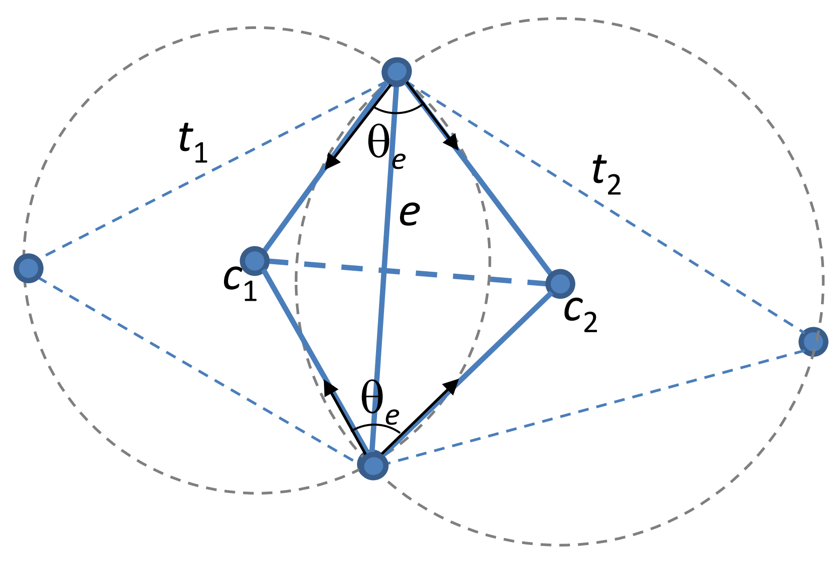
And we refer to the triangles defined by the edge *e* and the circumcenters as the kite of *e*.



Preliminaries

Circumcircles and Kites:

If we rotate the tangents at the end-points of the edge e by $\pi/2$, they will point towards the circumcenters.

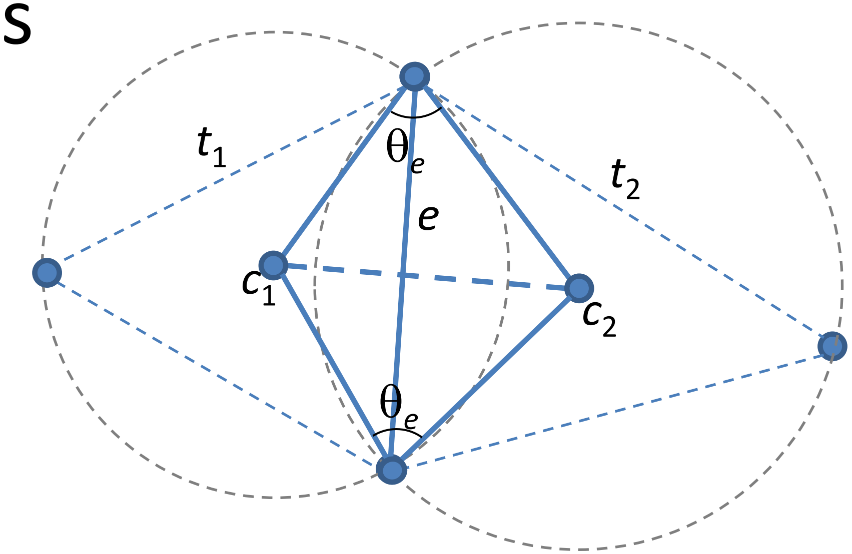


Preliminaries

Circumcircles and Kites:

If we rotate the tangents at the end-points of the edge *e* by $\pi/2$, they will point towards the circumcenters.

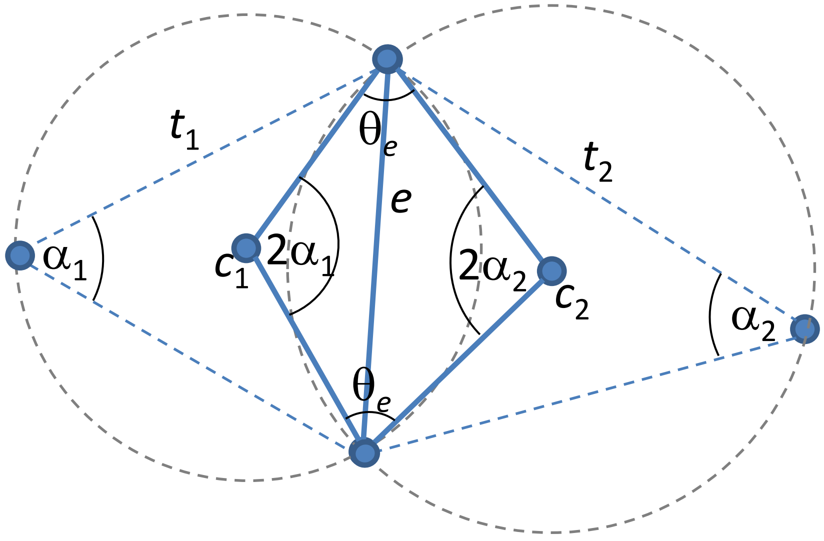
Since the rotated tangents align with the kite sides, the angles of the kite at end-points of *e* are e_θ.



Preliminaries

Circumcircles and Kites:

We also know that the angles at the circumcenters are twice the angles opposite *e*.



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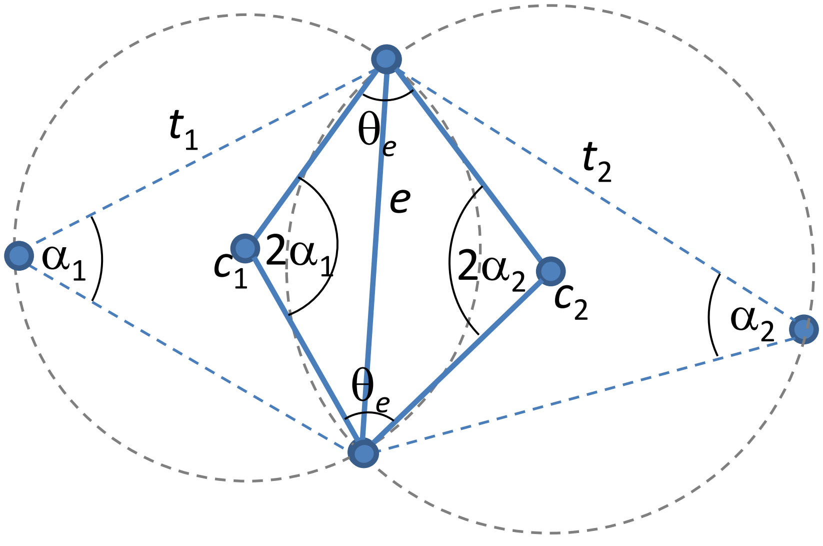
We also know that the angles at the circumcenters are twice the angles opposite *e.*

Since the total angle inside the kite is 2π, we must have:

$$2\theta_e + 2\alpha_1 + 2\alpha_2 = 2\pi$$

$$\Updownarrow$$

$$\theta_e = \pi - \alpha_1 - \alpha_2$$



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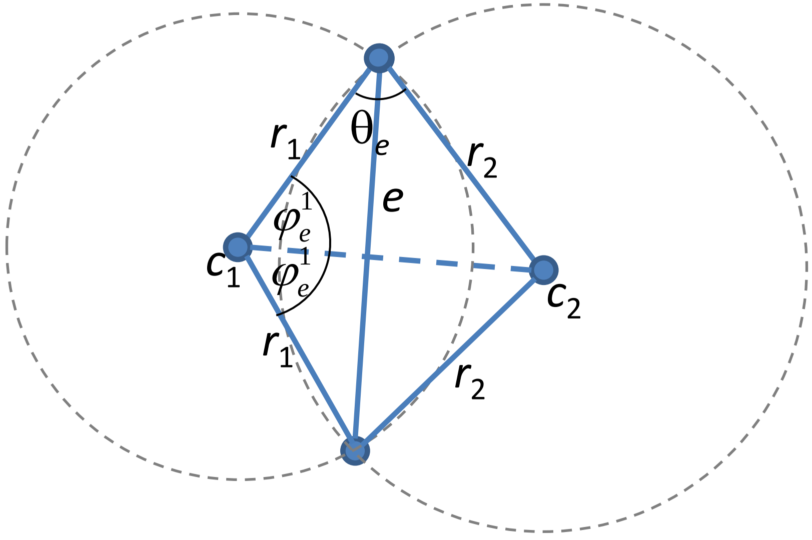
<u>Circumcircles and Kites</u>:

If we know the angle θ_e and we know the radii of the circumcircles, we can figure out the other half-angles of the kite.



