FFTs in Graphics and Vision

Spherical Convolution
and
Axial Symmetry Detection

Outline

- Math Review
  - Symmetry
  - General Convolution
- Spherical Convolution
- Axial Symmetry Detection

Math Review

Symmetry:

Given a unitary representation of a group G on a vector space V, we say that a vector $v \in V$ is invariant under the action of G if for all $g \in G$:

$$\rho_g(v) = v$$

Math Review

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The set of G-invariant vector V_G is a vector space.

Math Review

Symmetry:

The linear map π_G is a projection onto V_G, if:

- $\pi_G(v) \in V_G$ for all $v \in V$
- $\pi_G(v) = v$ for all $v \in V_G$
- $v - \pi_G(v)$ is perpendicular to every G-invariant vector.

Math Review

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- $v - \pi_G(v)$ is perpendicular to every G-invariant vector.

The map π_G is the map sending a vector v to the closest G-invariant vector.

Math Review

Symmetry:

The measure of symmetry of a vector *v* with respect to the group *G* is the size of its projection onto the space of *G*-invariant vectors:

$$\text{Sym}^2(v,G) = \left\|\pi_G(v)\right\|^2$$

Math Review

Convolution:

Given two functions *f*(*p*) and *g*(*p*), we define the convolution of the two functions to be:

$$(f * g)(q) = \int f(p) \cdot g(q-p)\,dp$$

Math Review

Convolution:

If we hold the function *g* fixed we can define a map from the space of functions back into itself:

$$C_g(f) = f * g$$

Math Review

<u>Convolution</u>:

If we hold the function g fixed we can define a map from the space of functions back into itself:

$$C_g(f) = f * g$$

<u>Claim</u>:

The map C_g is a linear operator.

Math Review

Convolution:

If we hold the function *g* fixed we can define a map from the space of functions back into itself:

$$C_g(f) = f * g$$

Claim:

Given functions *f* and *h* and scalars α and β:

$$C_g(\alpha f + \beta h)(q) = \int \left(\alpha f(p) + \beta h(p)\right) \cdot g(q-p)\,dp$$

Math Review

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$$= \alpha \int f(p) \cdot g(q-p)dp + \beta \int h(p) \cdot g(q-p)dp$$

Math Review

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Given functions f and h and scalars α and β:

$$\begin{aligned} C_g(\alpha f + \beta h)(q) &= \int \left(\alpha f(p) + \beta h(p)\right) \cdot g(q-p)\,dp \\ &= \alpha \int f(p) \cdot g(q-p)\,dp + \beta \int h(p) \cdot g(q-p)\,dp \\ &= \alpha C_g(f) + \beta C_g(h) \end{aligned}$$

Math Review

Convolution:

Assume that the function *g* is real-valued and radial, i.e. the value of *g* at a point *p* is completely determined by the distance of *p* from the origin:

$$g(p) = \tilde{g}\left(|p|\right)$$

Math Review

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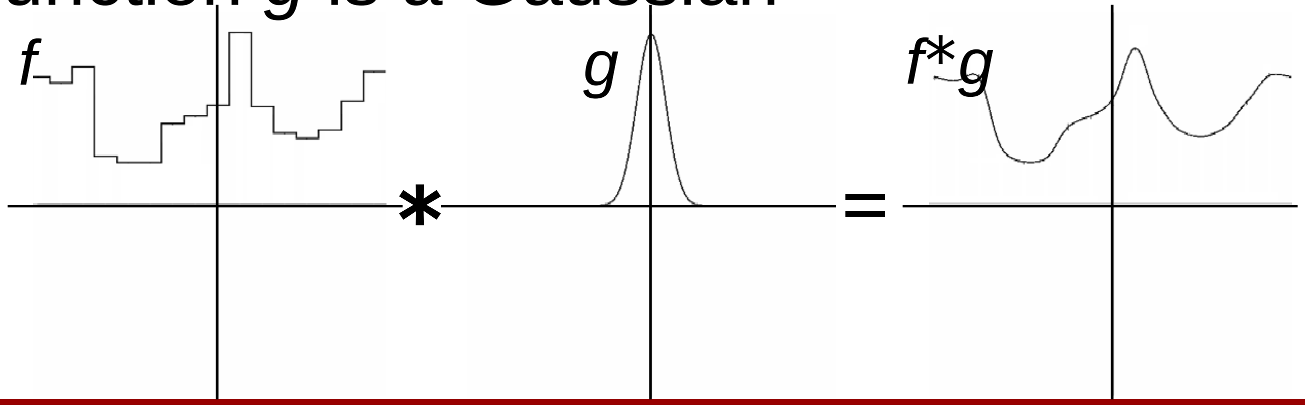
Example:

The function *g* is a Gaussian



Math Review

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Assume that the function *g* is real-valued and radial, i.e. the value of *g* at a point *p* is completely determined by the distance of *p* from the origin:

$$g(p) = \tilde{g}\left(|p|\right)$$

Claim:

In this case, C_g is self-adjoint (i.e. symmetric).

Math Review

Convolution:

Proof:

We need to show that for any functions *f* and *h*:

$$\left\langle C_g(f), h \right\rangle = \left\langle f, C_g(h) \right\rangle$$

Math Review

Convolution:

Proof:

We need to show that for any functions *f* and *h*:

$$\left\langle C_g(f), h \right\rangle = \left\langle f, C_g(h) \right\rangle$$

Expanding the left side, we get:

$$\left\langle C_g(f), h \right\rangle = \int C_g(f)(p)\overline{h(p)}dp$$

Math Review

Convolution:

Proof:

We need to show that for any functions *f* and *h*:

$$\left\langle C_g(f), h \right\rangle = \left\langle f, C_g(h) \right\rangle$$

Writing out the operator C_g, we get:

$$\left\langle C_g(f), h \right\rangle = \int C_g(f)(p)\overline{h(p)}dp$$

$$\left\langle C_g(f), h \right\rangle = \int (f * g)(p)\overline{h(p)}dp$$

Math Review

Convolution:

Proof:

We need to show that for any functions *f* and *h*:

$$\langle C_g(f), h\rangle = \langle f, C_g(h)\rangle$$

Expressing the convolution as an integral gives:

$$\langle C_g(f), h\rangle = \int \left(f * g\right)(p)\overline{h(p)}\,dp$$

$$\langle C_g(f), h\rangle = \int \left(\int f(q)g(p-q)\,dq\right)\overline{h(p)}\,dp$$

Math Review

<u>Convolution</u>:

<u>Proof</u>:

