Discrete Differential Geometry
(600.657)

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Outline

• Why discrete differential geometry?
• What will we cover?
Why Discrete Differential Geometry?

Differential Geometry:

• Surface Evolution
  – Mean-curvature flow
  – Willmore flow
Why Discrete Differential Geometry?

Differential Geometry:

• Surface Evolution
  – Mean-curvature flow
  – Willmore flow

• Dynamical Systems
  – Twisting rods
  – Smoke
  – Fluids
Why Discrete Differential Geometry?

Discretized Geometry:
One approach is to discretize the system, breaking it up into discrete time steps and using differencing to approximate differentiation.
Why Discrete Differential Geometry?

Example (Conservation of Energy):
Dropping a ball from an initial height of $y_0$, we have a system with:

- $y(t)$: height at time $t$
- $v(t) := y'(t)$, velocity at time $t$
- $a(t) := v'(t)$, acceleration (constant=$-g$)

We know that the value:

$$E(t) = \frac{1}{2}mv'(t) + mg y(t)$$

should be constant.
Why Discrete Differential Geometry?

Example (Conservation of Energy):

Discretizing with time step $\Delta t$, we set:

\[ a_k = \left( v_k - v_{k-1} \right) / \Delta t \]
\[ v_k = \left( y_{k+1} - y_k \right) / \Delta t \]
Why Discrete Differential Geometry?

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which gives:

$$a_k = -g$$
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$$v_k = -g \Delta t$$
Why Discrete Differential Geometry?

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Discretizing with time step $\Delta t$, we set:

\[
\begin{align*}
    a_k &= \left( v_k - v_{k-1} \right) / \Delta t \\
    v_k &= \left( y_{k+1} - y_k \right) / \Delta t
\end{align*}
\]

which gives:

\[
\begin{align*}
    a_k &= -g \\
    v_k &= -g \, \Delta t \, k
\end{align*}
\]

\[
y_k = y_0 - \frac{(g \, \Delta t \, k)^2}{2} + \frac{g \, \Delta t^2 \, k}{2}
\]
Why Discrete Differential Geometry?

Example (Conservation of Energy):

\[ a_k = -g \]
\[ v_k = -g \Delta t k \]
\[ y_k = y_0 - \frac{(g \Delta t k)^2}{2} + \frac{g \Delta t^2 k}{2} \]

Plugging this into the equation for the energy, we get:

\[ E_k = mgy_0 + \frac{mg^2 \Delta t^2 k}{2} \]
Why Discrete Differential Geometry?

Example (Conservation of Energy):
So our discretized system is wrong in two ways. First, we do not get the “correct” solution

\[ y_k = y_0 - \frac{(g \Delta t k)^2}{2} + \frac{g \Delta t^2 k}{2} \]
Why Discrete Differential Geometry?

Example (Conservation of Energy):
So our discretized system is wrong in two ways.
First, we do not get the "correct" solution

\[ y_k = y_0 - \frac{(g \Delta t k)^2}{2} + \frac{g \Delta t^2 k}{2} \]

More importantly, the system gains energy:

\[ E_k = mgy_0 + \frac{mg^2 \Delta t^2 k}{2} \]
Why Discrete Differential Geometry?

Discrete Geometry:
Although we expect the results of the finite approximation to be imprecise, we would like to construct it so that the invariants are preserved.
Why Discrete Differential Geometry?

Discrete Geometry:
In the case of the dropping ball, we would like to have energy preservation:

\[ \frac{1}{2}mv_k^2 + mgy_k = \frac{1}{2}mv_{k+1}^2 + mgy_{k+1} \]
Why Discrete Differential Geometry?

Discrete Geometry:

In the case of the dropping ball, we would like to have energy preservation:

\[ \frac{1}{2} m v_k^2 + m g y_k = \frac{1}{2} m v_{k+1}^2 + m g y_{k+1} \]

which forces us to define the relation between changes in height and velocities differently:

\[ y_{k+1} - y_k = \frac{v_{k+1}^2 - v_k^2}{-2g} \]
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In the case of the dropping ball, we would like to have energy preservation:

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With this new discrete derivative, our system is guaranteed to be energy preserving.
What Will We Cover?

- Differential Geometry of Curves/Surfaces
- What we can measure
- Discrete Exterior Calculus
- Physical Modeling
- Conformal Geometry
- Surface and Volume Meshing