Differential Geometry:
Curves, Tangents, and Curvature

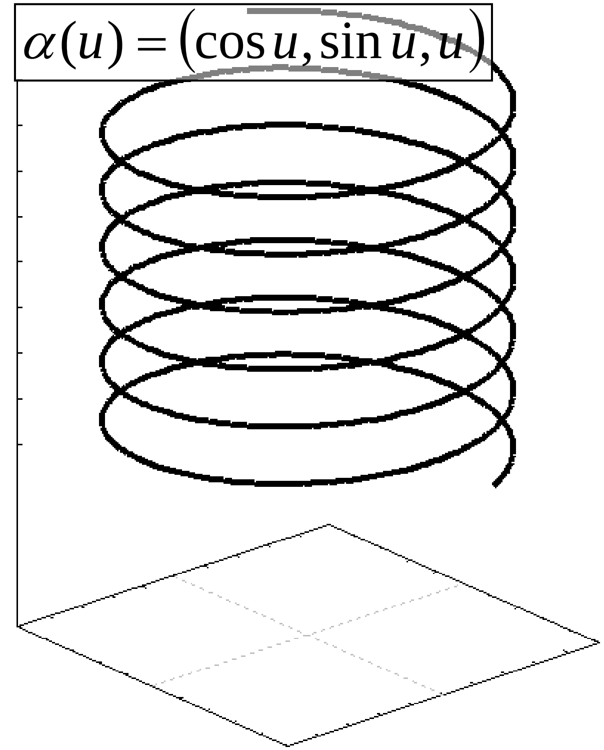
Differential Geometry of Curves and Surfaces, Do Carmo (Chapter 1)

Differentiable Curves

Definition:

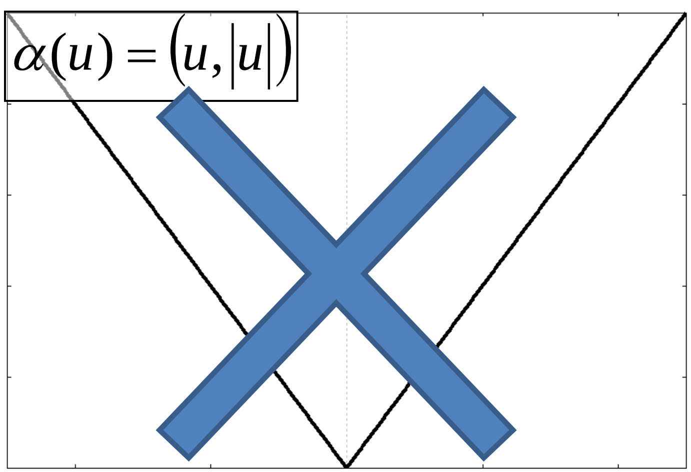
A *parameterized differentiable curve* is a differentiable map $\alpha: I \to \mathbf{R}^n$ of an open interval $I=(a,b)$ of the real line $\mathbf{R}$ into $\mathbf{R}^n$.

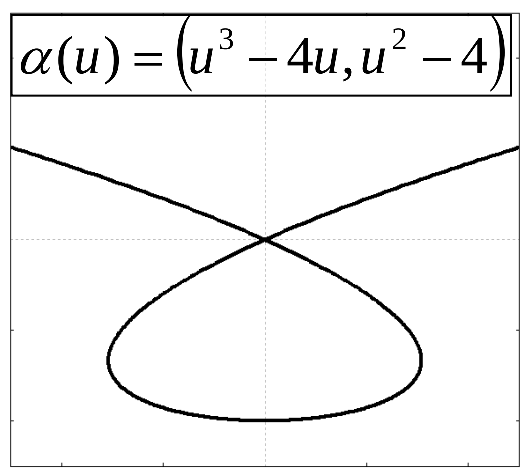
Differentiable Curves

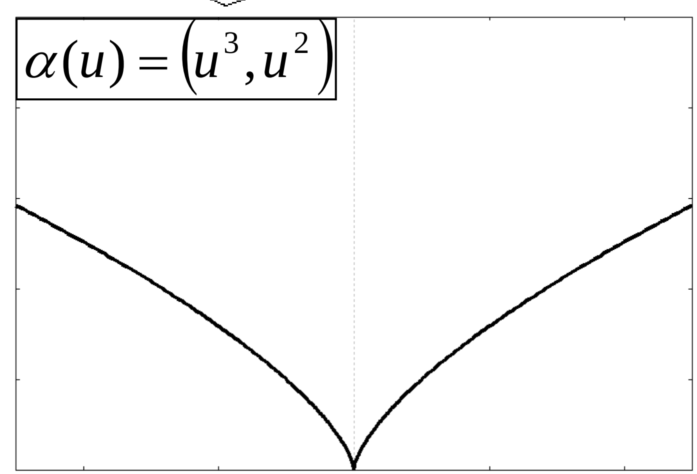


<u>Examples</u>:

- $\alpha(u)=(\cos u , \sin u , u)$
- $\alpha(u)=(u , |u|)$
- $\alpha(u)=(u^3-4u , u^2-4)$
- $\alpha(u)=(u^3 , u^2)$



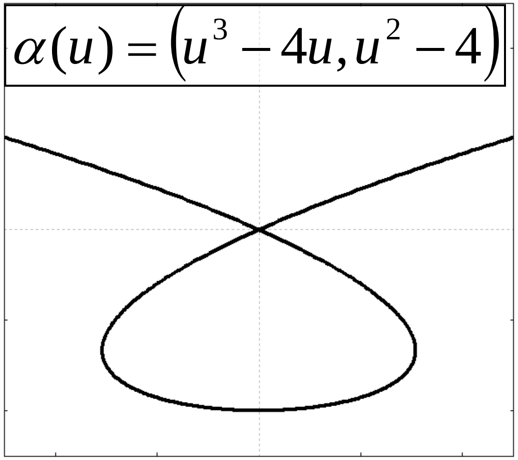


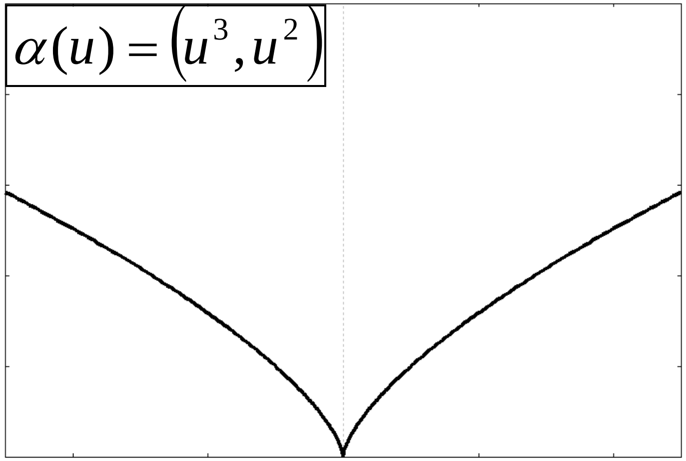


Differentiable Curves

<u>Note</u>:

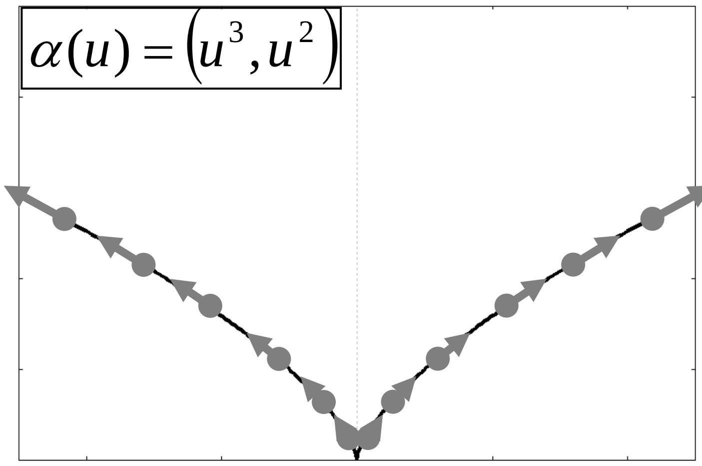
If $\phi(u)$ is a differentiable map from $[a',b']$ to $[a,b]$, then the function $\alpha \circ \phi : [a',b'] \rightarrow \boldsymbol{R}^n$ is also a parameterized differentiable curve tracing out the same points.





Regular Curves

Given a parameterized curve, $\alpha(u)$, the *tangent* of α at a point *u* is the vector defined by the derivative of the parameterization, $\alpha'(u)$.