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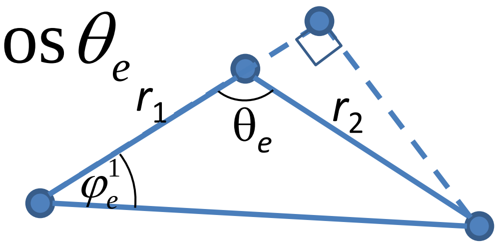
If we know the angle θ_e and we know the radii of the circumcircles, we can figure out the other half-angles of the kite:

$$\tan\left(\phi_e^1\right) = \frac{\sin(\pi - \theta_e) r_2}{r_1 + \cos(\pi - \theta_e) r_2} = \frac{\sin\theta_e}{r_1 / r_2 - \cos\theta_e}$$



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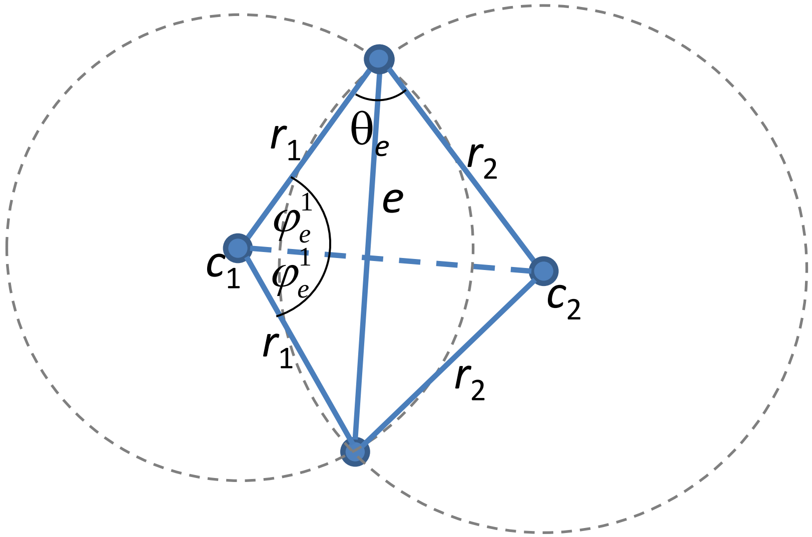
If we know the angle θ_e and we know the radii of the circumcircles, we can figure out the other half-angles of the kite:

$$\tan\left(\phi_e^1\right) = \frac{\sin\theta_e}{r_1 / r_2 - \cos\theta_e}$$

and from that we can get the lengths of the edges:

$$|e| = 2r_1 \sin\left(\phi_e^1\right)$$



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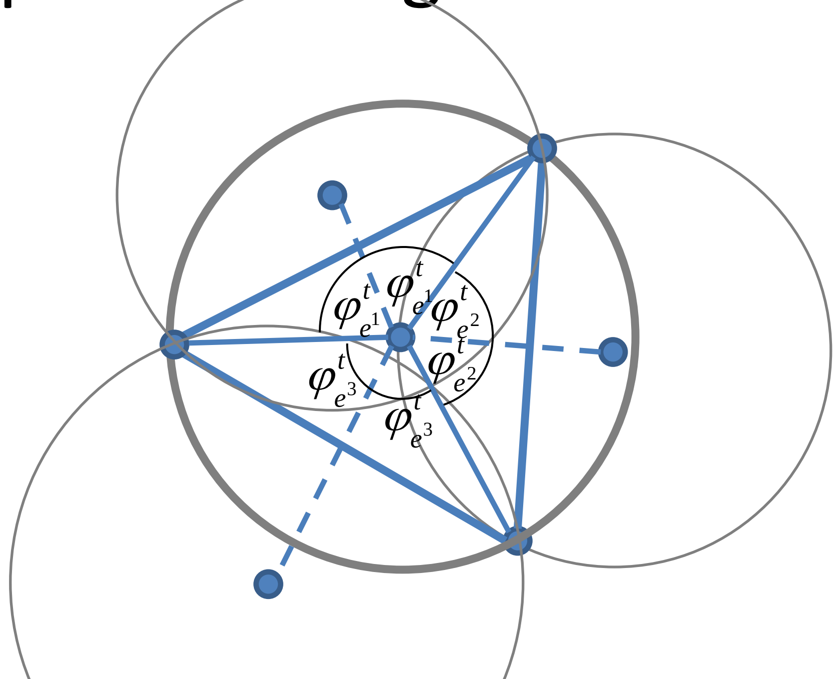
<u>Circumcircles and Kites</u>:

Although any assignment of radii allows us to compute the half-angles of the kite from the angles θ_e, not all give planar triangulations.



Preliminaries

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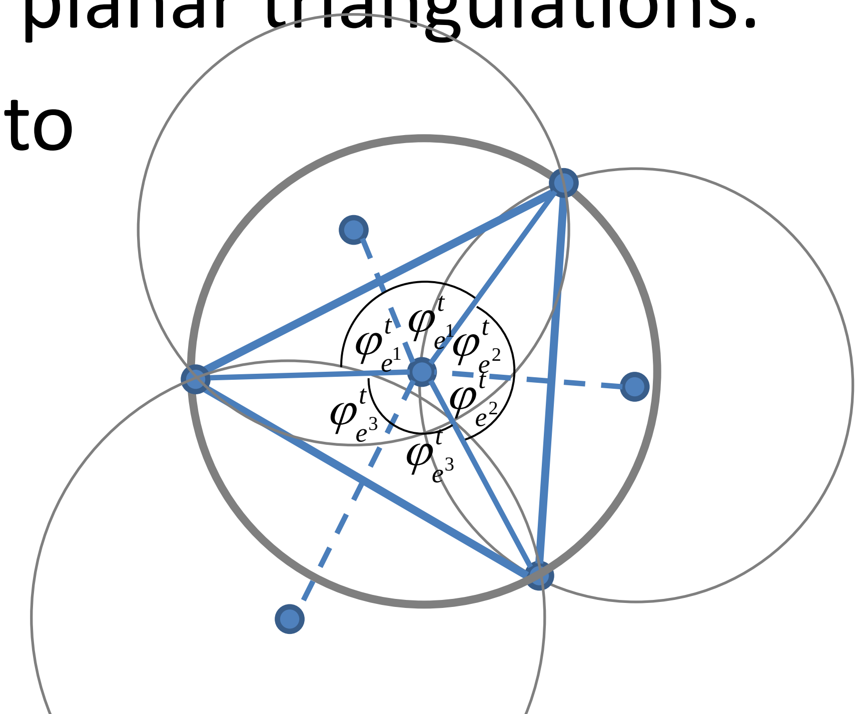
Although any assignment of radii allows us to compute the half-angles of the kite from the angles θ_e, not all give planar triangulations.

For the triangulation to be planar, we require that the sum of the kite angles about a circumcenter is 2π:

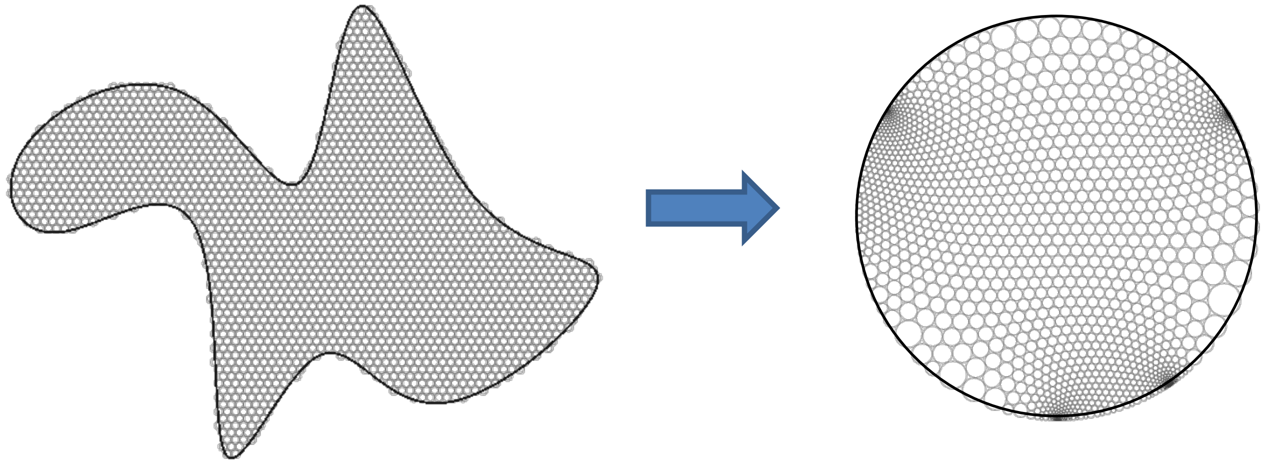
$$2\pi = \sum_{e\in t} 2\phi_e^t \qquad \forall t \in T$$



