FFTs in Graphics and Vision

Characters of Irreducible Representations
Outline

• Uniqueness of Eigenvalue Decomposition

• Simultaneous Diagonalizability

• Characters of (Irreducible) Representations
Uniqueness of E. Decomposition

Consider the decomposition of a vector space into 1D eigenspaces with respect to a self-adjoint operator $A$:

$$V = \bigoplus_{i} V_i$$

with $V_i = \text{Span}\{v_i\}$ and $A(v_i) = \lambda_i v_i$. 
Uniqueness of E. Decomposition

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This decomposition is unique only as long as the eigenvalues are all different.
Uniqueness of E. Decomposition

If $\lambda_i = \lambda_j$ for some $i \neq j$, then for any $\theta$ setting:

$$w_i = \cos(\theta)v_i + \sin(\theta)v_j$$

$$w_j = -\sin(\theta)v_i + \cos(\theta)v_j$$

we get a different decomposition of $V$ into 1D eigenspaces.
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• Uniqueness of Eigenvalue Decomposition
• Simultaneous Diagonalizability
• Characters of (Irreducible) Representations
Simultaneous Diagonalizability

If we have a finite-dimensional vector space $V$ and a self-adjoint operator $A: V \rightarrow V$, we know that there exists an orthonormal basis $\{v_1, \ldots, v_n\}$ with:

$$A v_i = \lambda_i v_i$$
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What if we have two self-adjoint operators?
Simultaneous Diagonalizability

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$$Av_i = \lambda_i v_i$$

What if we have two self-adjoint operators?

Is there an orthonormal basis $\{v_1, \ldots, v_n\}$ with:

$$Av_i = \lambda_i v_i \quad \text{and} \quad Bv_i = \mu_i v_i$$
Simultaneous Diagonalizability

Schur’s Lemma:

If $A$ and $B$ commute, (i.e. $AB=BA$) then the operators are simultaneously diagonalizable.
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Proof:

Let $\lambda$ be an eigenvalue of $A$, and consider the operator $(A- \lambda I)$. 
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Proof:

Let $\lambda$ be an eigenvalue of $A$, and consider the operator $(A - \lambda \text{Id})$.

This operator has a (non-trivial) kernel which is the space of all vectors in $V$ which are eigenvectors of $A$ with eigenvalue $\lambda$:

$$\text{Kernel } (A - \lambda \text{Id}) \supseteq \{ \mathbf{v} \in V \mid A\mathbf{v} = \lambda \mathbf{v} \}$$
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Proof:

Since the operators commute, we know that if $v$ is in the kernel of $(A-\lambda \text{Id})$ then:

$$ (A - \lambda \text{Id}) B v = B (A - \lambda \text{Id}) v $$
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Since the operators commute, we know that if \( \nu \) is in the kernel of \( (A - \lambda \text{Id}) \) then:

\[
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Proof:

Since the operators commute, we know that if $v$ is in the kernel of $(A - \lambda \text{Id})$ then:

$$ (A - \lambda \text{Id}) Bv = 0 $$

So $Bv$ must also be in the kernel.
Simultaneous Diagonalizability

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If $A$ and $B$ commute, (i.e. $AB=BA$) then the operators are simultaneously diagonalizable.

Proof:

If we denote by $V_\lambda$ the space of eigenvectors of $A$ with eigenvalue $\lambda$, then we must have:

$$B V_\lambda = V_\lambda$$
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If \( A \) and \( B \) commute, (i.e. \( AB = BA \)) then the operators are simultaneously diagonalizable.

Proof:

Since \( B \) is self-adjoint and maps \( V_\lambda \) to itself, we know that there exists an orthonormal basis \( \{v_{1,\lambda}, \ldots, v_{n_\lambda,\lambda}\} \) for \( V_\lambda \) such that:

\[
B (v_{i,\lambda}) = \mu_{i,\lambda} v_{i,\lambda}
\]
Simultaneous Diagonalizability

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If $A$ and $B$ commute, (i.e. $AB=BA$) then the operators are simultaneously diagonalizable.

Proof:

Since the $V_\lambda$ are orthogonal spaces, the vectors:

$$\bigcup_{\lambda} v_{\lambda, \lambda}, \ldots, v_{n_{\lambda}, \lambda}$$

are an orthogonal basis for $V$. 

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But these vectors also satisfy:

$$Av_{i,\lambda} = \lambda v_{i,\lambda} \quad \text{and} \quad Bv_{i,\lambda} = \mu_{i,\lambda} v_{i,\lambda}$$
Uniqueness of Diagonalizability

Question:

Does this imply that if \( v \) is an eigenvector of \( A \), then it is also an eigenvector of \( B \)?
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Answer:

No! Consider \( A \) is some self-adjoint operator and \( B \) is the identity.
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The problem is that the eigenvector \( \lambda \) may have non-trivial multiplicity – i.e. the space \( V_\lambda \) may be more than one dimensional.
Question:

Does this imply that if $v$ is an eigenvector of $A$, then it is also an eigenvector of $B$?

Answer:

We can think of $V_\lambda$ as the sub-space of $V$ consisting of (1D) spaces on which the operator $A$ acts identically.
Uniqueness of Diagonalizability

Question:

Does this imply that if \( v \) is an eigenvector of \( A \), then it is also an eigenvector of \( B \)?

Answer:

We can think of \( V_\lambda \) as the sub-space of \( V \) consisting of (1D) spaces on which the operator \( A \) acts identically.