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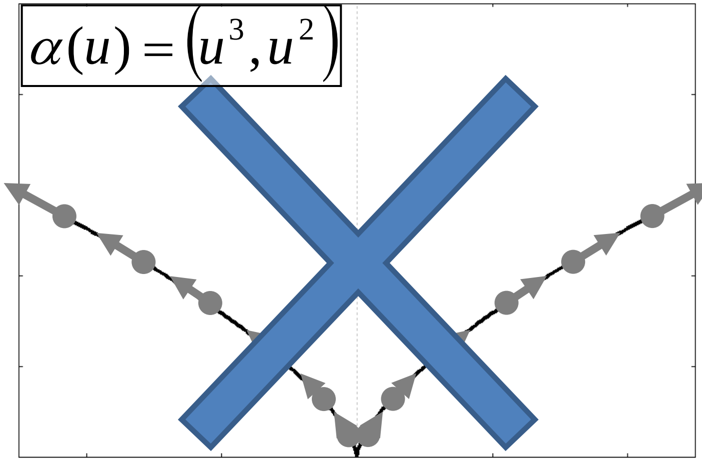
The parameterized curve $\alpha(u)$ is *regular* if $\alpha'(u) \neq 0$ for all $u \in I$.



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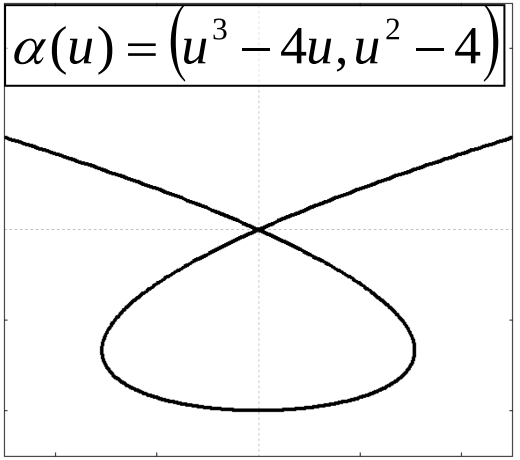
The parameterized curve $\alpha(u)$ is *regular* if $\alpha'(u) \neq 0$ for all $u \in I$. This implies:

– There is a "well-defined" tangent line at each point on the curve.

– The map $\alpha(u)$ is "injective"



Regular Curves

We say that two parameterized, differentiable, regular, curves α:[*a*,*b*]$\rightarrow$***R***^{n} and β:[*a'*,*b'*]$\rightarrow$***R***^{n} define the same *differentiable curve* if there exists a differentiable map ϕ:[*a'*,*b'*]$\rightarrow$[*a*,*b*], with $\phi'(u)\neq0$, such that $\beta\circ\phi=\alpha$.

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We say that they define the same *oriented differentiable curve* if, in addition, $\phi'(u)>0$.

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From now on, we will talk about (oriented) curves as equivalence classes, and parameterized curves as instances of a class.

Regular Curves

Given a parameterized curve $\alpha(v)$, and given $u \in I$, the *arc-length* from the point u_0 is:

$$s(u) = \int_{u_0}^{u} \left|\alpha'(v)\right| dv$$

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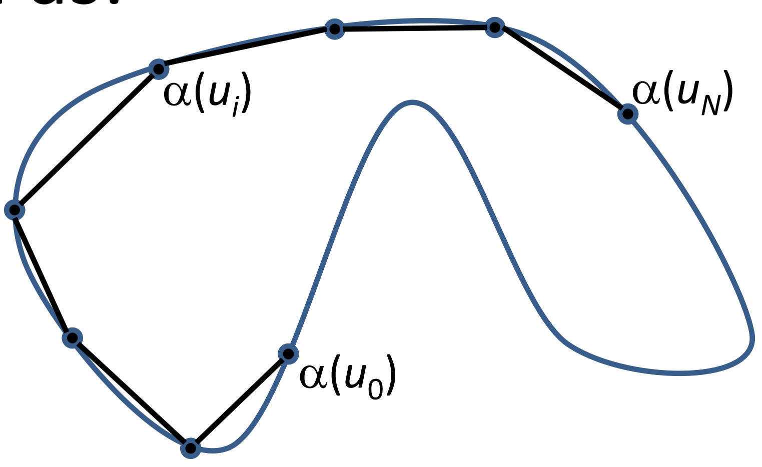
If we partition the interval $[u_0, u]$ into N sub-intervals, setting $\Delta u = (u - u_0)/N$ and $u_i = u_0 + i\Delta u$, we can approximate the integral as:

$$\begin{aligned} s(u) &= \lim_{N\to\infty} \sum_{i=0}^{N-1} \left|\alpha'(u_i)\right| \Delta u \\ &= \lim_{N\to\infty} \sum_{i=0}^{N-1} \frac{\left|\alpha(u_{i+1}) - \alpha(u_i)\right|}{\left|\Delta u\right|} \Delta u \\ &= \lim_{N\to\infty} \sum_{i=0}^{N-1} \left|\alpha(u_{i+1}) - \alpha(u_i)\right| \frac{\Delta u}{\left|\Delta u\right|} \end{aligned}$$



Regular Curves

Using the arc-length, we can integrate functions defined over the curve.

Given a function $f\!:\!\boldsymbol{R}^n \rightarrow \boldsymbol{R}$, the integral of f over the curve α can be obtained by integrating over the parameterization domain I:

$$\int_{p\in\alpha} f(p)\,dp = \int_a^b f\big(\alpha(u)\big)\big|\alpha'(u)\big|\,du$$

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Note: If the parameterizations α and β define the same curve, they define the same integral. So integrals are properties of curves.

Regular Curves

Given a regular differentiable curve α, we can define the unit-tangent vector at a point $\alpha(u)$ as:

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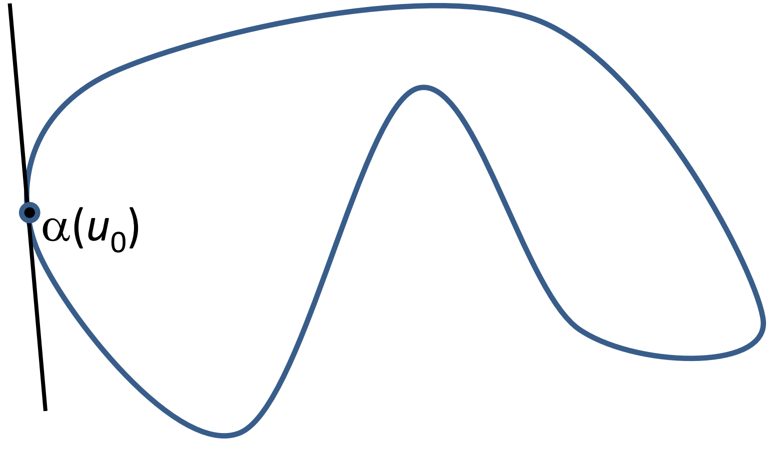
Tangent lines are properties of (un-oriented) curves.

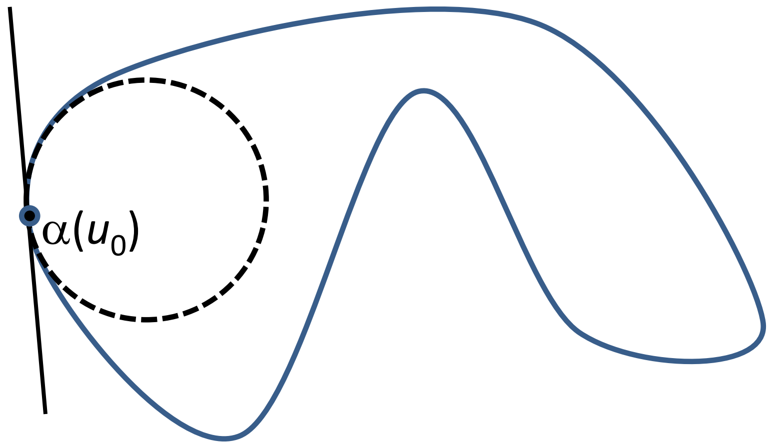
Curvature

The tangent line at a point $\alpha(u_0)$ on the curve describes the best-fitting line to the curve.



Curvature

The tangent line at a point $\alpha(u_0)$ on the curve describes the best-fitting line to the curve. We would like to extend this to finding the best-fit circle at a point.