Circle Packing

<u>Recall</u>:

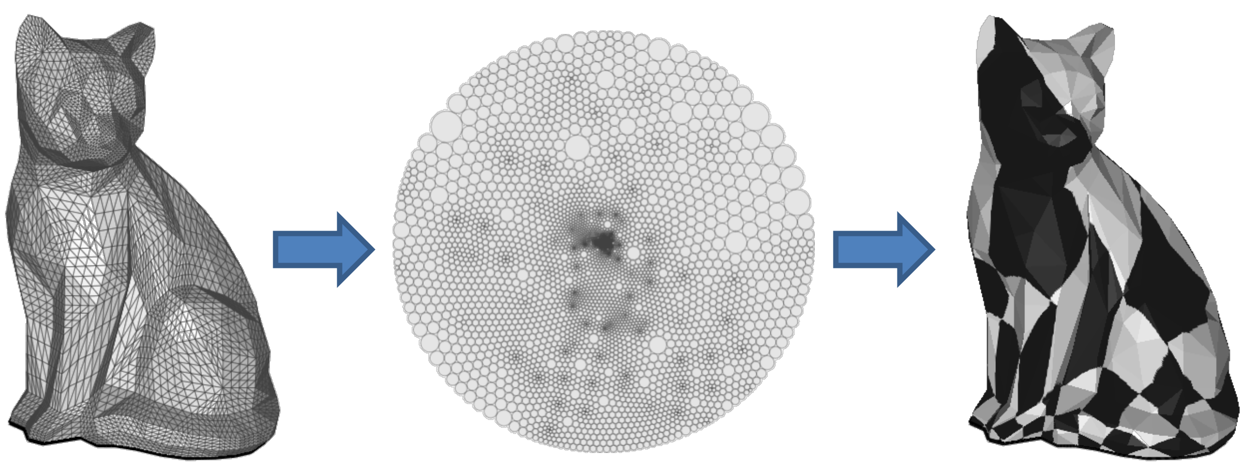
We can get a conformal map (in the limit) by using a regular hexagonal lattice to pack the inside of curve with circles and then mapping the packing into the disk.



Circle Packing

Recall:

However, this method only took the combinatorics of the triangulation into account and ignored geometry, so it cannot be used to generate a conformal map for meshes.



Circle Packing

Questions:

- Can we use circle packings to conformally map triangulations of the plane into the plane?
- In general, can we preserve angles when mapping a mesh to the plane?

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  - No, since we would still be ignoring the geometry.
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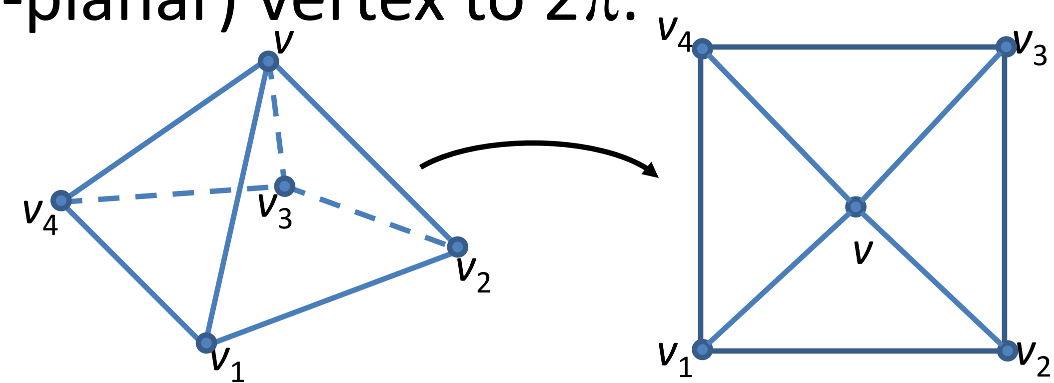
Circle Packing

<u>Questions</u>:

- Can we use circle packings to conformally map triangulations of the plane into the plane?
- In general, can we preserve angles when mapping a mesh to the plane?
  - No since we would have to map the angles at a (non-planar) vertex to 2π.



Circle Packing

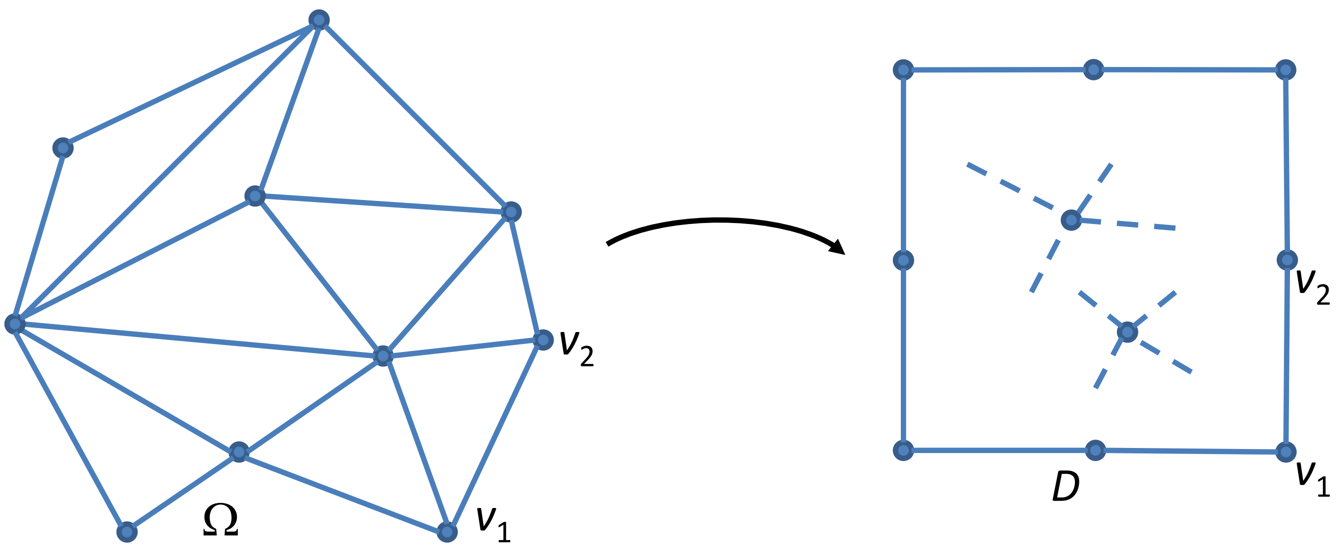
Approach:

1. We will incorporate geometric information to generate conformal mappings of planar triangulation into the plane.
2. We will minimize the non-conformal angular distortion arising from mapping meshes into the plane.