We need to show that for any functions *f* and *h*:

$$\langle C_g(f), h \rangle = \langle f, C_g(h) \rangle$$

Changing the order of integration, we get:

$$\langle C_g(f), h \rangle = \int \left(\int f(q) g(p-q) dq \right) \overline{h(p)} dp$$

$$\langle C_g(f), h \rangle = \int \int f(q) g(p-q) \overline{h(p)} dp dq$$

Math Review

Convolution:

Proof:

We need to show that for any functions *f* and *h*:

$$\langle C_g(f), h\rangle = \langle f, C_g(h)\rangle$$

Using the fact that *g* is real-valued and radial:

$$\langle C_g(f), h\rangle = \iint f(q)\, g(p-q)\, \overline{h(p)}\, dp\, dq$$

$$\langle C_g(f), h\rangle = \iint f(q)\, \overline{h(p)}\, \overline{g(q-p)}\, dp\, dq$$

Math Review

Convolution:

Proof:

We need to show that for any functions *f* and *h*:

$$\left\langle C_g(f),h\right\rangle = \left\langle f,C_g(h)\right\rangle$$

Changing the order of integration again gives:

$$\left\langle C_g(f),h\right\rangle = \iint f(q)\overline{h(p)}\,\overline{g(q-p)}\,dp\,dq$$

$$\left\langle C_g(f),h\right\rangle = \int f(q)\left(\int \overline{h(p)}\,\overline{g(q-p)}\,dp\right)dq$$

Math Review

Convolution:

Proof:

We need to show that for any functions *f* and *h*:

$$\left\langle C_g(f), h \right\rangle = \left\langle f, C_g(h) \right\rangle$$

Using the properties of complex conjugates gives:

$$\left\langle C_g(f), h \right\rangle = \int f(q) \left(\int \overline{h(p)}\,\overline{g(q-p)}\,dp \right) dq$$

$$\left\langle C_g(f), h \right\rangle = \int f(q) \overline{\left(\int h(p) g(q-p)\,dp \right)} dq$$

Math Review

Convolution:

Proof:

We need to show that for any functions *f* and *h*:

$$\left\langle C_g(f), h \right\rangle = \left\langle f, C_g(h) \right\rangle$$

Using the equation for convolution, we get:

$$\left\langle C_g(f), h \right\rangle = \int f(q) \overline{\left(\int h(p) g(q-p) dp \right)} dq$$

$$\left\langle C_g(f), h \right\rangle = \int f(q) \overline{(h * g)(q)} dq$$

Math Review

Convolution:

Proof:

We need to show that for any functions *f* and *h*:

$$\left\langle C_g(f), h \right\rangle = \left\langle f, C_g(h) \right\rangle$$

Using the equation for C_g, we get:

$$\left\langle C_g(f), h \right\rangle = \int f(q)\overline{\left(h * g\right)(q)}\,dq$$

$$\left\langle C_g(f), h \right\rangle = \int f(q)\overline{C_g(h)(q)}\,dq$$

Math Review

Convolution:

Proof:

We need to show that for any functions *f* and *h*:

$$\left\langle C_g(f), h \right\rangle = \left\langle f, C_g(h) \right\rangle$$

And finally, using the equation for the dot-product:

$$\left\langle C_g(f), h \right\rangle = \int f(q)\overline{C_g(h)(q)}dq$$

$$\Downarrow$$

$$\left\langle C_g(f), h \right\rangle = \left\langle f, C_g(h) \right\rangle$$

Outline

- Math Review
- Spherical Convolution
- Axial Symmetry Detection

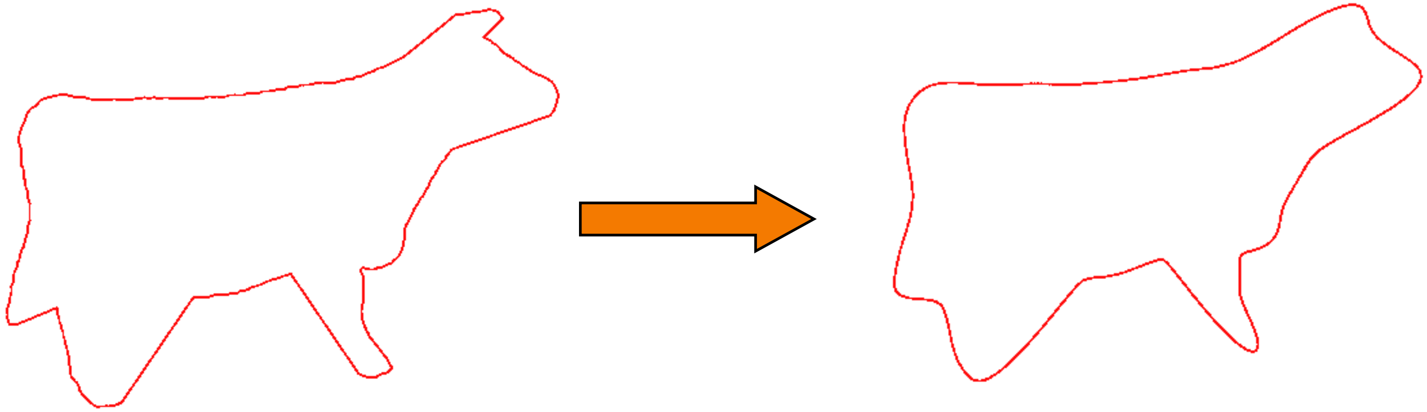
Spherical Convolution/Correlation

In the case of circle we used convolution / correlation for two different tasks:

Spherical Convolution/Correlation

In the case of circle we used convolution / correlation for two different tasks:

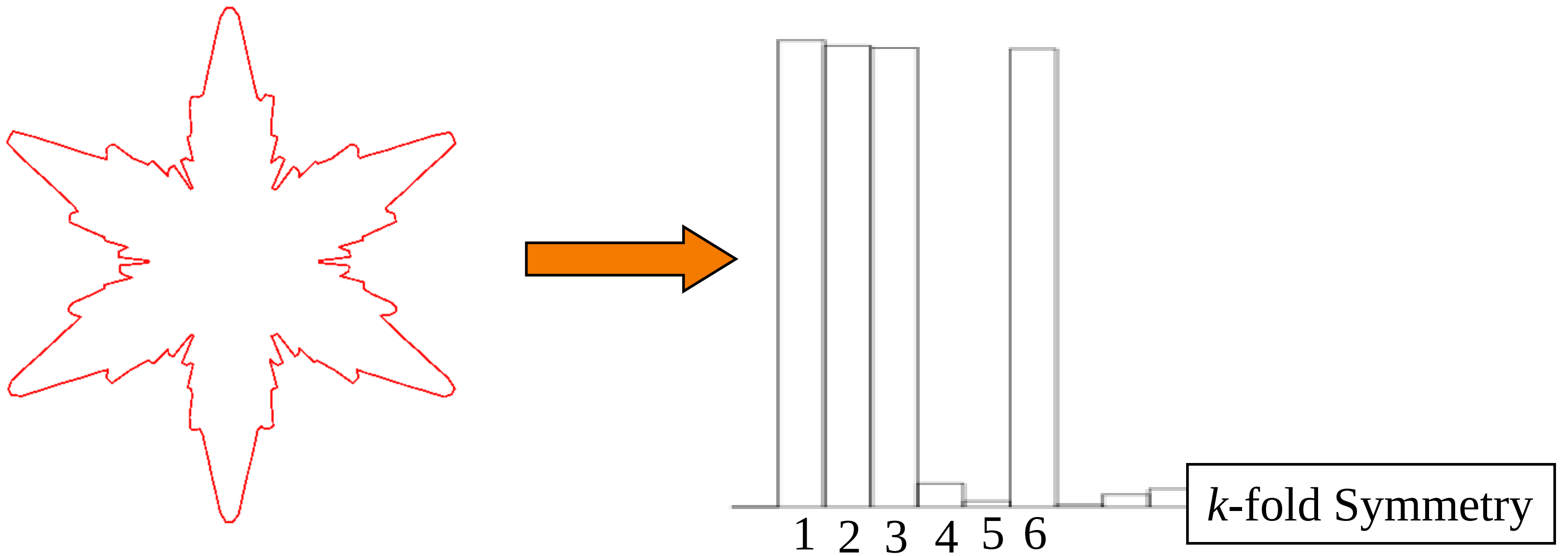
1. We used convolution for operations like smoothing circular functions



Spherical Convolution/Correlation

In the case of circle we used convolution / correlation for two different tasks:

1. We used convolution for operations like smoothing circular functions
2. We used correlation for operations like alignment and symmetry detection



Spherical Convolution/Correlation

Up to now, we thought of these two operations as essentially the same.

The situation changes as we move to functions on a sphere.

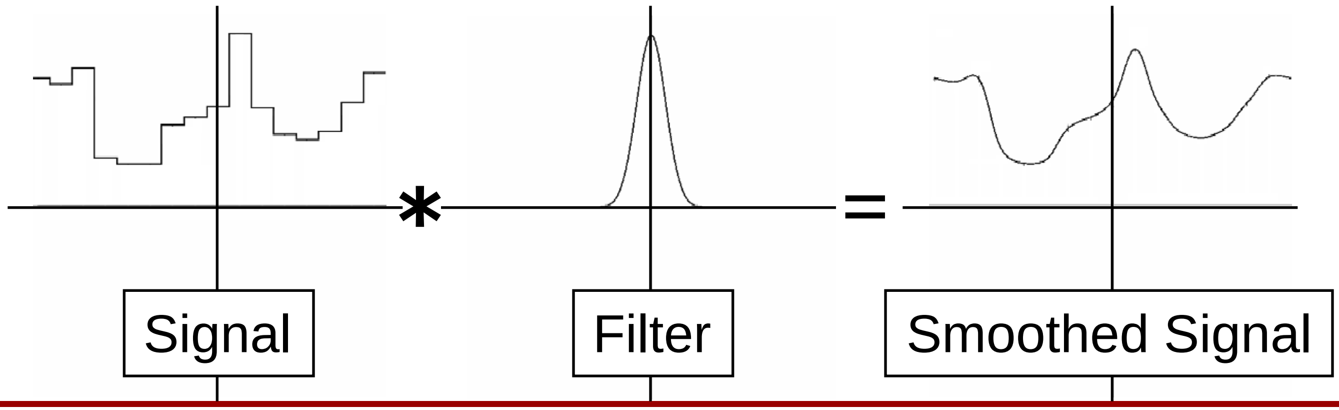
Spherical Convolution/Correlation

When we perform an operation like smoothing, the input is:

- A function on the circle defining the signal, and
- A function on the circle defining the smoothing filter

The output of the operation is:

- A function on the circle



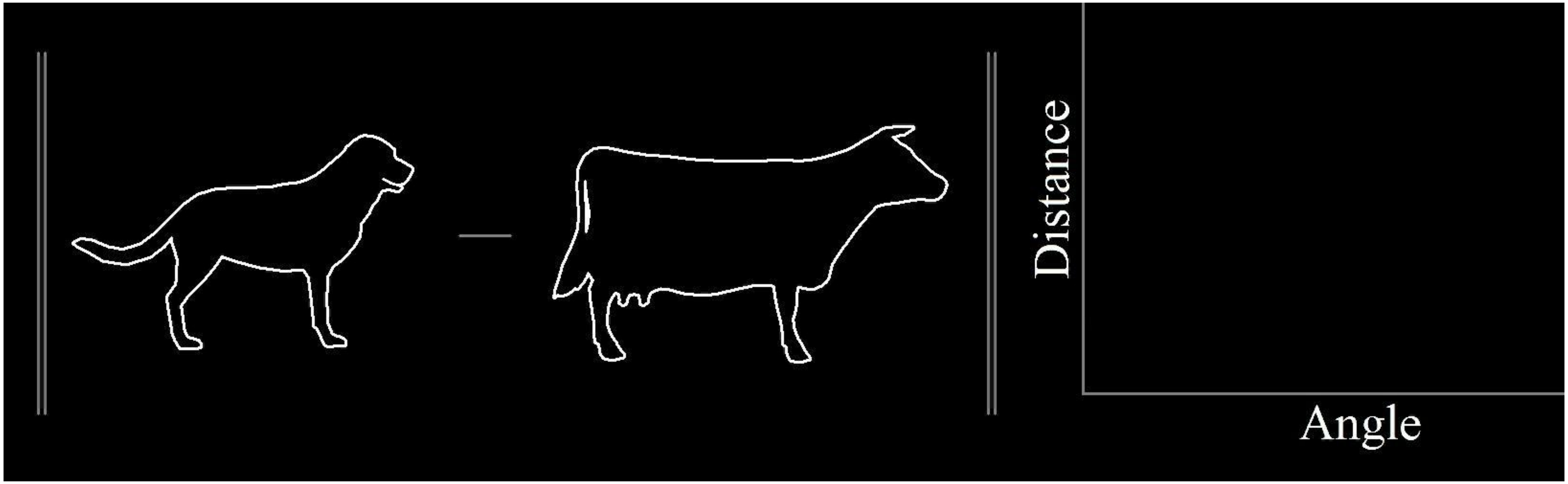
Spherical Convolution/Correlation

When we perform an operation like alignment, the input is:

- Two functions on a circle

The output is:

- A function on the space of 2D rotations



Spherical Convolution/Correlation

In the case of a circle, the situation is simpler because the space of rotations is itself a circle:

- There is a one-to-one mapping from points on a circle to rotations, with a point on a circle with angle θ corresponding to a rotation by an angle of θ.