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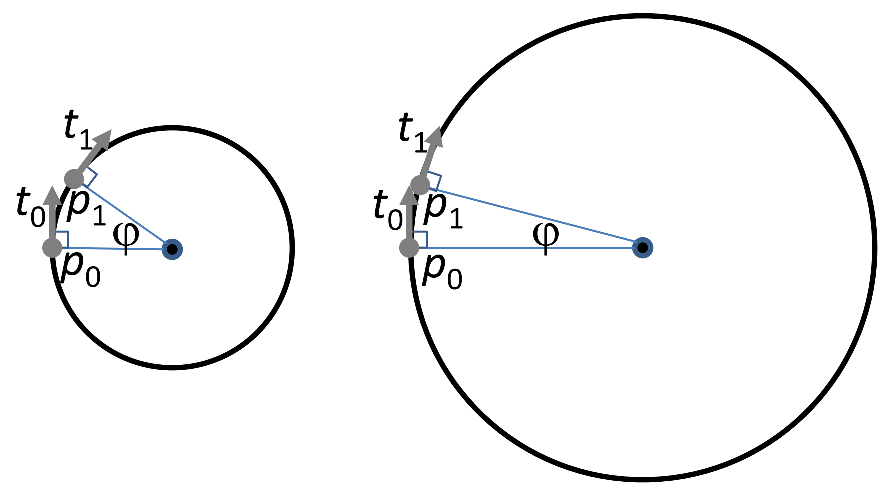

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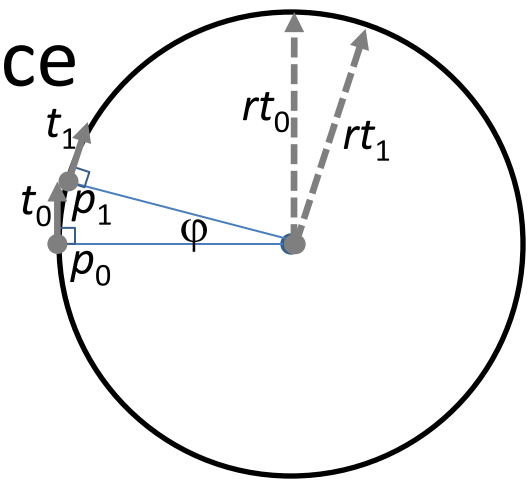
We use the fact that the length of the chord between p_0 and p_1 equals the distance between the tangents, multiplied by the circle's radius:

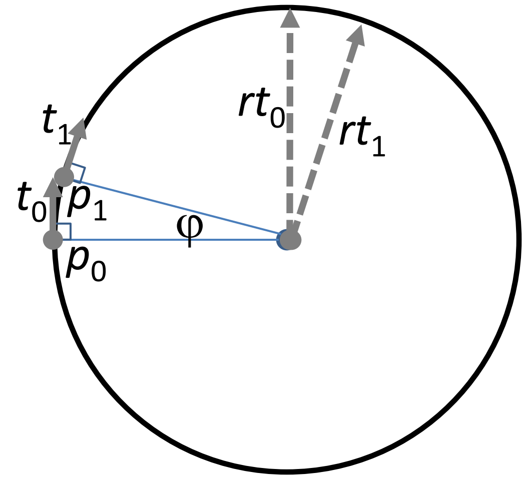


$$r\left|t_1 - t_0\right| = \left|p_1 - p_0\right|$$

Curvature

Thus, we can express the radius as:

$$\frac{1}{r} = \frac{|t_1 - t_0|}{|p_1 - p_0|}$$



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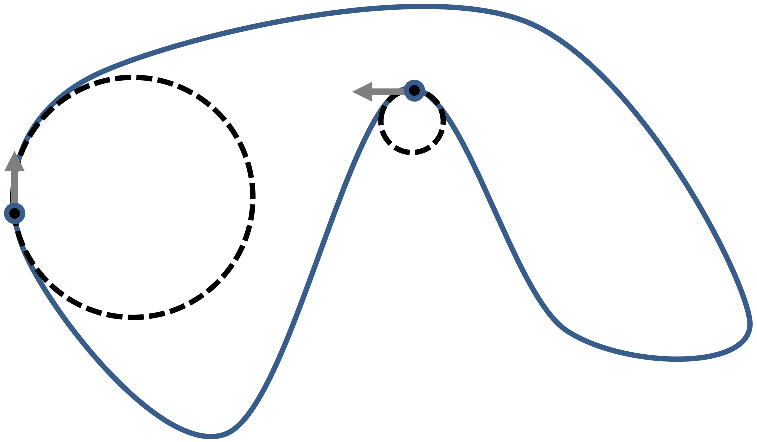
Thus, we can express the radius as:

$$\frac{1}{r} = \frac{|t_1 - t_0|}{|p_1 - p_0|}$$

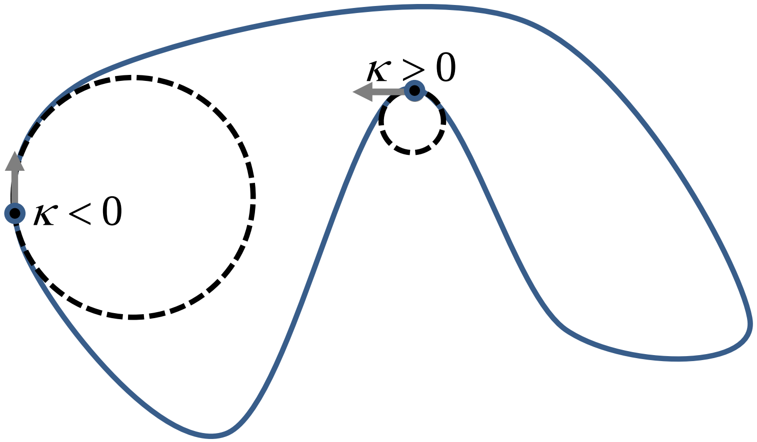
In the case of the curve, we can compute the curvature by looking at the limit as u_1 goes to u_0:

$$\kappa(u_0) = \lim_{u_1 \to u_0} \frac{|t(u_1) - t(u_0)|}{|\alpha(u_1) - \alpha(u_0)|} = \frac{|t'(u_0)|}{|\alpha'(u_0)|}$$



Curvature (2D)

For the case of an oriented curve in 2D, we can assign a sign to the curvature, making it positive if the center of the circle is to the left of the tangent and negative if it is to the right.



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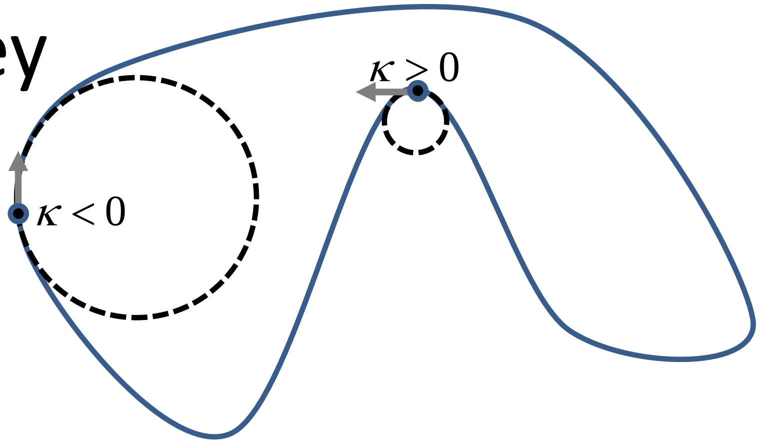


Curvature (2D)

Theorem:

For a closed oriented curve $\alpha \subset \mathbf{R}^2$, the integral of the curvature over α is an integer multiple of 2π:

$$\int_{p\in\alpha} \kappa(p)\,dp = 2\pi l$$

Curvature (2D)

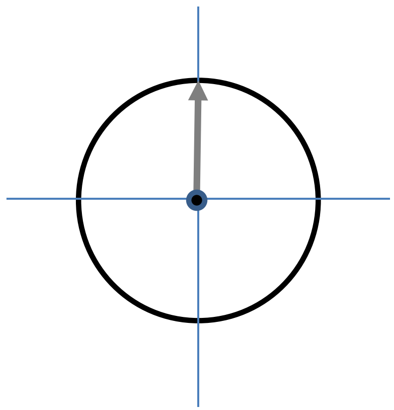
Rough Proof:

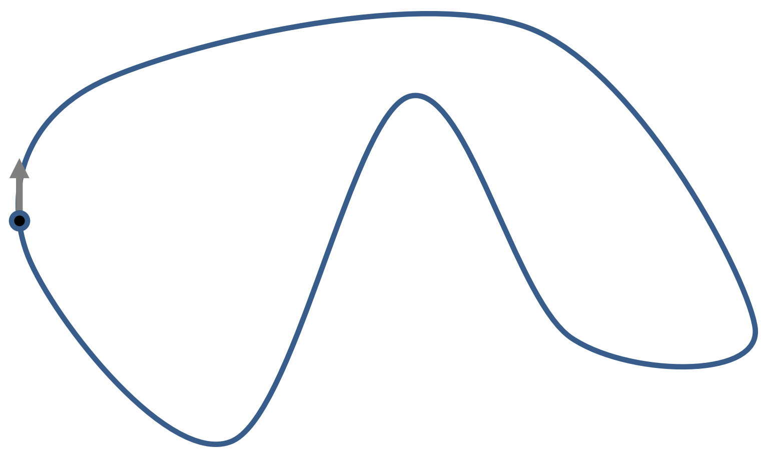
To prove this, we show that the integral of the curvature is equal to the integral of the change of angle traced out by the tangent indicatrix.