Planar Triangulation

Goal:

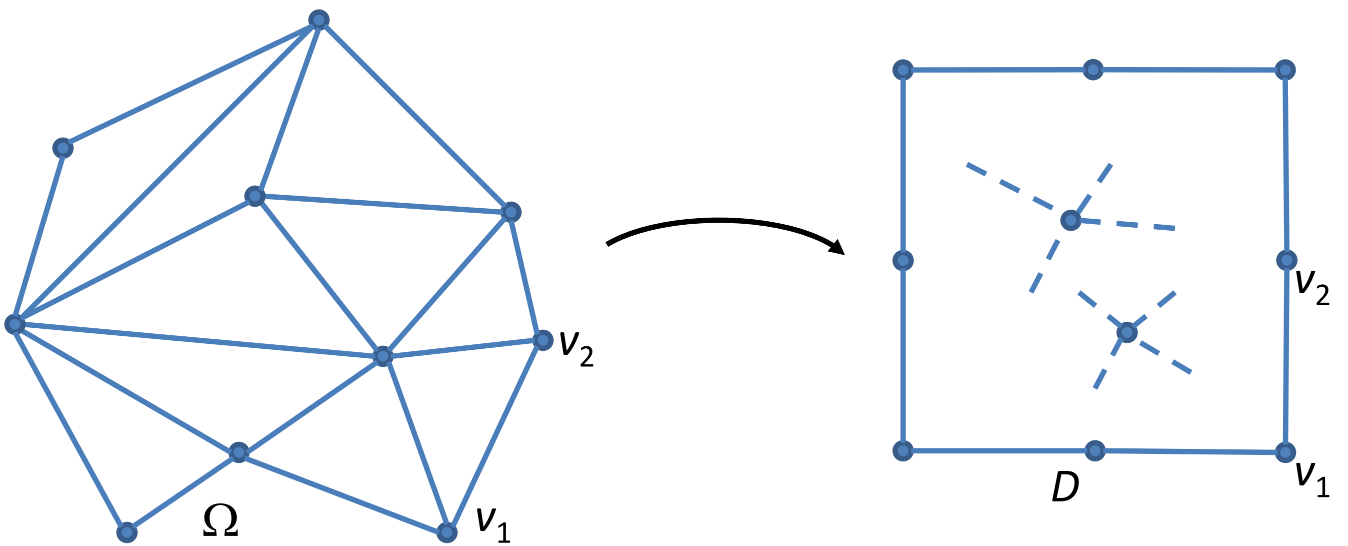
Given a planar triangulation of a (non-convex) domain Ω, and given a target domain D, we want to find a conformal map sending Ω to D.



Planar Triangulation

<u>Aside</u>:

Since Ω need not be convex, we cannot assume that the triangulation is Delaunay, but we will require that it is locally Delaunay.



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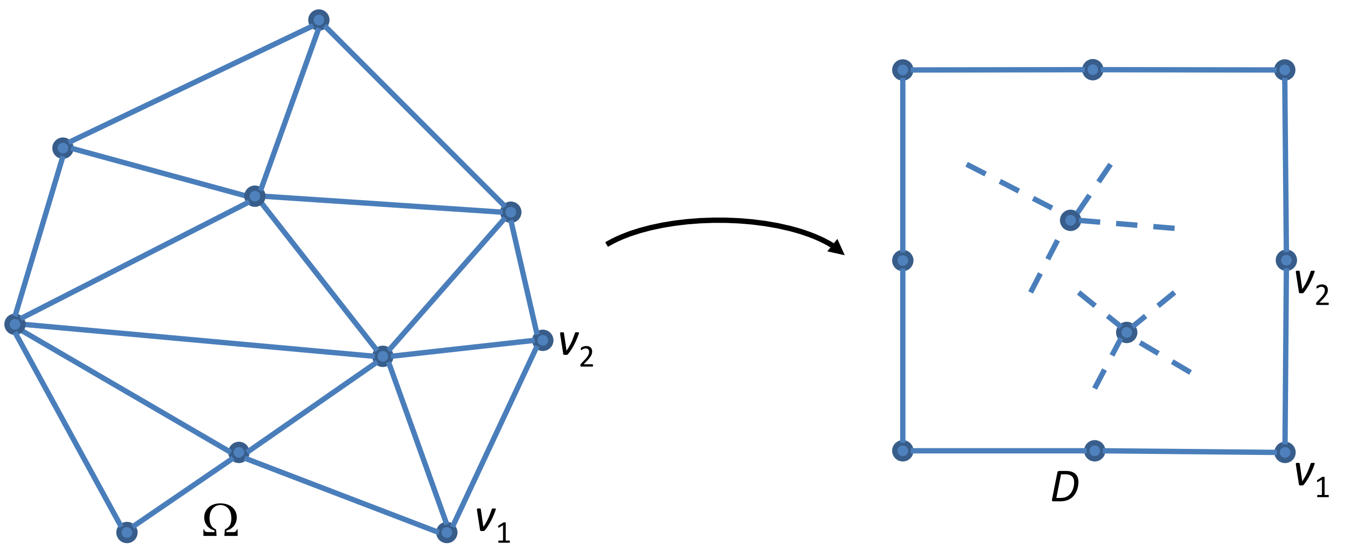

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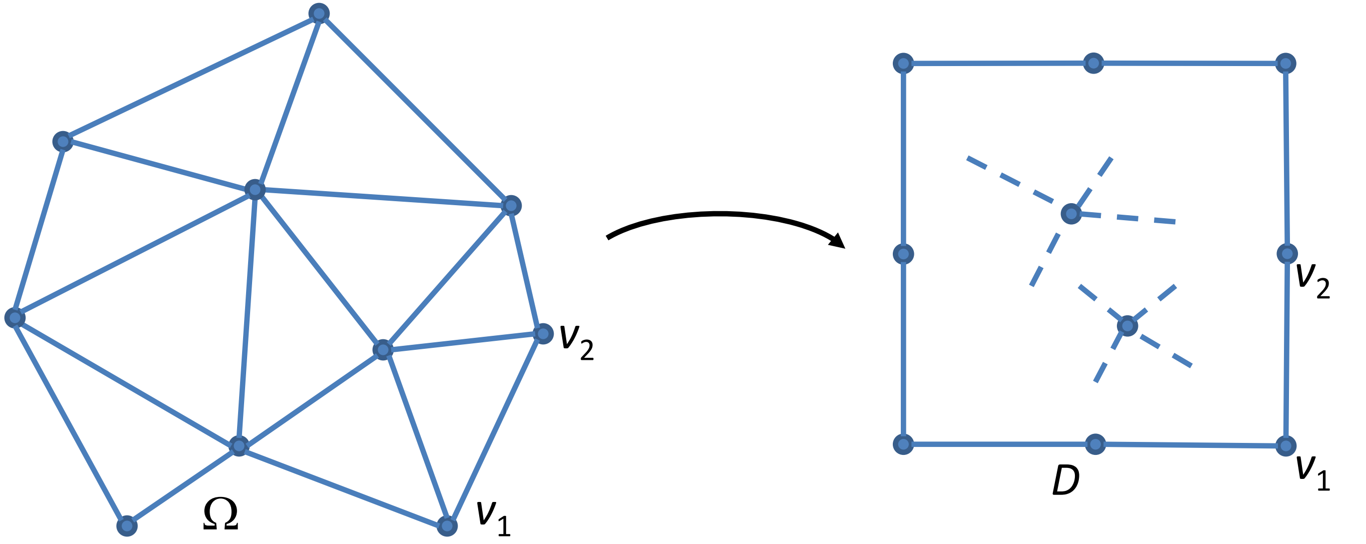
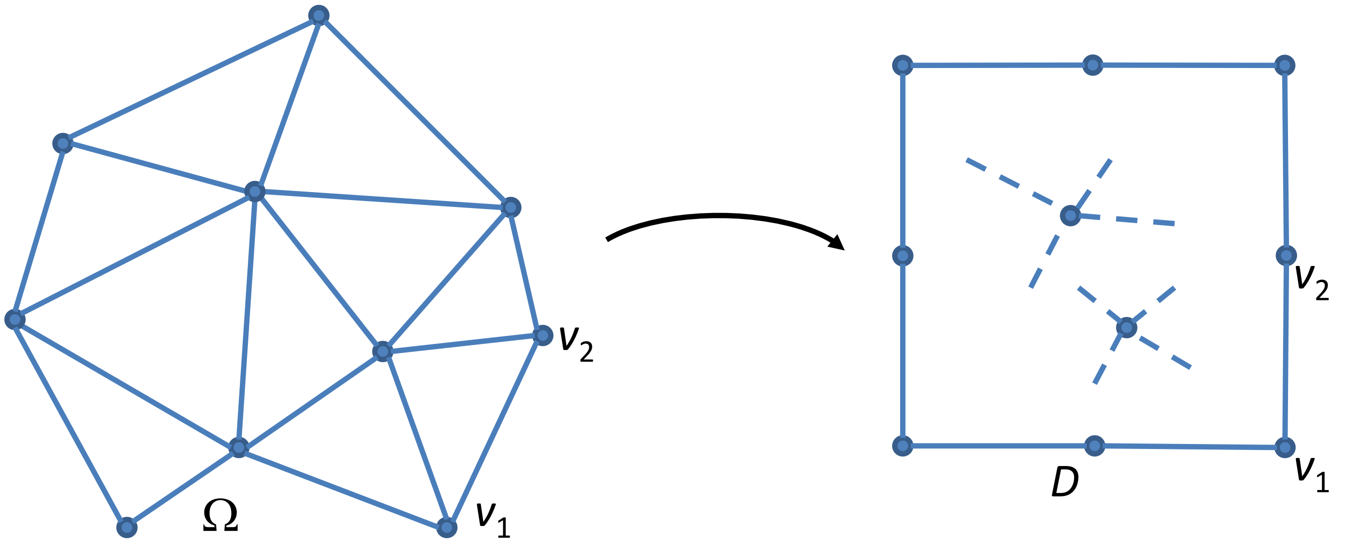


Planar Triangulation

<u>Elements of Conformality</u>:

For the study of conformality, our principal building blocks will be *circles*.