Spherical Convolution/Correlation

In the case of a circle, the situation is simpler because the space of rotations is itself a circle:

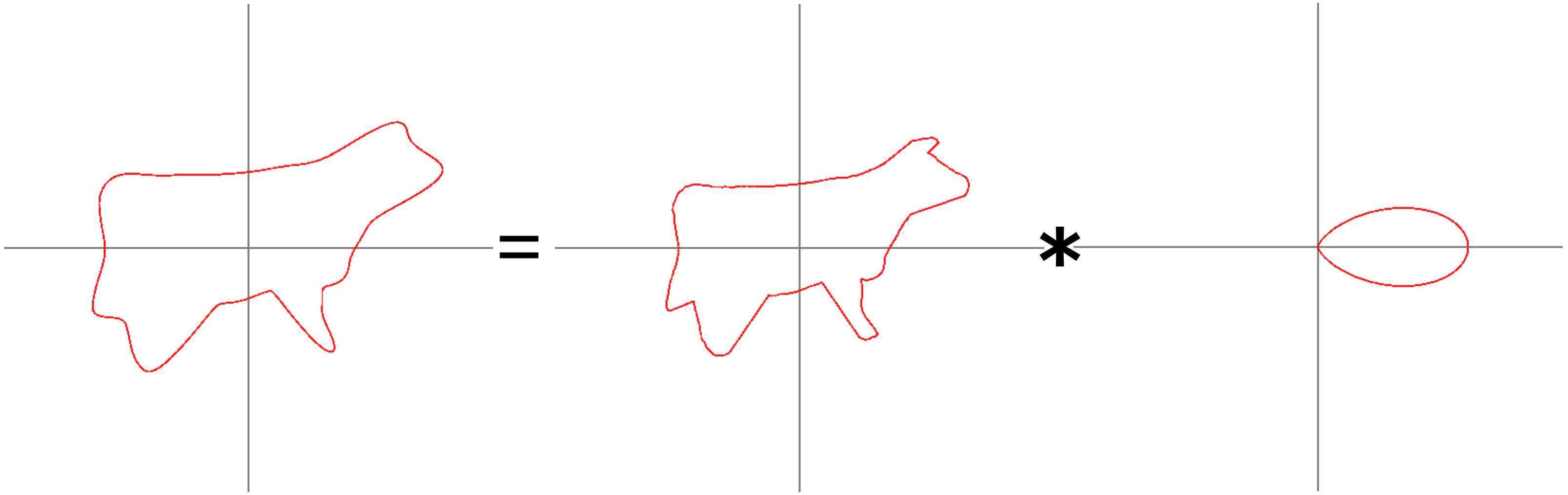
- There is a one-to-one mapping from points on a circle to rotations, with a point on a circle with angle θ corresponding to a rotation by an angle of θ.

In the case of the sphere, the situation becomes more complicated:

- The sphere is a 2D space while the rotations are a 3D space, so there can't be a one-to-one mapping.

Spherical Convolution

In the case of a circle, we compute the value of the smoothed function at *p* by rotating the filter so that (1,0) maps to *p* and then we compute the inner product of the signal with the rotated filter.



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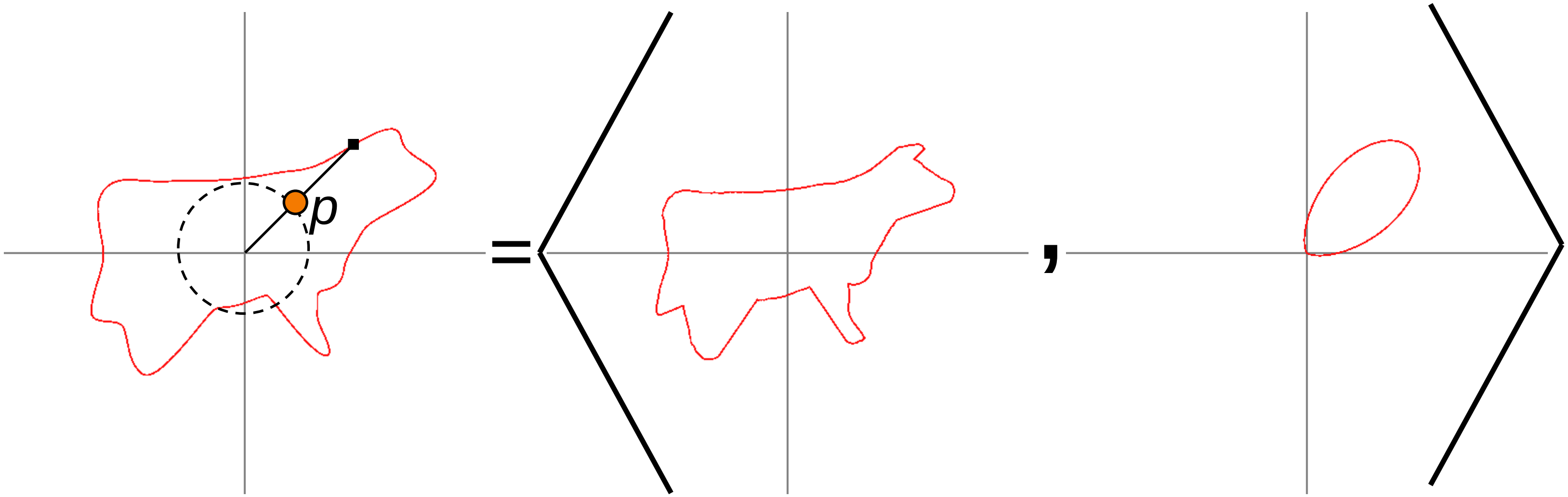
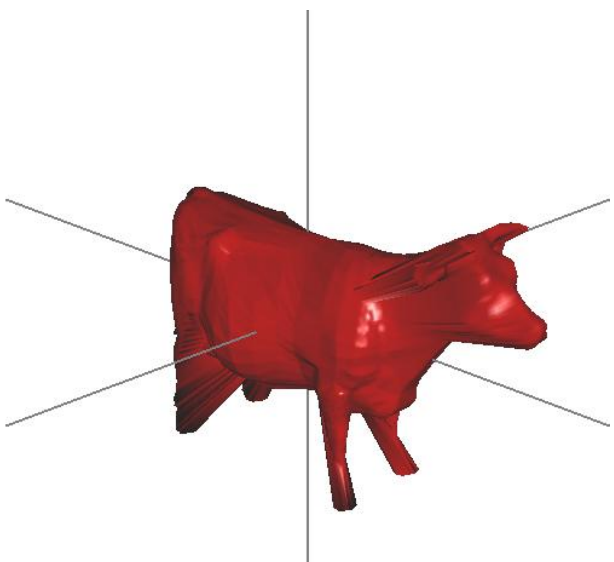
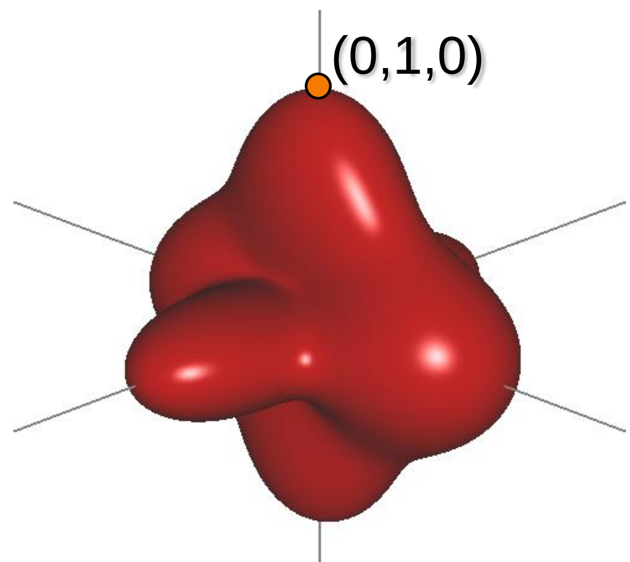