Curvature (2D)

Definition:

Given an oriented differentiable curve $\alpha: I \rightarrow \boldsymbol{R}^2$, we can define the *tangent indicatrix*, a map from the interval I to the unit-circle defined by the tangent.





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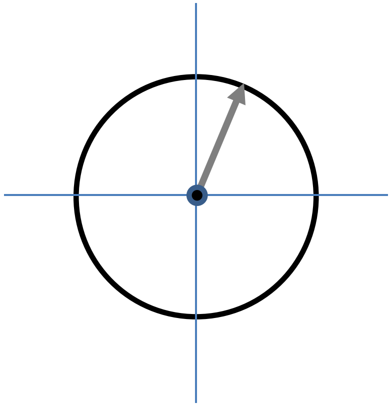

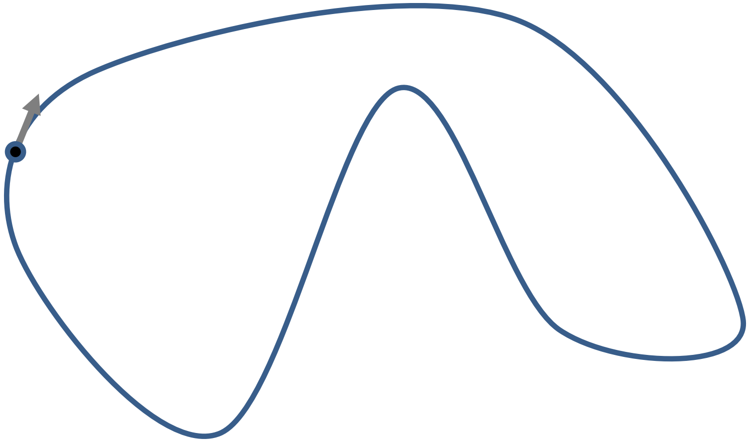

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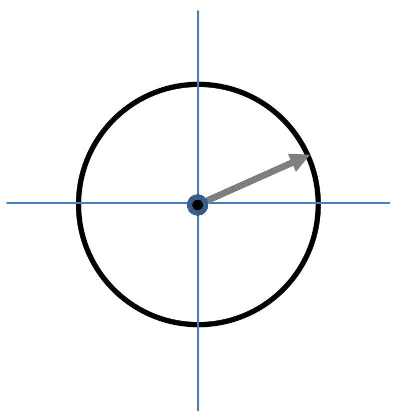

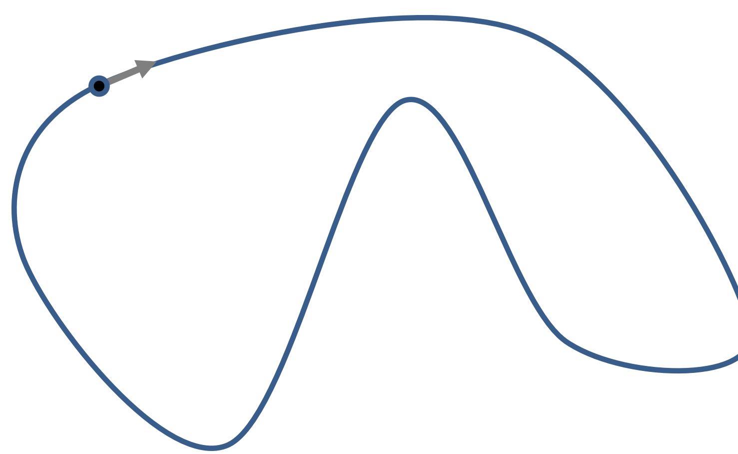

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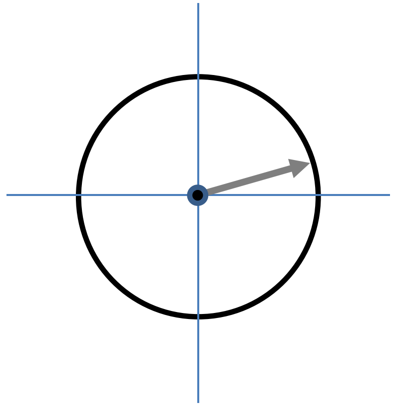

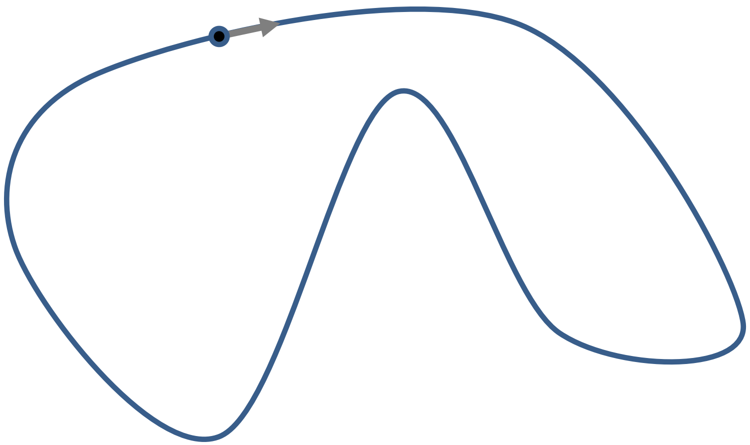

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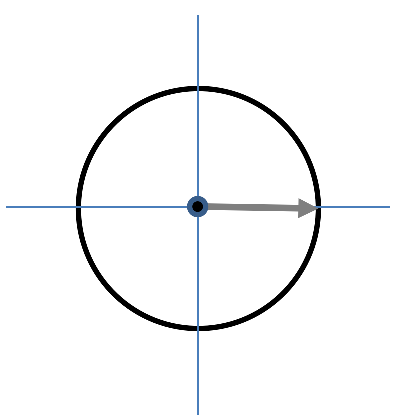

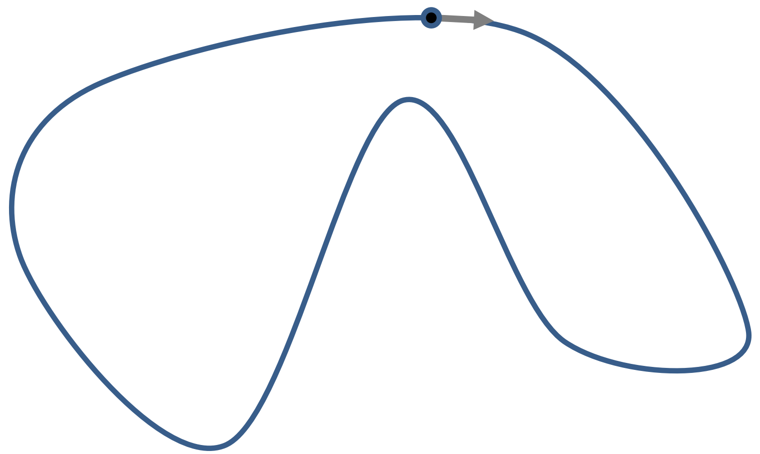

Curvature (2D)

Rough Proof:

1. Using the tangent indicatrix, we can assign an angle to each point on the curve, (the angle of the tangent vector).

Curvature (2D)

Rough Proof:

2. With a bit of manipulation, we can show that the change in the angle, as a function of arc-length, is equal to the curvature.

