Discrete Differential Geometry (600.657)

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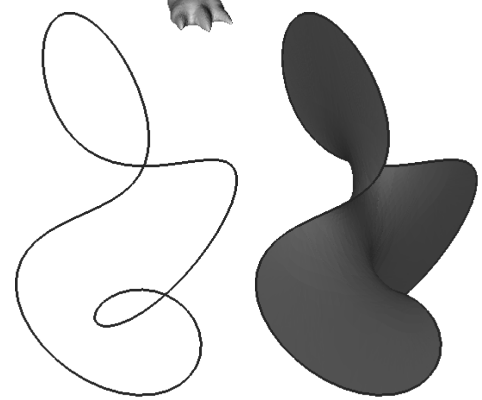
Outline

- Why discrete differential geometry?
- What will we cover?

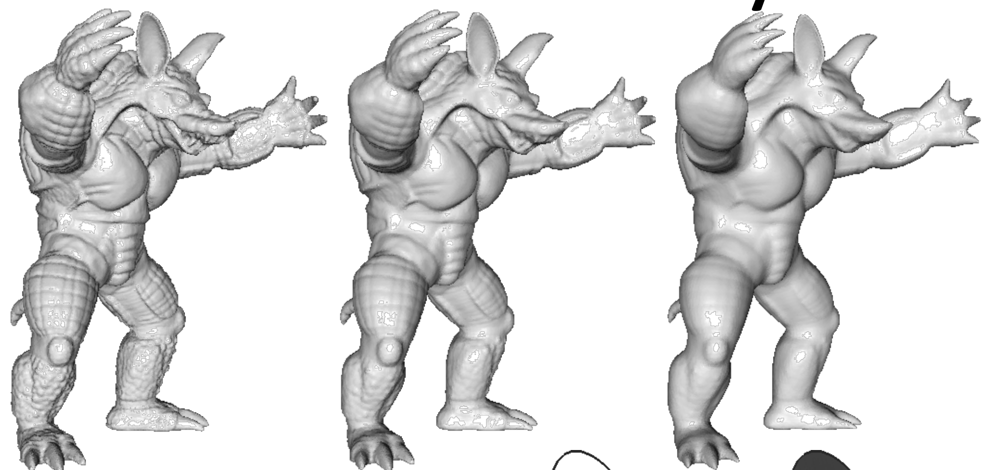
Why Discrete Differential Geometry?

<ins>Differential Geometry</ins>:

- Surface Evolution
  - Mean-curvature flow
  - Willmore flow

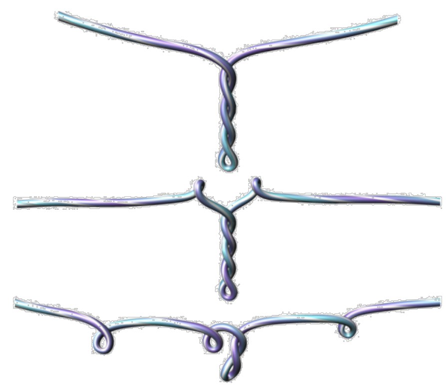


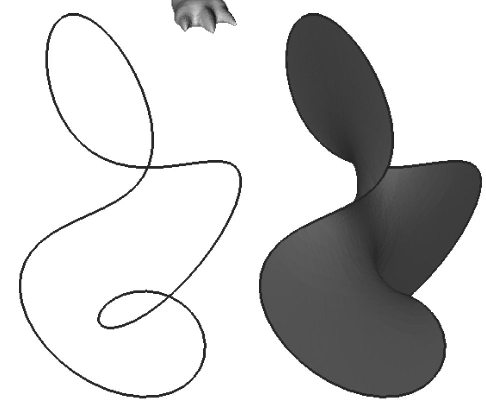
Why Discrete Differential Geometry?