Spherical Convolution

We can try an apply the same type of approach to the case of spherical functions.



Signal

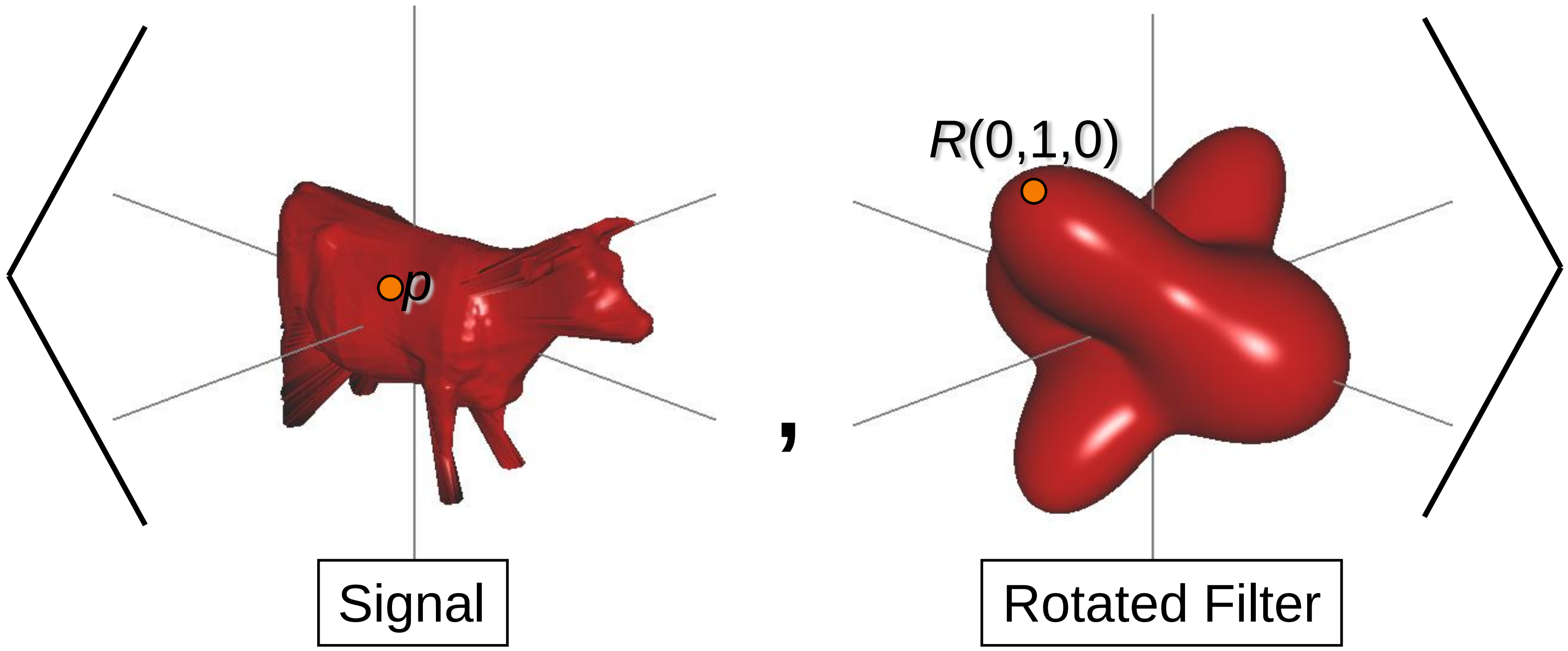


Filter

Spherical Convolution

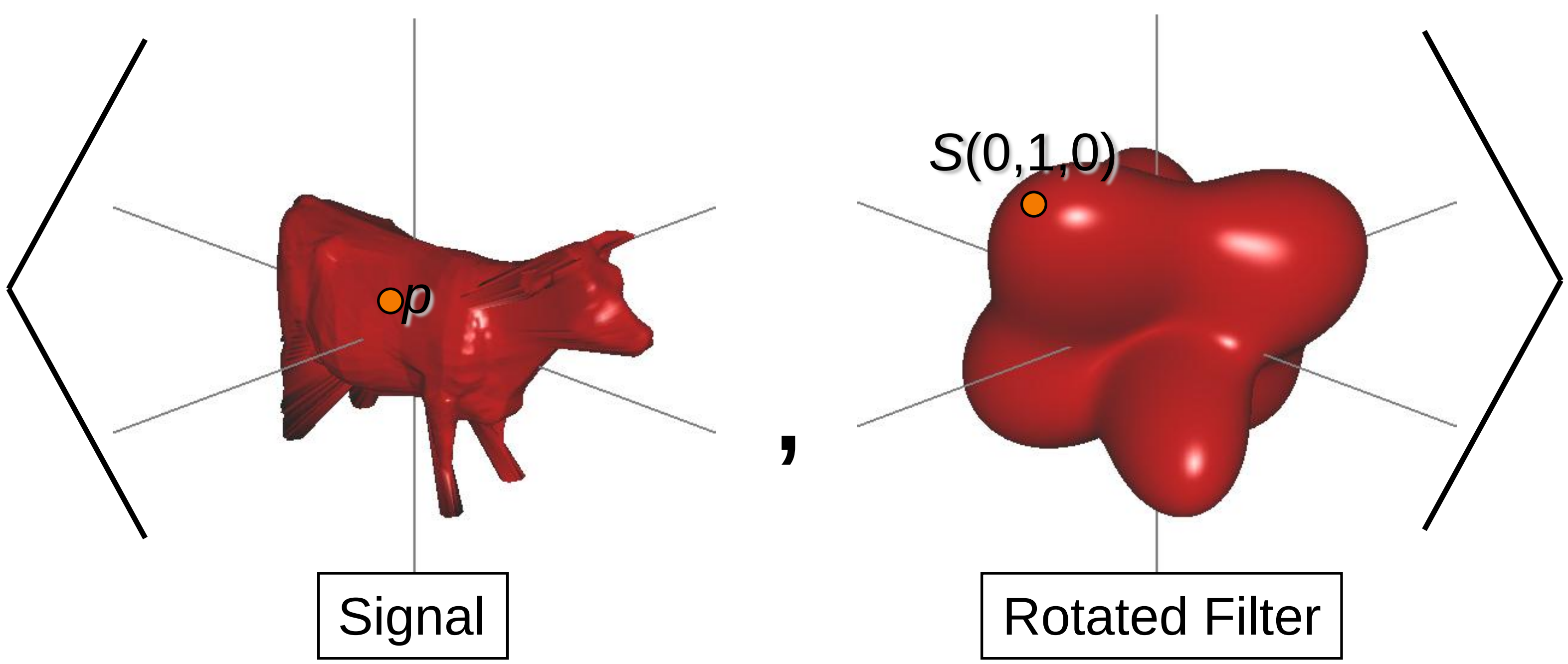
We would like to define a new function on the sphere whose value at the point *p* is obtained by:

Finding a rotation *R* that maps the North pole to *p* and then compute the inner product of the signal with the rotated filter.