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Recall that the (signed) curvature is measured as the (signed) change in tangent length, divided by the change in arc-length.

$$\kappa(u_0) = \pm \frac{|t'(u_0)|}{|\alpha'(u_0)|}$$

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Rough Proof:

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Recall that the (signed) curvature is measured as the (signed) change in tangent length, divided by the change in arc-length.

For small step sizes, the change in tangent length is equal to the change in angle:

$$\sin\varphi \approx \varphi$$

Curvature (2D)

Rough Proof:

3. Since the integral is the accumulation of changes of angle, weighted by arc-length, and since we start and end with the same angle, the accumulated angle must be an integer multiple of 2π.

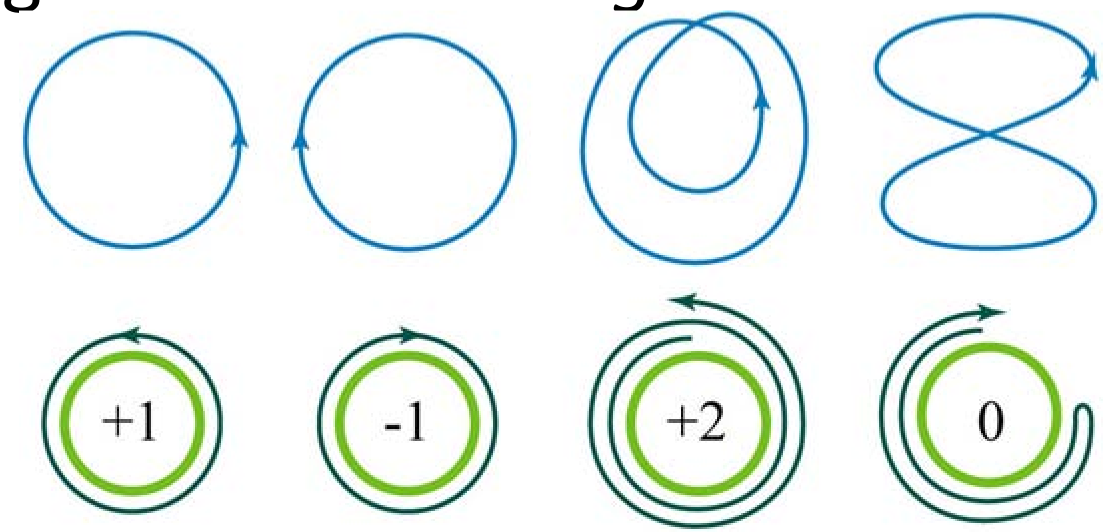
Curvature (2D)

Theorem:

For a closed curve $\alpha \subset \boldsymbol{R}^2$, the integral of the curvature over α is an integer multiple of 2π:

$$\int_{p\in\alpha} \kappa(p)dp = 2\pi l$$

The integer *l* is the *winding index* of the curve.

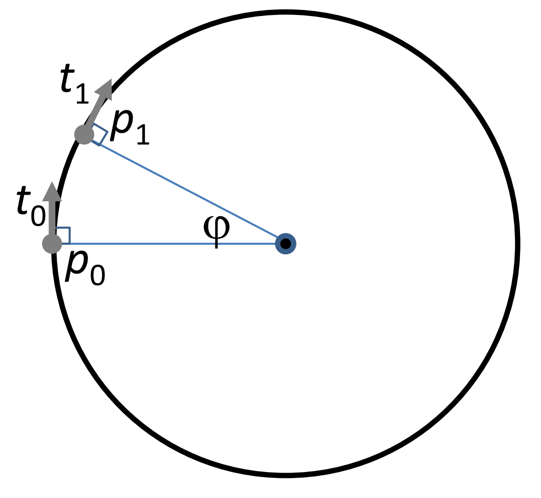


Curvature (2D)

In defining the curvature, we implicitly used two different definitions:

1. Curvature is the (signed) distance between tangents divided by the distance between the corresponding points on the circle.

$$\kappa(p_0) = \lim_{p_1 \to p_0} \frac{|t_1 - t_0|}{|p_1 - p_0|}$$



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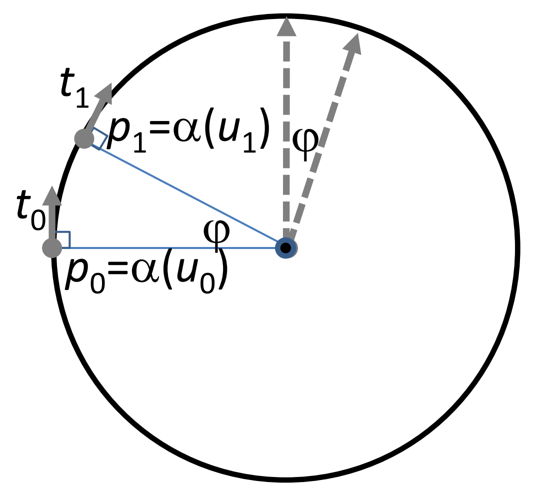
In defining the curvature, we implicitly used two different definitions:

2. Curvature is the (signed) angle between tangents divided by the arc-length between the corresponding points on the circle.

$$\kappa(p_0) = \lim_{u_1 \to u_0} \frac{\varphi}{\int_{u_0}^{u_1} |\alpha'(v)| dv}$$

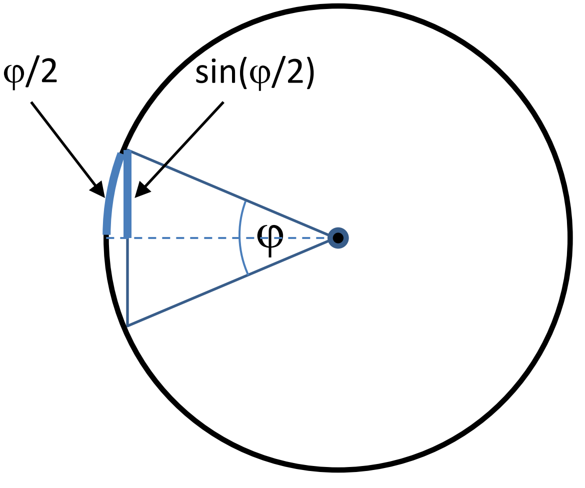


Curvature (2D)

In defining the curvature, we implicitly used two different definitions.

In the limit, the two definitions are the same since the ratio of chord-length to arc-length goes to one as points get closer:

$$2\sin\frac{\varphi}{2} = \frac{\varphi}{2} + O(\varphi^3)$$



Curvature (2D)

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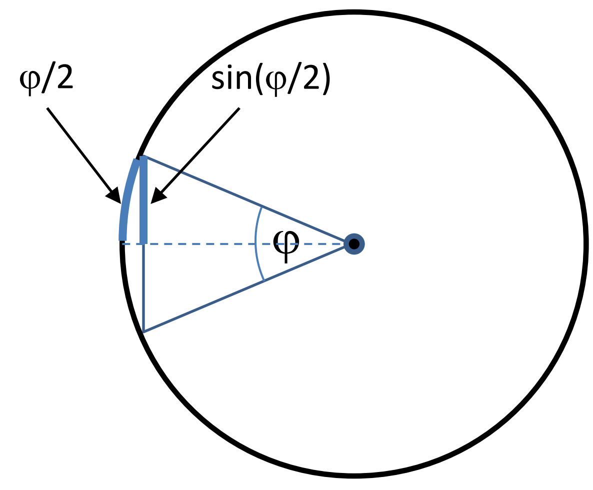
In the limit, the two definitions are the same since the ratio of chord-length to arc-length goes to one as points get closer.

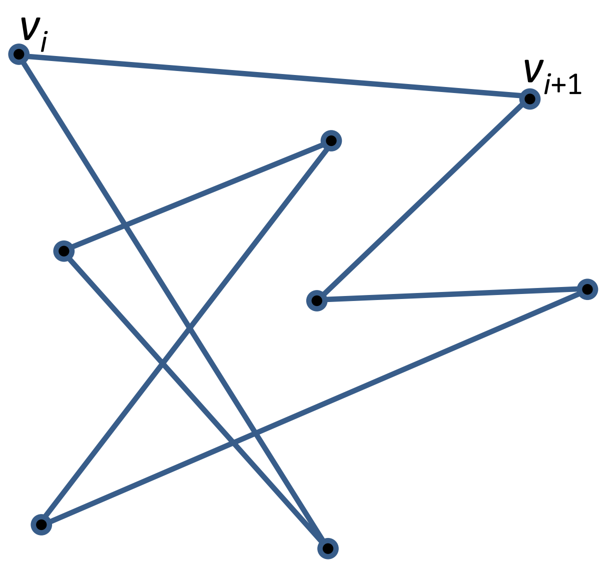
But in the discrete setting, they gives rise to two different notions of curvature.