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For the study of conformality, our principal building blocks will be *circles*.

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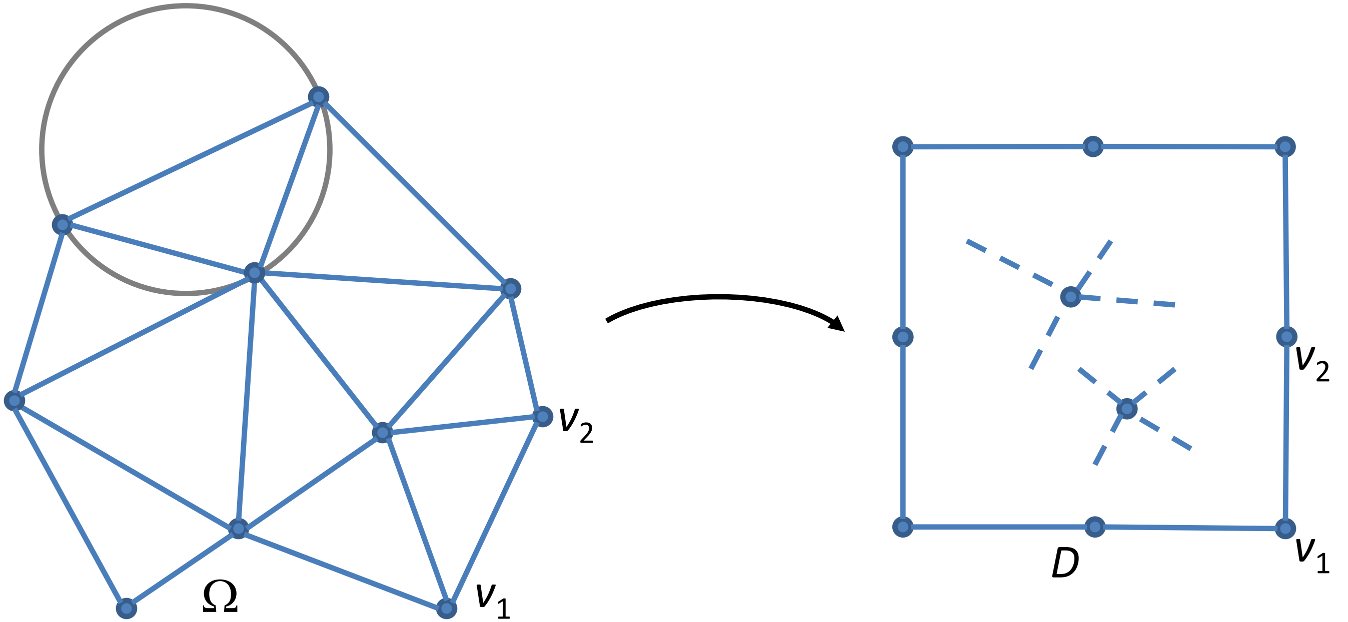

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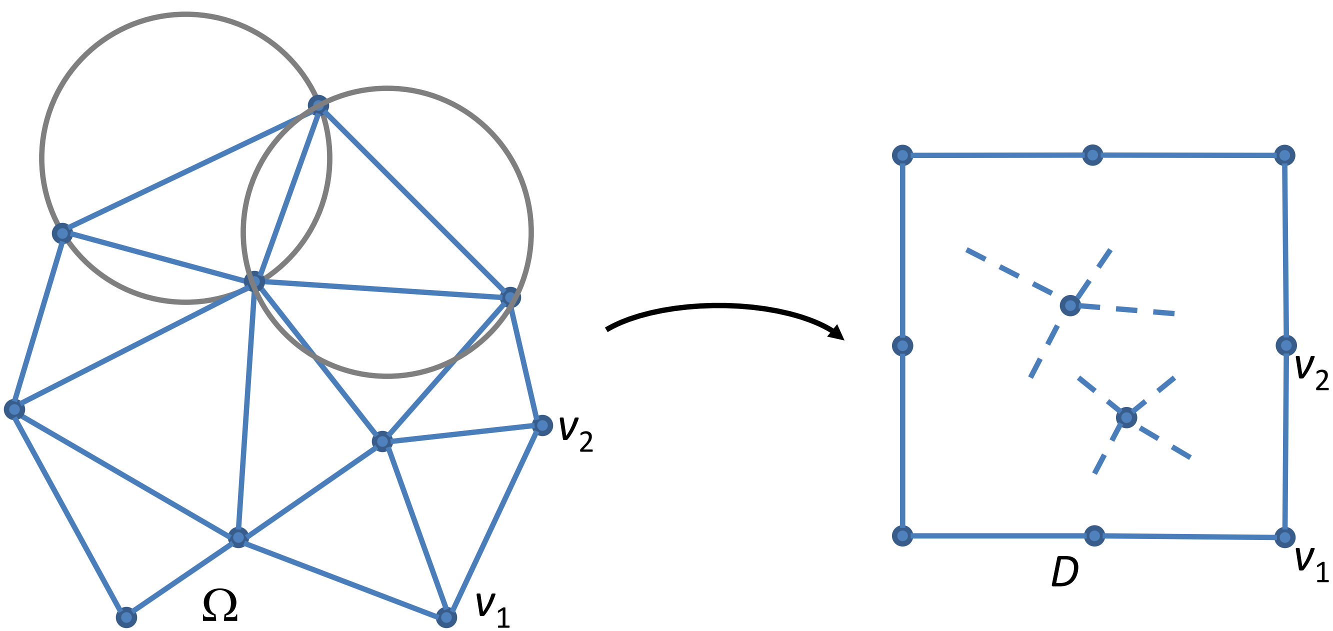

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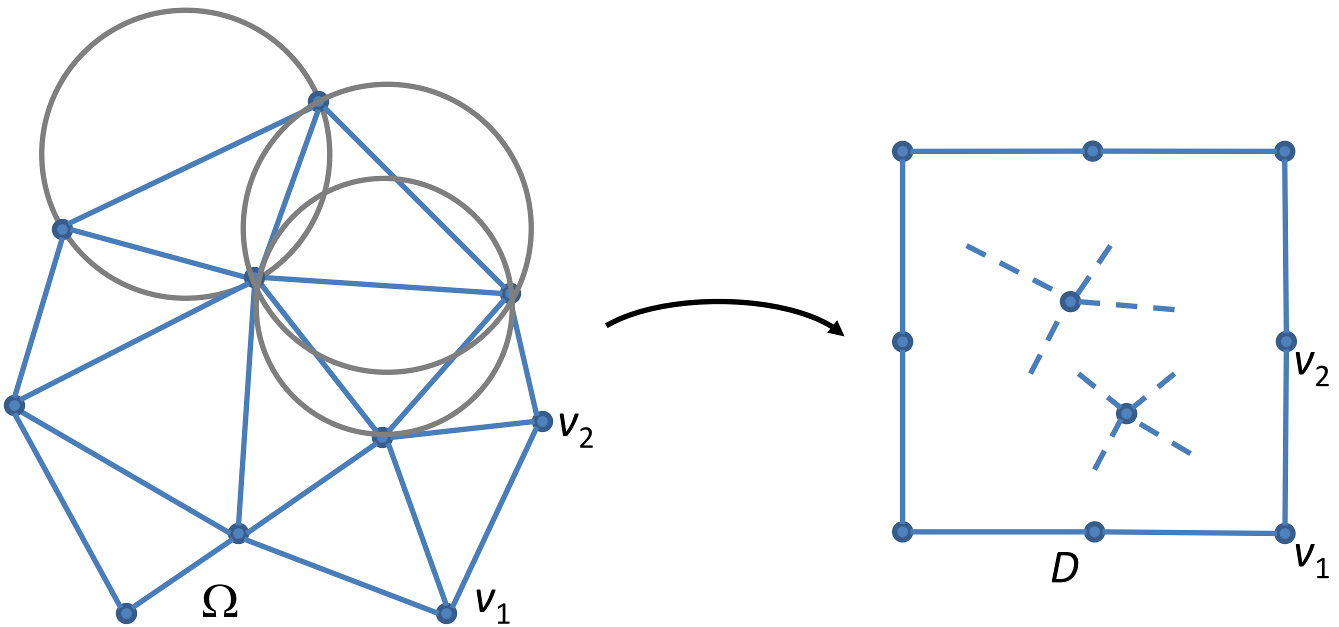