Spherical Convolution

The problem is that there are many different rotations that send the North pole to the point *p*, so this does not lead to a well-defined notion of smoothing.

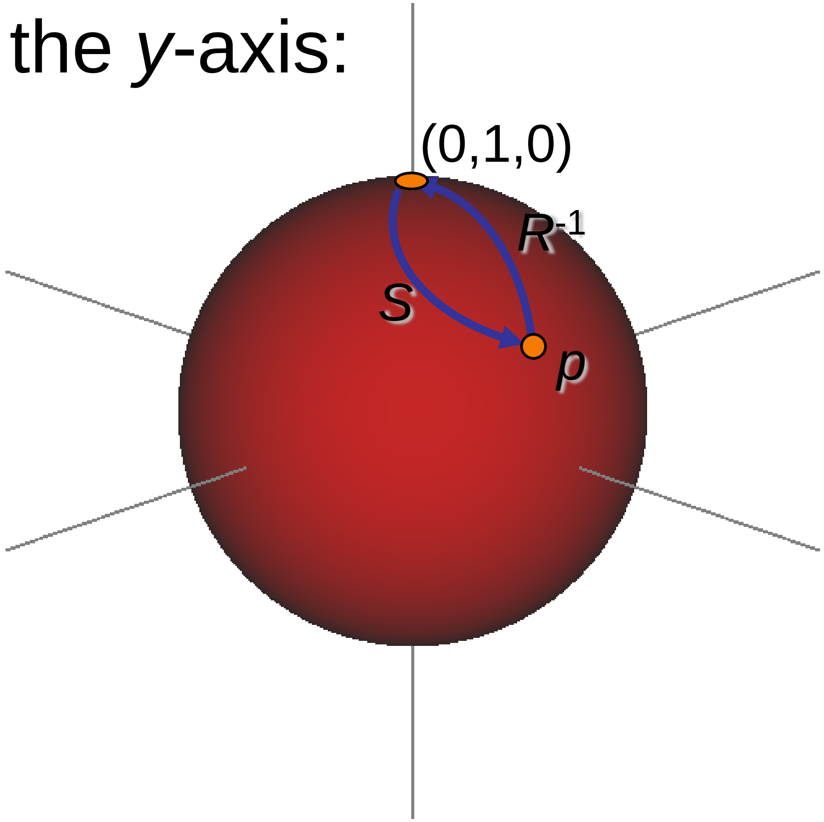


Spherical Convolution

Recall:

If we have two rotations *R* and *S* mapping the North pole to the point *p*, the rotations must differ by an initial rotation about the *y*-axis:

$$S = R \cdot R_y(\psi)$$



Spherical Convolution

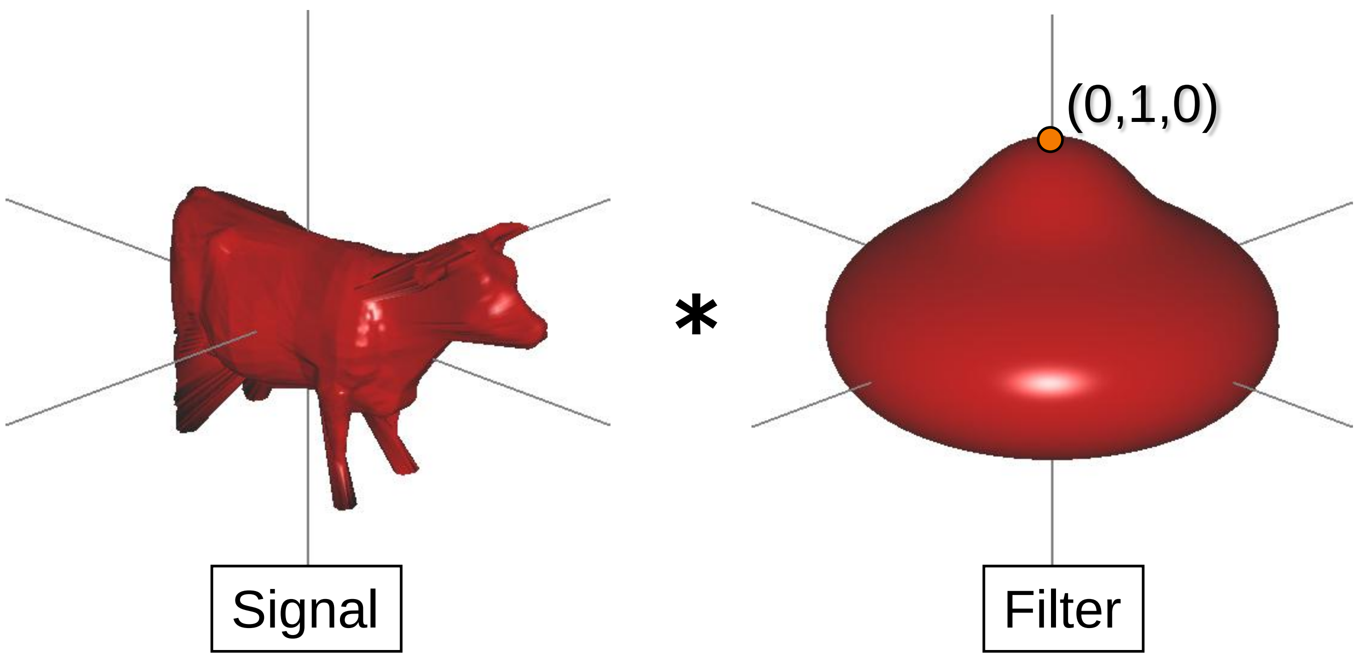
Recall:

Thus, we can make the notion of smoothing well-defined by ensuring that the initial rotation about the *y*-axis does not change the filter.

Spherical Convolution

Recall:

This means that we can extend the circular notion of smoothing to the sphere if we ensure that the filter is symmetric about the *y*-axis:



Spherical Convolution

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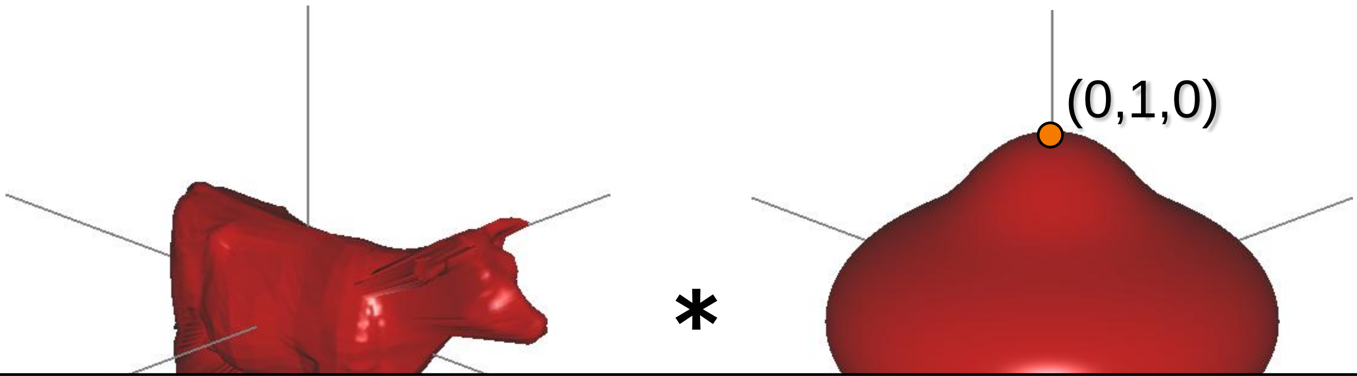


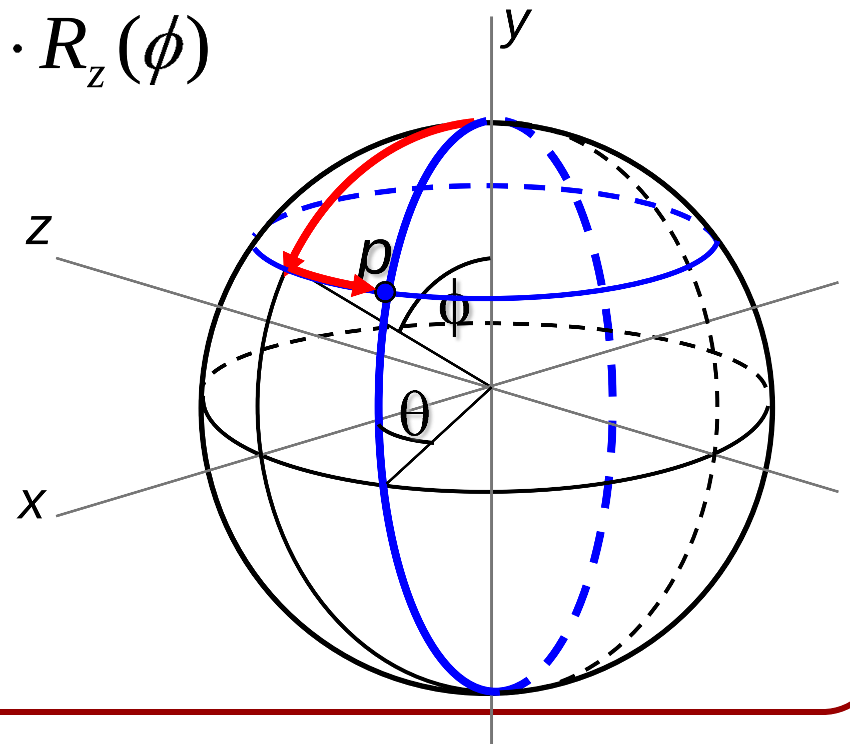
If *R* and *S* are rotations mapping the North pole to *p*, then the rotation of the filter by either *R* or *S* will give the same spherical function!