Discrete Curves

<u>Definition</u>:

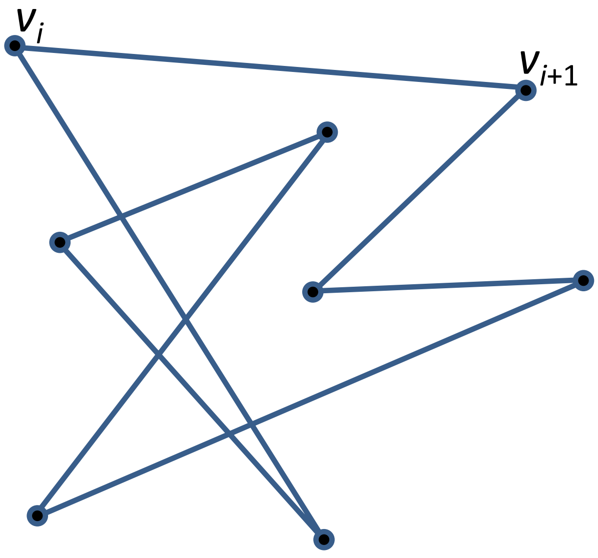
A *discrete curve* is the set of points on the line segments between consecutive vertices in an ordered set of vertices $\{v_0,\ldots,v_N\}\subset\mathbf{R}^n$, such that for any index i, $v_i \neq v_{i+1}$.



Discrete Curves

The *length* of the discrete curve between vertex v_{i_0} and v_i is:

$$l_i = \sum_{j=i_0}^{i-1} \left| v_{j+1} - v_j \right|$$



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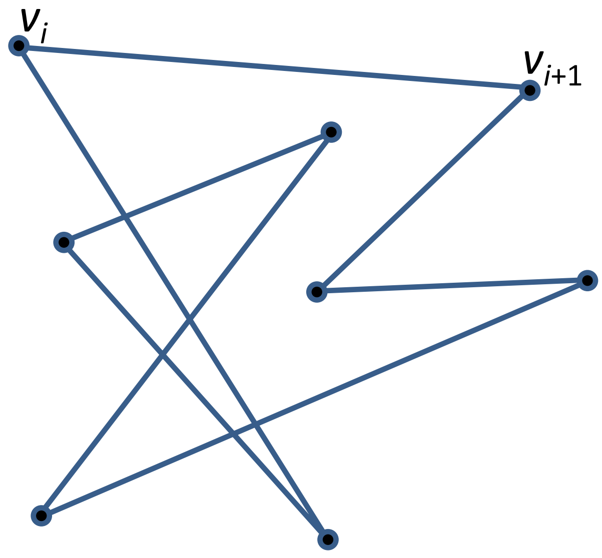
Given a function assigning a value f_i to edge $\overline{v_i v_{i+1}}$, we can compute the integral of the function as:

$$F = \sum_{i=0}^{N-1} \left| v_{i+1} - v_i \right| f_i$$



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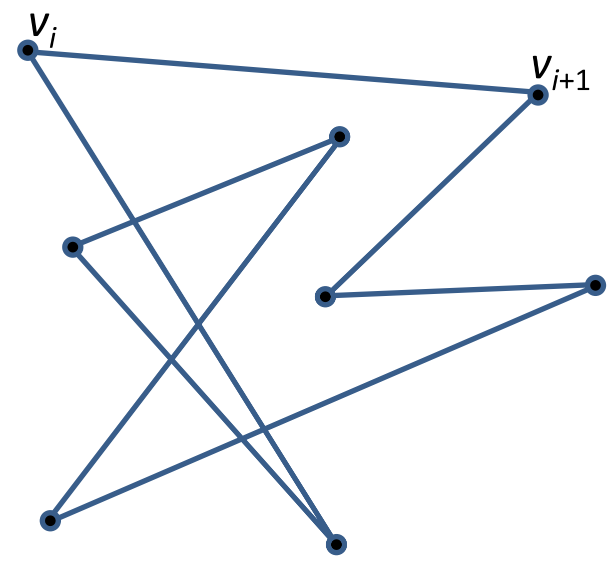
Similarly, given a function assigning a value f_i to vertex v_i, we can compute the integral of the function as:

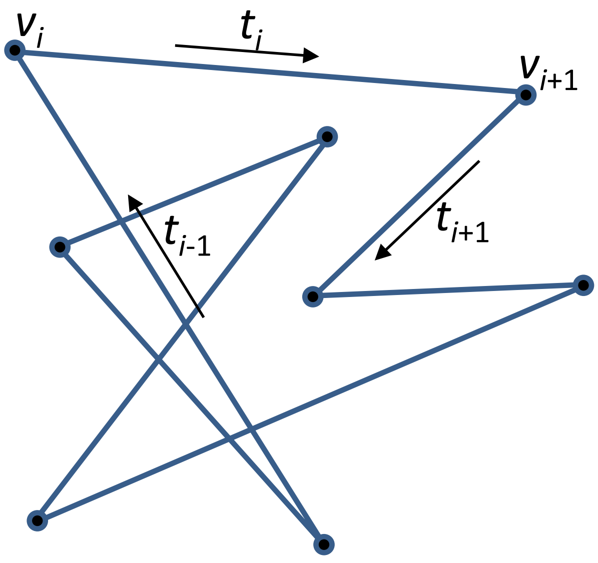
$$F = \sum_{i=0}^{N-1} \frac{\left| v_{i+1} - v_i \right| + \left| v_i - v_{i-1} \right|}{2} f_i$$



Discrete Curves

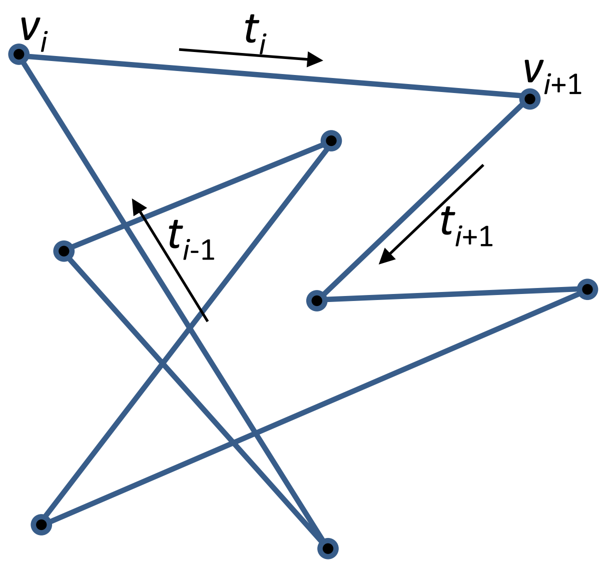
At any (point on an) edge $\overline{v_i v_{i+1}}$, we can define the tangent vector t_i as the unit vector pointing from v_i to v_{i+1}:

$$t_i = \frac{v_{i+1} - v_i}{\left|v_{i+1} - v_i\right|}$$



Discrete Curves

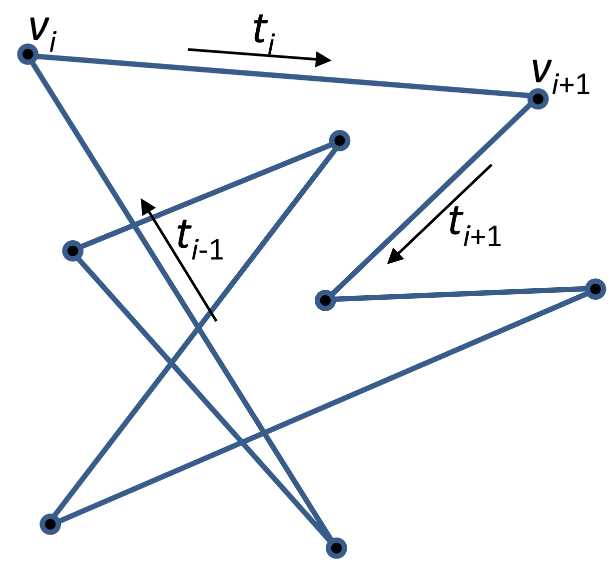
Q: Where and how should we define curvature?