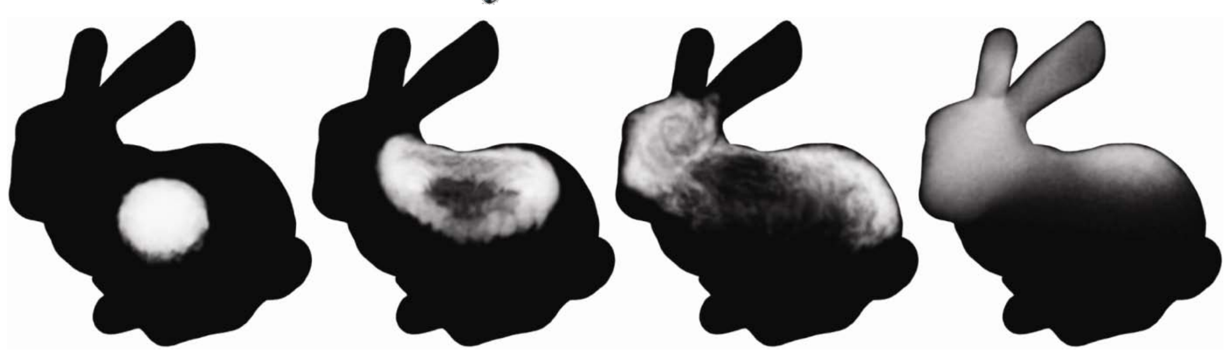


Differential Geometry:

- Surface Evolution
  - Mean-curvature flow
  - Willmore flow
- Dynamical Systems
  - Twisting rods
  - Smoke
  - Fluids







Why Discrete Differential Geometry?

Discretized Geometry:

One approach is to discretize the system, breaking it up into discrete time steps and using differencing to approximate differentiation.

Why Discrete Differential Geometry?

Example (Conservation of Energy):

Dropping a ball from an initial height of y_0, we have a system with:

$y(t)$:= height at time t
$v(t)$:=$y'(t)$, velocity at time t
$a(t)$:=$v'(t)$, acceleration (constant=-g)

We know that the value:

$$E(t) = \frac{1}{2} m v^t(t) + m g y(t)$$

should be constant.

Why Discrete Differential Geometry?

<u>Example (Conservation of Energy)</u>:

Discretizing with time step Δt, we set:

$$a_k = (v_k - v_{k-1}) / \Delta t$$
$$v_k = (y_{k+1} - y_k) / \Delta t$$

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$$y_k = y_0 - \frac{(g \, \Delta t \, k)^2}{2} + \frac{g \, \Delta t^2 \, k}{2}$$

Plugging this into the equation for the energy, we get:

$$E_k = mgy_0 + \frac{mg^2 \Delta t^2 k}{2}$$

Why Discrete Differential Geometry?

<u>Example (Conservation of Energy)</u>:

So our discretized system is wrong in two ways.

First, we do not get the "correct" solution

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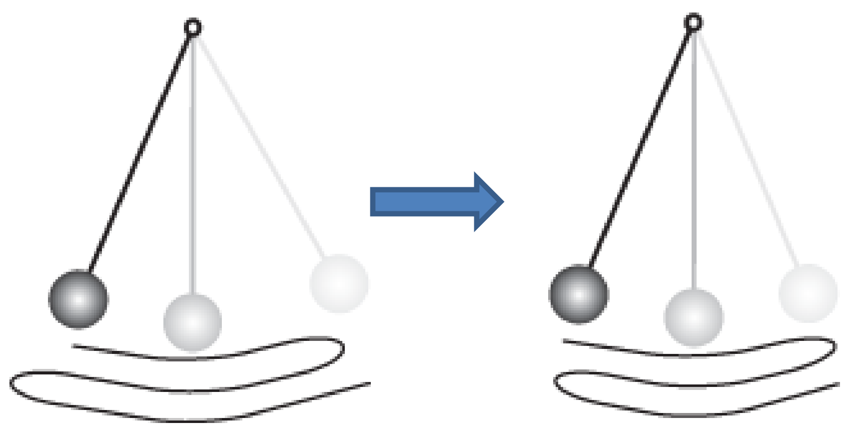
More importantly, the system gains energy:

$$E_k = mgy_0 \boxed{+ \frac{mg^2\Delta t^2 k}{2}}$$

Why Discrete Differential Geometry?

<u>Discrete Geometry</u>:

Although we expect the results of the finite approximation to be imprecise, we would like to construct it so that the invariants are preserved.



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<u>Discrete Geometry</u>:

In the case of the dropping ball, we would like to have energy preservation:

$$\frac{1}{2}mv_k^2 + mgy_k = \frac{1}{2}mv_{k+1}^2 + mgy_{k+1}$$

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which forces us to define the relation between changes in height and velocities differently:

$$y_{k+1} - y_k = \frac{v_{k+1}^2 - v_k^2}{-2g}$$

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With this new discrete derivative, our system is guaranteed to be energy preserving.

What Will We Cover?

- Differential Geometry of Curves/Surfaces
- What we can measure
- Discrete Exterior Calculus
- Physical Modeling
- Conformal Geometry
- Surface and Volume Meshing