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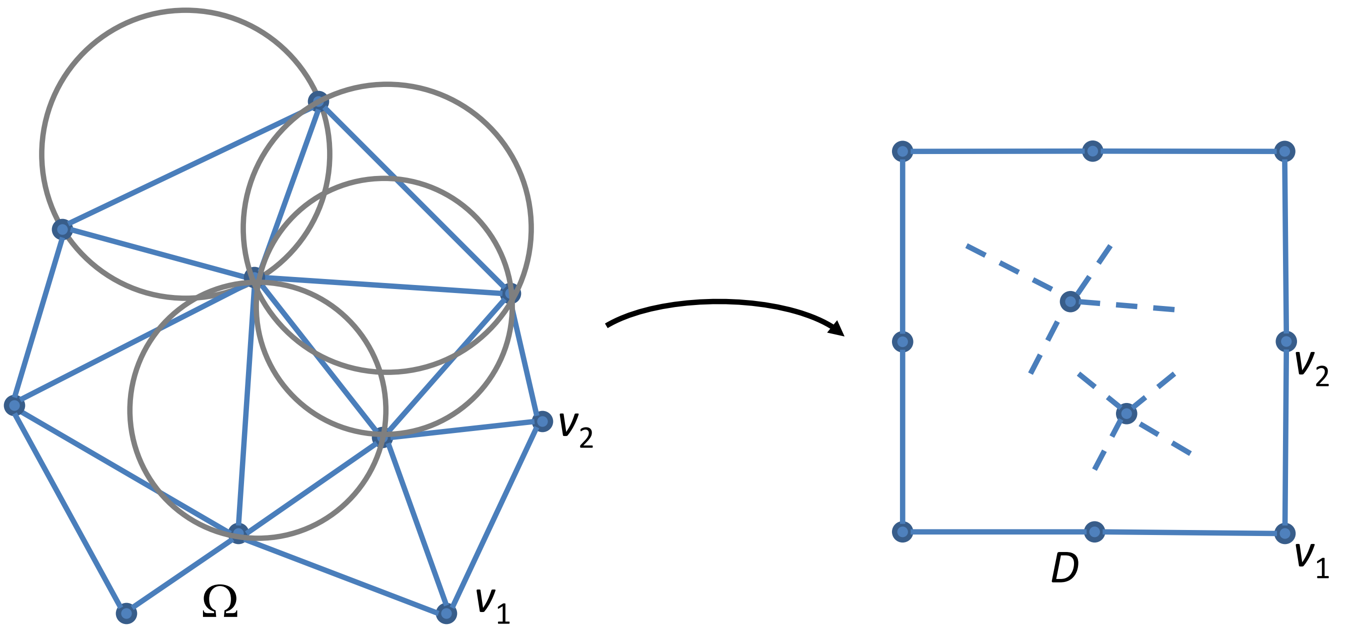

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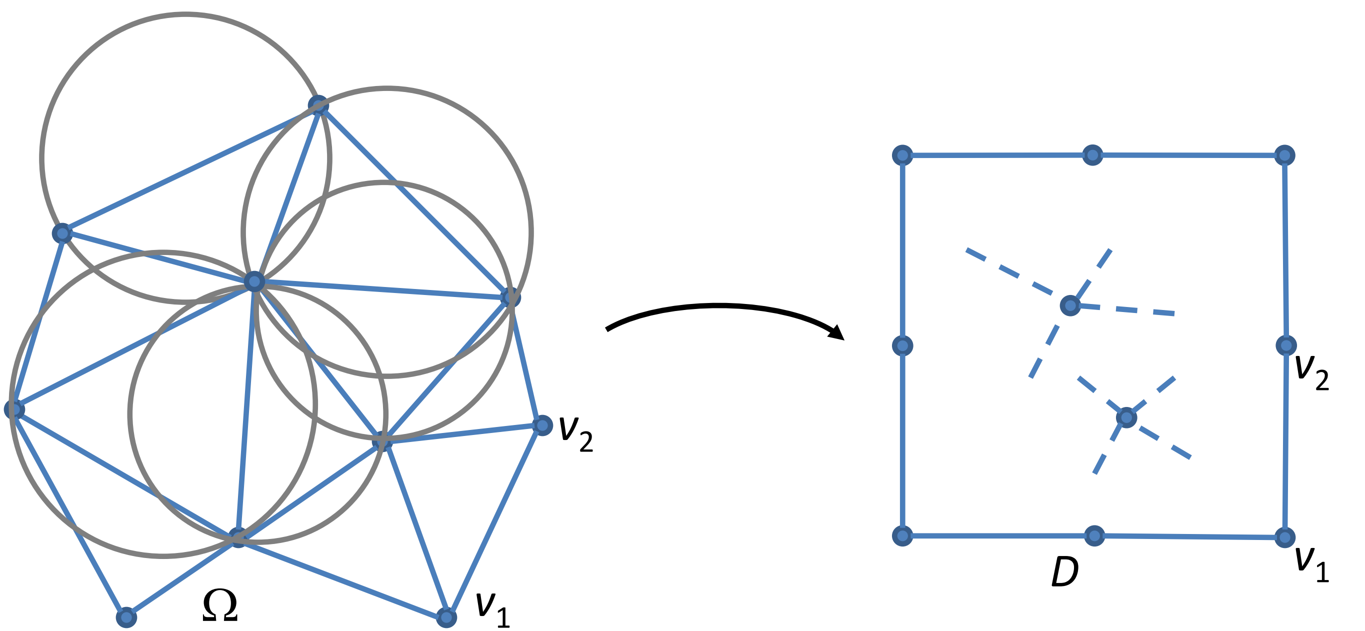

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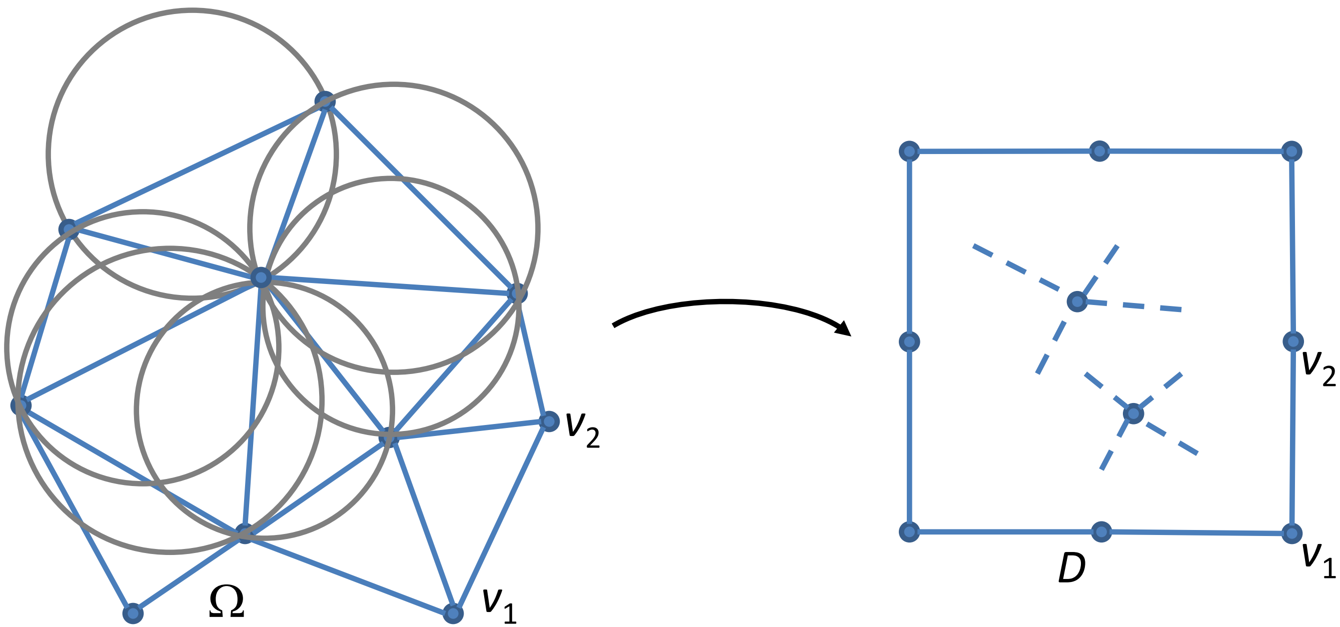