Discrete Curves

Q: Where and how should we define curvature?

A: Since the tangent only changes at vertices we should define the curvature as a vertex value.

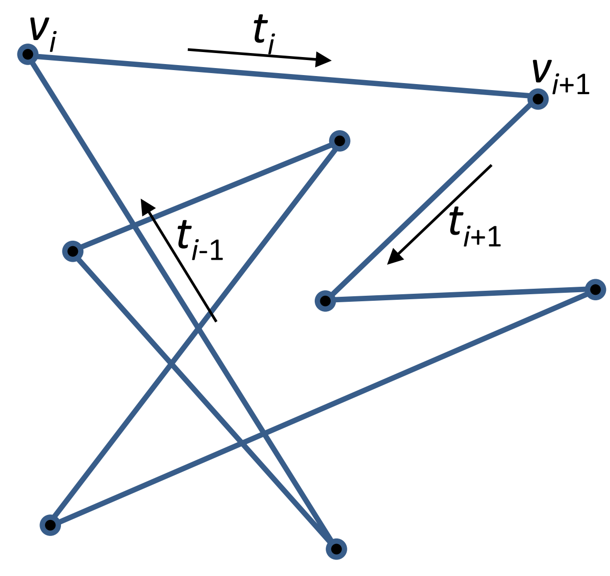


Discrete Curves

Q: Where and how should we define curvature?

A: Since the tangent only changes at vertices we should define the curvature as a vertex value.

We should define the value of the curvature as the change in the tangent as we move through the vertex.

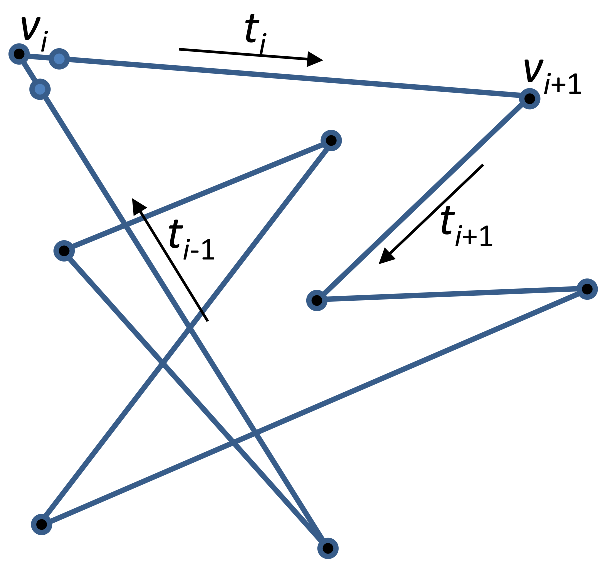


Discrete Curves

Note that we cannot define the tangent as the result of a limiting process:

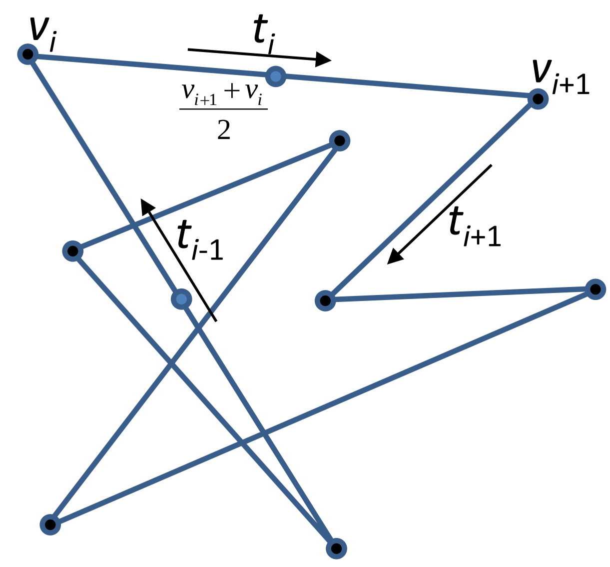
$$\kappa_i = \lim_{\Delta t \to 0} \frac{\left|t_i - t_{i-1}\right|}{\left\|\left[(1-\Delta t)v_i + \Delta t v_{i+1}\right] - \left[(1-\Delta t)v_i + \Delta t v_{i-1}\right]\right\|}$$

$$= \lim_{\Delta t \to 0} \frac{\left|t_i - t_{i-1}\right|}{\Delta t \left|v_{i+1} - v_{i-1}\right|}$$



Discrete Curves

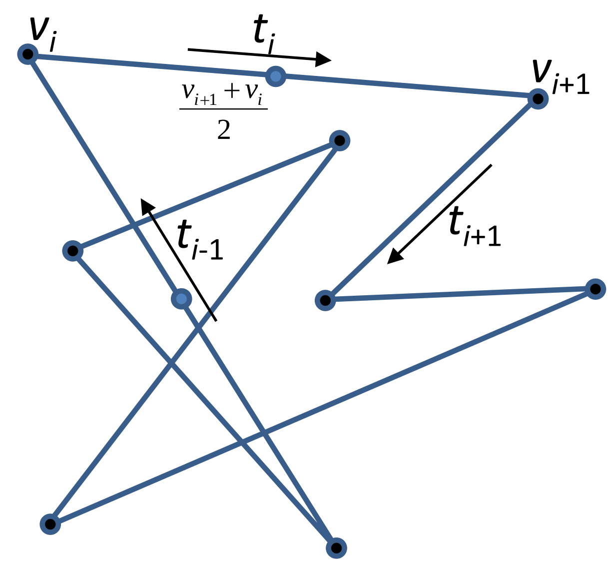
However, we can estimate it using the finite-differences using the edge centers $(v_i+v_{i+1})/2$.



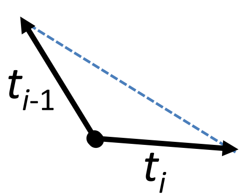
Discrete Curves

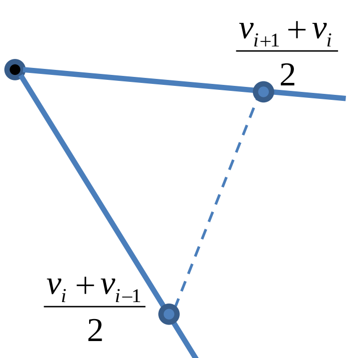
However, we can estimate it using the finite-differences using the edge centers $(v_i+v_{i+1})/2$.

1. Define the curvature in terms of change in tangents divided by the distance between edge centers:



$$\kappa_i = \frac{|t_i - t_{i-1}|}{\left\|\left[(v_i + v_{i+1})/2\right] - \left[(v_{i-1}+v_i)/2\right]\right\|}$$



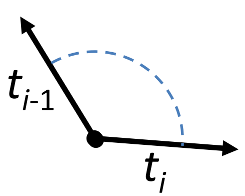


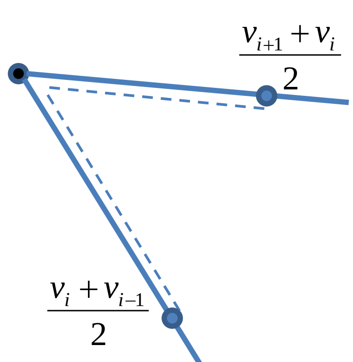
Discrete Curves

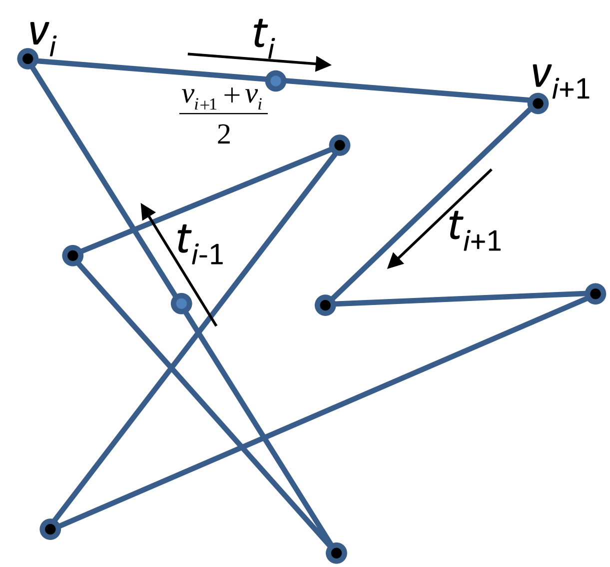
However, we can estimate it using the finite-differences using the edge centers $(v_i+v_{i+1})/2$.