We can distinguish \( V_\lambda \) from \( V_{\lambda'} \), because the \( A \) acts on these sub-spaces differently, but we can’t distinguish between the 1D components of \( V_\lambda \).
Outline

• Uniqueness of Eigenvalue Decomposition
• Simultaneous Diagonalizability
• Characters of (Irreducible) Representations
Characters of a Representations

For irreducible representations, there is a concept analogous to “eigenvalue” called the character.
Characters of a Representation

Given a representation of a group $G$ on a vector space $V$, each element $g \in G$ defines a unitary transformation $\rho(g)$ on $V$. 
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If we choose a basis $\{v_1, \ldots, v_n\}$ for $V$, we can express $\rho(g)$ as a matrix w.r.t this basis:

$$
\rho(g) = \begin{pmatrix}
\lambda_{11}(g) & \cdots & \lambda_{n1}(g) \\
\vdots & \ddots & \vdots \\
\lambda_{1n}(g) & \cdots & \lambda_{nn}(g)
\end{pmatrix}
$$
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\end{pmatrix}$$

The elements of the matrix depend on the basis. The trace does not.
Characters of a Representation

This allows us to associate a (complex valued) function, \( \chi_\rho : G \rightarrow \mathbb{C} \) with every representation \( \rho \):

\[
\chi_\rho (g) = \text{Trace} \left( \begin{pmatrix}
\lambda_{11}(g) & \cdots & \lambda_{n1}(g) \\
\vdots & \ddots & \vdots \\
\lambda_{1n}(g) & \cdots & \lambda_{nn}(g)
\end{pmatrix} \right) = \sum_{i=1}^{n} \lambda_{ii}(g)
\]

This is the character of the representation.
Characters of a Representation

As with eigenspaces of a symmetric matrix, the group acts identically on two irreducible representations if their characters are the same.

Otherwise, their characters are orthogonal.
Characters of a Representation

As with eigenspaces of a symmetric matrix, the group acts identically on two irreducible representations if their characters are the same. Otherwise, their characters are orthogonal.

Similarly, the decomposition into irreducible representations is only unique up to:

\[ V = \bigoplus_{\chi} W_{\chi} \]

where \( W_{\chi} \) is the sum of all the irreducible representations with the same character \( \chi \).
Irreducible Representations

Example:

We know that if $G$ is the group of 2D rotations, $G=SO(2)$, and $V$ is the space of functions on a circle, we have:

$$V = \bigoplus_{k=-\infty}^{\infty} V_k$$

where $V_k$ is the 1D space of functions spanned by the complex exponentials:

$$V_k = \text{Span} \ e^{ik\theta}$$
Irreducible Representations

Example:

For a fixed rotation $g \in G$, we know that $g$ defines a unitary transformation $\rho_k(g)$ on $V_k$. 
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Denoting by $g_\theta$ the rotation by $\theta$ degrees, we get:

$$\rho_k \begin{pmatrix} g_\theta \end{pmatrix} = \begin{pmatrix} e^{-ik\theta} \end{pmatrix}$$
Irreducible Representations

Example:

So the character of this representation is:

$$\chi_{\rho_k} \circ \theta = \text{Trace} e^{-ik\theta} = e^{-ik\theta}$$
Irreducible Representations

Example:

So the character of this representation is:

$$\chi_{\rho_k} g_\theta \equiv \text{Trace}(-i\kappa \theta) = e^{-i\kappa \theta}$$

Note that the characters are orthogonal:

$$\langle \chi_{\rho_k}, \chi_{\rho_{k'}} \rangle = 0$$

whenever $k \neq k'$. 
Irreducible Representations

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So the character of this representation is:

$$\chi_{\rho_k} \left( e^{i\theta} \right) = \text{Trace}(e^{-ik\theta}) = e^{-ik\theta}$$

Note that the characters are orthogonal:

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whenever \(k \neq k'\).

Thus, each irreducible representation only occurs once and the decomposition into irreducible representations is unique.
Irreducible Representations

Example:

Similarly, the decomposition of the space of spherical functions into irreducible representations spanned by the spherical harmonics is also unique.
Relationship to Symmetric Operators

Suppose we have a representation \( \rho \) of a group \( G \) on a vector space \( V \), the decomposition of \( V \) into irreducible representations is unique, and we have a self-adjoint operator \( A \) commuting with the action of \( G \):

\[
\rho(g)A = A\rho(g)
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Relationship to Symmetric Operators

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\[
\rho(g)A = A\rho(g)
\]

Since \( A \) is self-adjoint, we can decompose \( V \) into eigenspaces:

\[
V = \bigoplus_{\lambda} V_{\lambda}
\]
Relationship to Symmetric Operators

Since $A$ commutes with the action of $G$, the $V_\lambda$ must be sub-representations:

$$\rho(g)V_\lambda = V_\lambda$$
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where each $V_{\lambda,i}$ is irreducible.

Since this gives a decomposition of $V$ into irreducible representations, and we assumed the decomposition was unique, this implies that the irreducible representations are eigenspaces of $A$. 
Relationship to Symmetric Operators

In particular, if we define the operator $C_l$ to be the spherical convolution with the axially symmetric $l$-th frequency harmonic:

$$C_l(f)(\theta, \phi) = \left< f, \rho_R(\theta, \phi) Y_l^0 \right>$$
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$$C_l(f)(\theta, \phi) = \left\langle f, \rho_{R(\theta, \phi)}Y_l^0 \right\rangle$$

then the symmetry of $C_l$, combined with the fact that it commutes with rotation, implies that:

$$C_l(Y_l^m)(\theta, \phi) = \lambda_l Y_l^m(\theta, \phi)$$