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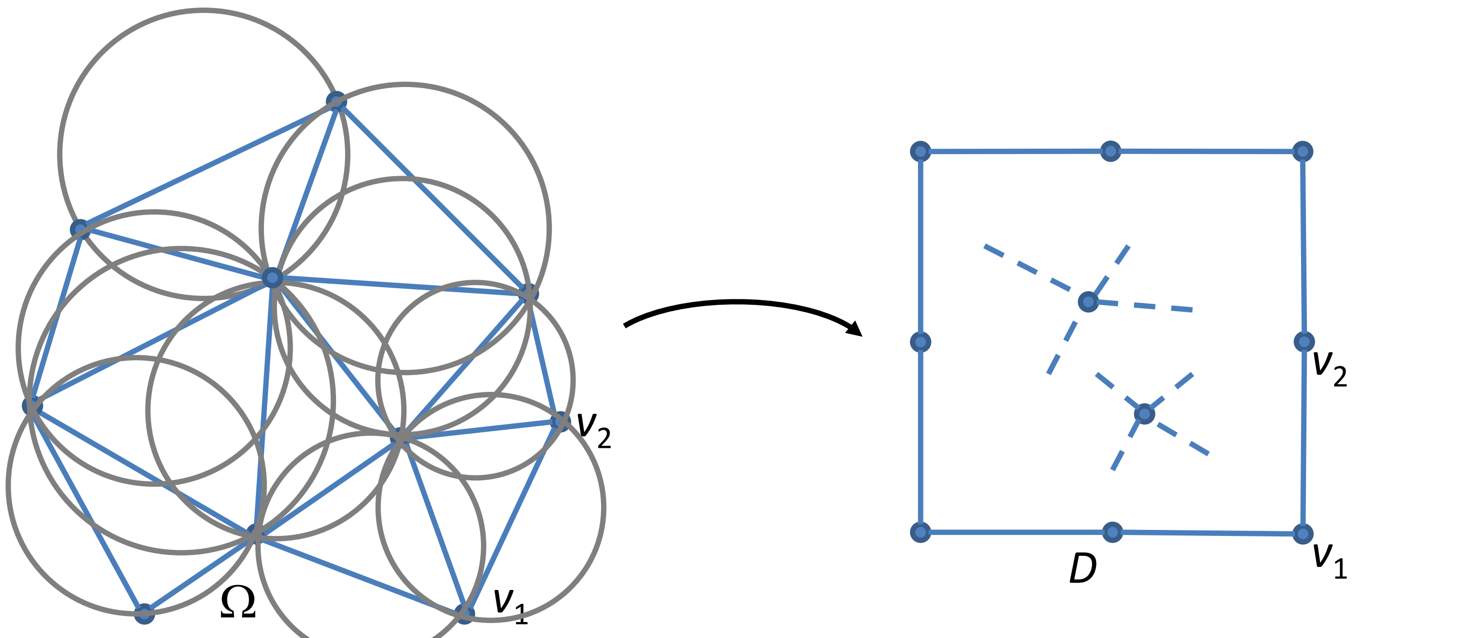
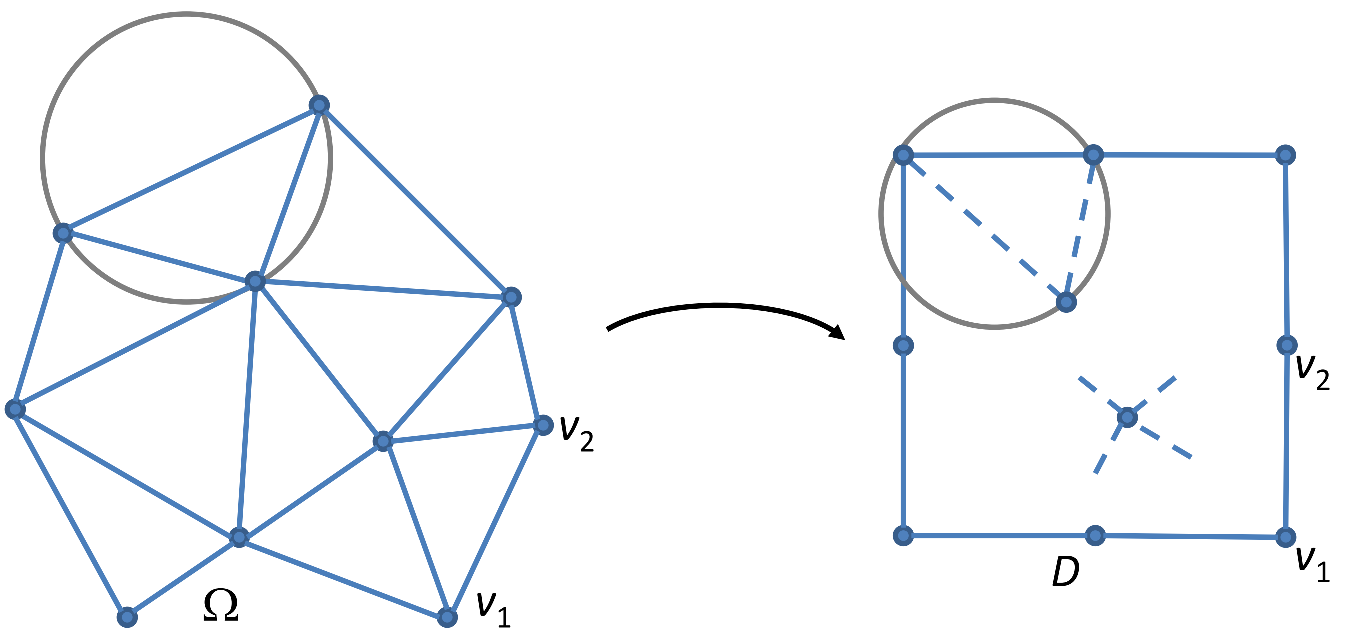


Planar Triangulation

<u>Elements of Conformality</u>:

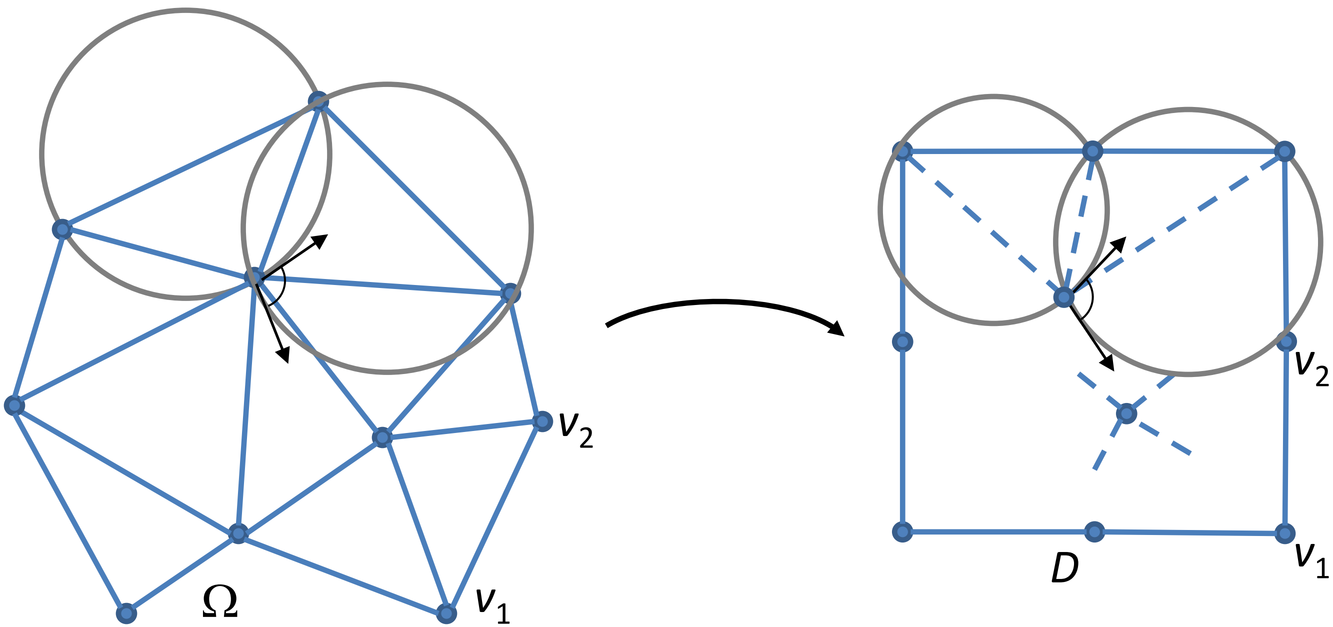
The mapping of Ω to D will automatically take circumcircles of triangles in Ω to circumcircles of triangles in D.