2. Define the curvature in terms of change in tangents angle divided by the arc-length between edge centers:

$$\kappa_i = \frac{\angle t_1 t_2}{\left|(v_{i+1} - v_i)/2\right| + \left|(v_i - v_{i-1})/2\right|}$$



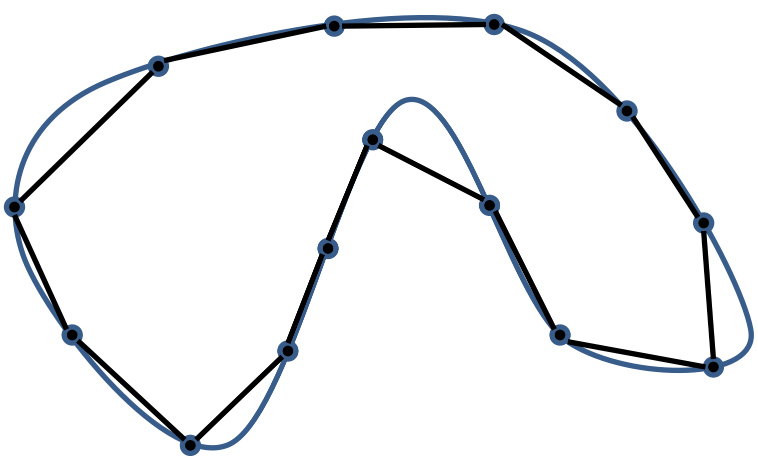




Discrete Curves

In the limit, as we refine the tessellation of a differentiable curve, both methods define the same curvature.

Does it make a difference which one we use?

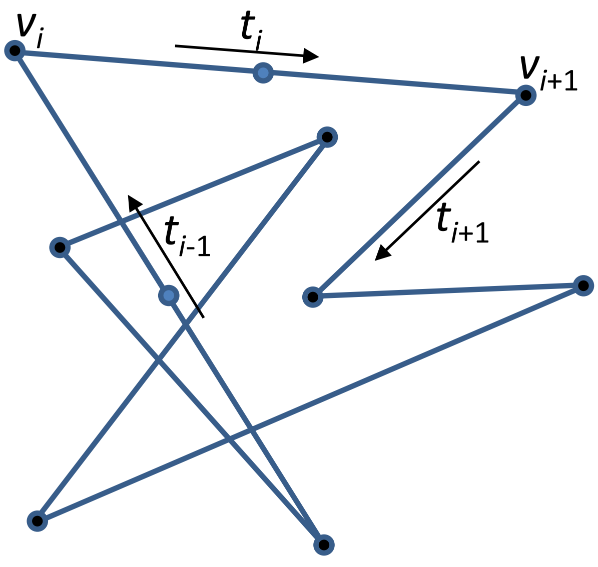


Discrete Curves

Does it make a difference which one we use?

Consider integrating the curvature over the closed curve:

$$K = \sum_{i=0}^{N-1} \frac{|v_{i+1} - v_i| + |v_i - v_{i-1}|}{2} \kappa_i$$



Discrete Curves

Does it make a difference which one we use?

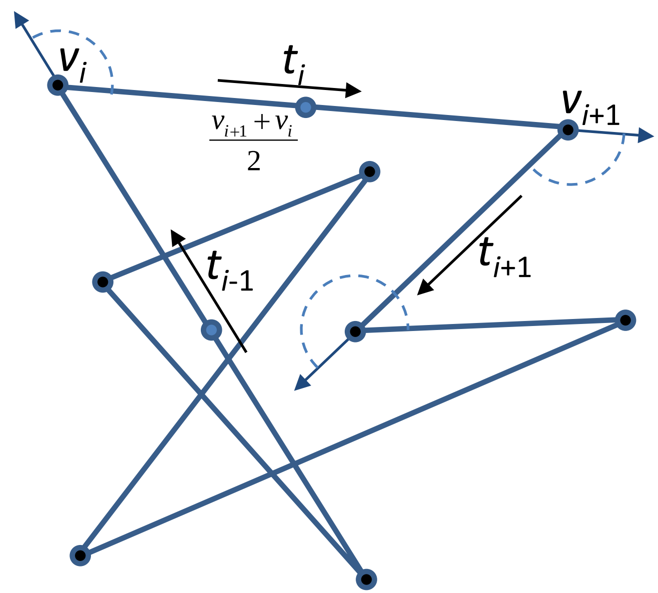
Consider integrating the curvature over the closed curve:

$$\mathrm{K} = \sum_{i=0}^{N-1} \frac{\left|v_{i+1} - v_i\right| + \left|v_i - v_{i-1}\right|}{2} \kappa_i$$

Using the second definition, we get:

$$\kappa_i = \frac{\angle t_1 t_2}{\left|(v_{i+1} - v_i)/2\right| + \left|(v_i - v_{i-1})/2\right|}$$

$$\mathrm{K} = \sum_{i=0}^{N-1} \angle t_1 t_2$$



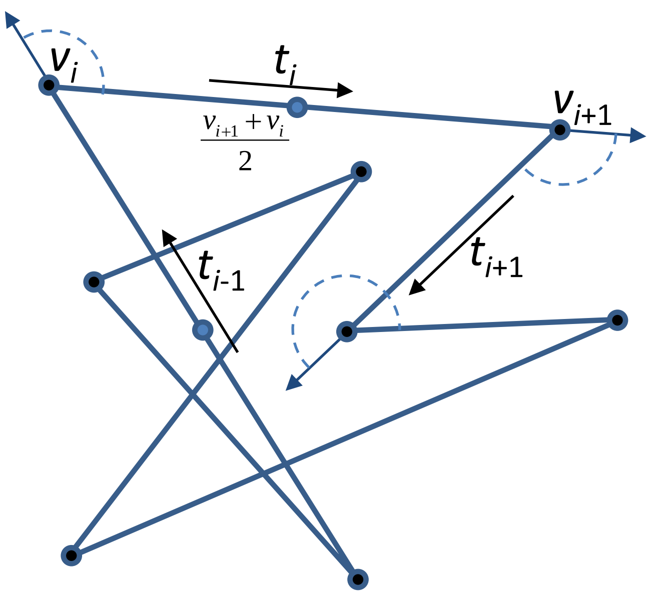
Discrete Curves

Does it make a difference which one we use?

Consider integrating the curvature over the closed curve:

$$\mathrm{K} = \sum_{i=0}^{N-1} \frac{|v_{i+1} - v_i| + |v_i - v_{i-1}|}{2} \kappa_i$$

Using the second definition, we get the sum of angular deficits over the vertices of a closed polygon.



Discrete Curves

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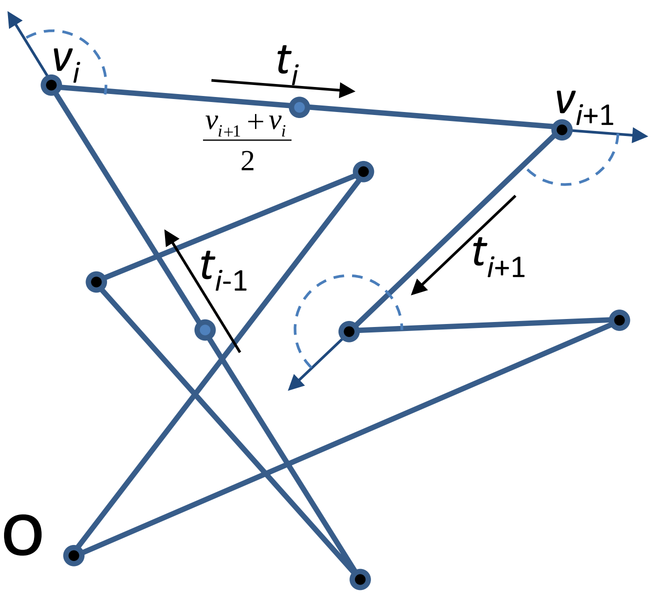

Using the second definition, we get the sum of angular deficits over the vertices of a closed polygon.

This <u>has</u> to be a multiple of 2π so this definition conforms to the winding index.

Discrete Curves

Does it make a difference which one we use?

Consider integrating the curvature over the closed curve:

$$\mathrm{K} = \sum_{i=0}^{N-1} \frac{|v_{i+1} - v_i| + |v_i - v_{i-1}|}{2} \kappa_i$$

Using the first definition we would get a value for the curvature integral that is not a multiple of 2π, so this definition does not conform to the winding index.