Planar Triangulation

<u>Elements of Conformality</u>:

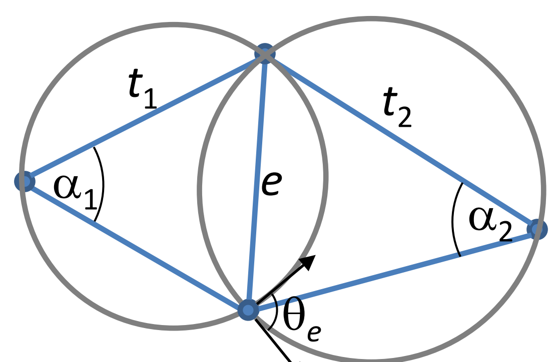
For the map to be conformal, we want it to preserve the angles between the circumcircles of edge-adjacent triangles.



Planar Triangulation

<u>Elements of Conformality</u>:

For triangles t_1 and t_2 meeting at edge e, the angle θ_e between the circumcircles is π-α_1-α_2.



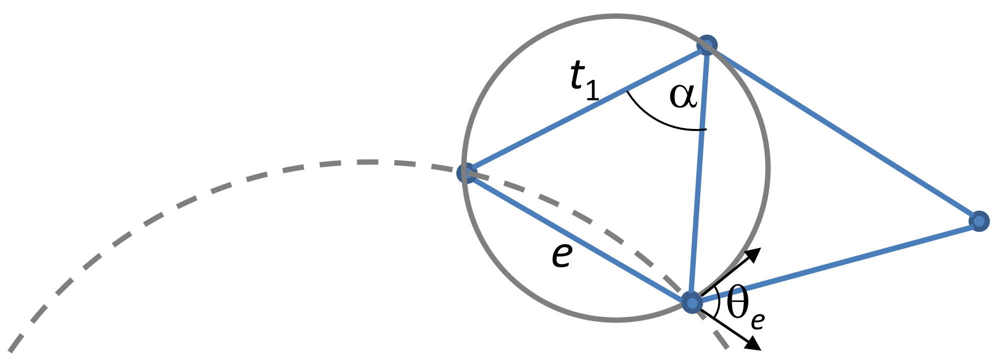
Planar Triangulation

<u>Elements of Conformality</u>:

For triangles t_1 and t_2 meeting at edge e, the angle θ_e between the circumcircles is π-α_1-α_2.

For a boundary triangle, we think of it as being adjacent to a triangle with a vertex at infinity.

Then the angle associated to the boundary edge becomes π-α.

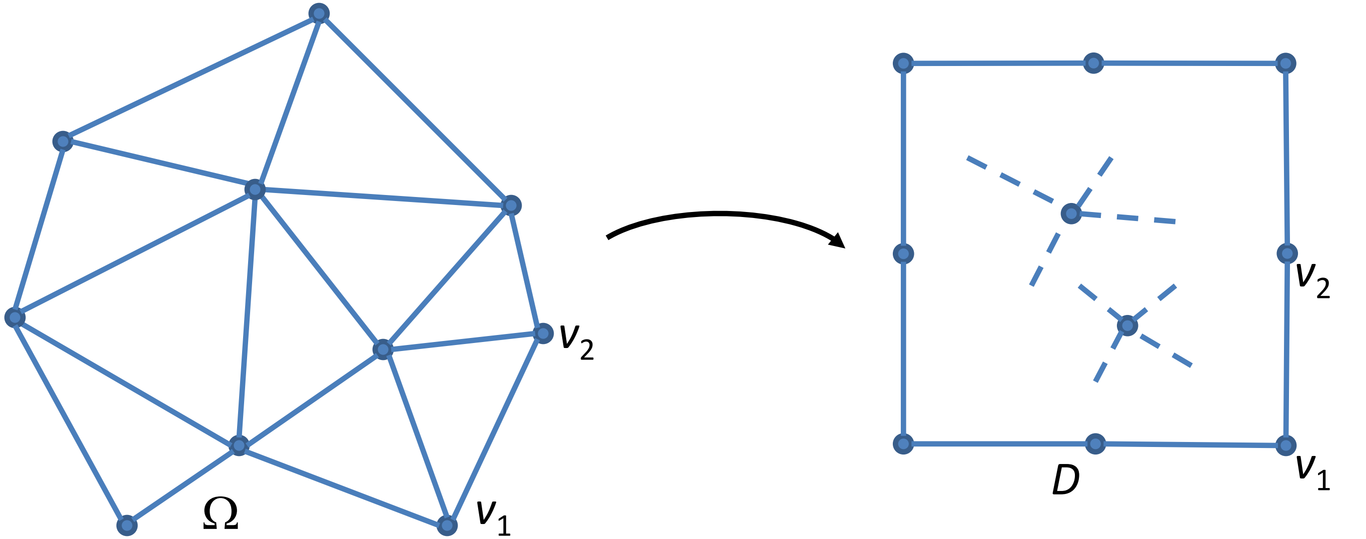


Planar Triangulation

Goal:

To find a mapping of the triangulation of Ω into a new triangulation that:

1. Satisfies the desired boundary constraints.
2. Preserves the angles θ_e.



Planar Triangulation

Defining a Mapping:

Q: What do we need to know in order to define a mapping?

Planar Triangulation

Defining a Mapping:

A: It suffices to know the radius of each circumcircle in the image.

Planar Triangulation

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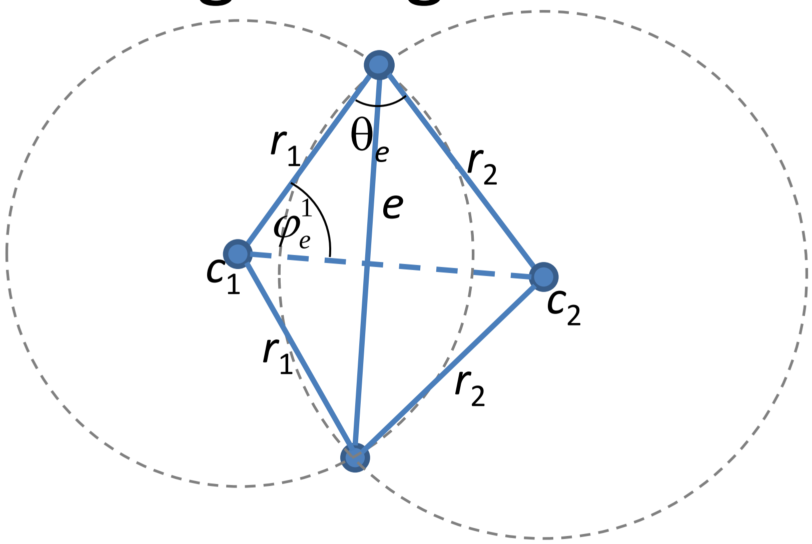
A: It suffices to know the radius of each circumcircle in the image.

If we know the radii and the angles θ_e, we can compute kite half-angles and edge lengths:

$$|e| = 2r_1 \sin\left(\phi_e^1\right)$$



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If we know the radii and the angles θ_e, we can compute kite half-angles and edge lengths:

$$|e| = 2r_1 \sin\left(\phi_e^1\right)$$

So we can layout the triangulation by placing the first edge down along the *x*-axis and then successively placing down the vertices across from the edge.