Spherical Convolution

Convolution:

Using the Euler angle representation, we know that the rotation taking the North pole to the point $p=\Phi(\theta,\phi)$ is the rotation:

$$R(\theta,\phi)=R_y(\theta)\cdot R_z(\phi)$$



Spherical Convolution

Convolution:

Thus, given

- A spherical function $f(\theta,\phi)$
- A spherical filter $g(\theta,\phi)$ that is rotationally-symmetric about the *y*-axis

The convolution of f with g at $p=\Phi(\theta,\phi)$ can be expressed by rotating g so the North pole gets mapped to p and computing the inner product:

$$(f * g)(\theta,\phi) = \left\langle f, \rho_{R(\theta,\phi)} g \right\rangle$$

Spherical Convolution

<u>Convolution</u>:

Expressing the spherical functions *f* and *g* in terms of the spherical harmonic basis, we get:

$$f(\theta,\phi)=\sum_l\sum_{m=-l}^{l}\hat{f}(l,m)Y_l^m(\theta,\phi)$$

$$g(\theta,\phi)=\sum_l\sum_{m=-l}^{l}\hat{g}(l,m)Y_l^m(\theta,\phi)$$

Spherical Convolution

Convolution:

Recall that the spherical harmonics can be expressed as a complex exponential in θ times a "polynomial" in $\cos\phi$:

$$Y_l^m(\theta,\phi) = P_l^m(\cos\phi)e^{im\theta}$$

Spherical Convolution

Convolution:

Recall that the spherical harmonics can be expressed as a complex exponential in θ times a "polynomial" in cosφ:

$$Y_l^m(\theta,\phi) = P_l^m(\cos\phi)e^{im\theta}$$

So a rotation by an angle of α degrees about the *y*-axis acts on the (*l*,*m*)-th spherical harmonics by:

$$\rho_{R_y(\alpha)}\left(Y_l^m\right) = e^{-im\alpha}Y_l^m$$

Spherical Convolution

Convolution:

Thus, if the filter g is rotationally symmetric about the y-axis, any rotation about the y-axis must not change g. That is, for all α we must have:

$$\rho_{R_y(\alpha)} g = g$$

Spherical Convolution

Convolution:

Thus, if the filter *g* is rotationally symmetric about the *y*-axis, any rotation about the *y*-axis must not change *g*. That is, for all α we must have:

$$\rho_{R_y(\alpha)} g = g$$

Or in terms of the spherical harmonics:

$$\sum_l \sum_{m=-l}^{l} \hat{g}(l,m) Y_l^m(\theta,\phi) = \sum_l \sum_{m=-l}^{l} \hat{g}(l,m) e^{-im\alpha} Y_l^m(\theta,\phi)$$

Spherical Convolution

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$$\Downarrow$$

$$\hat{g}(l,m) = \hat{g}(l,m) e^{-im\alpha}$$

Spherical Convolution

Convolution:

$$\boxed{\hat{g}(l,m) = \hat{g}(l,m)e^{-im\alpha}}$$

Thus, either:

- $e^{-im\alpha}=1$ for all α, which would imply that $m=0$, or
- $\hat{g}(l,m)=0$

Spherical Convolution

Convolution:

$$\hat{g}(l,m) = \hat{g}(l,m)e^{-im\alpha}$$

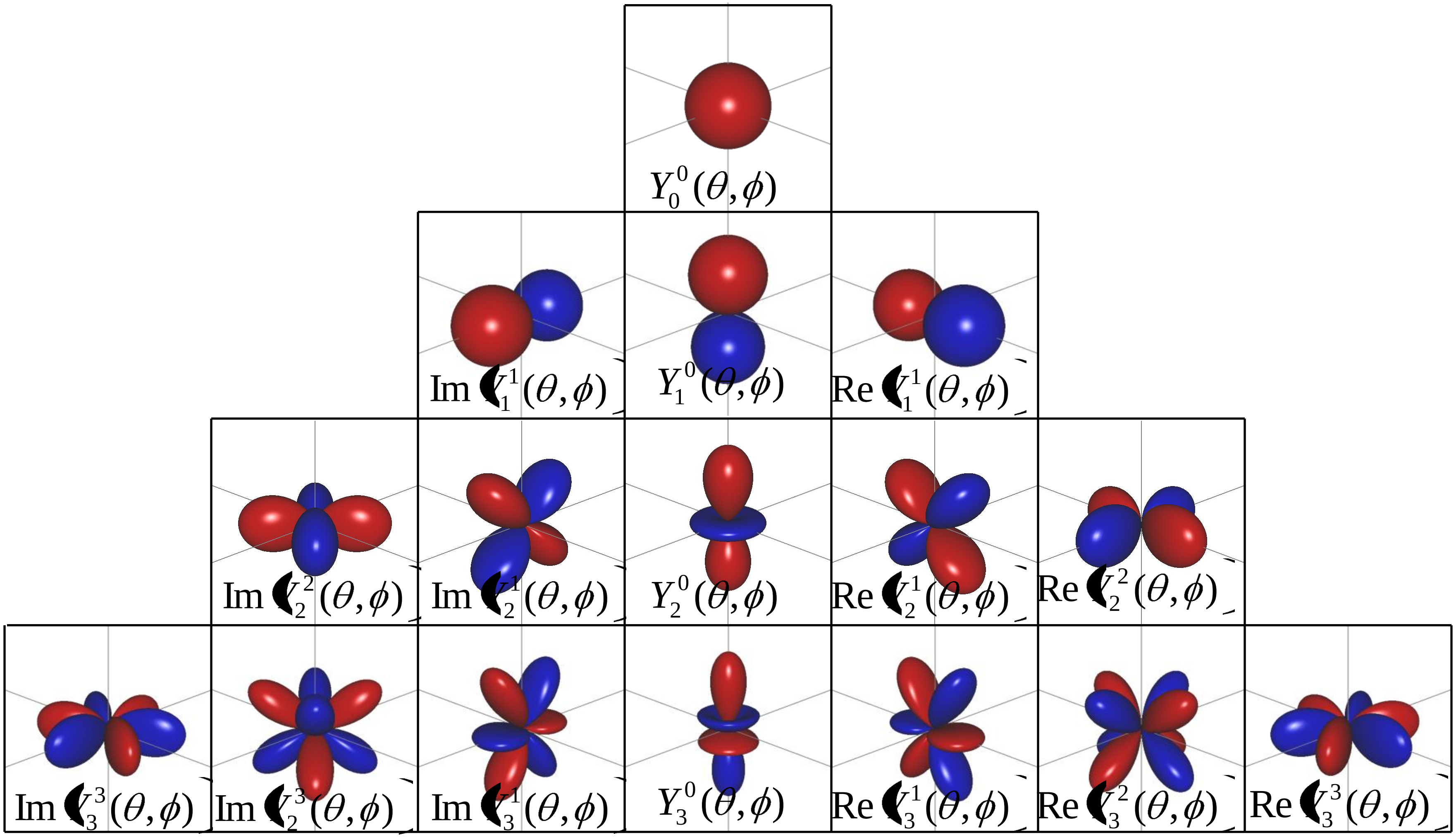
Thus, either:

- $e^{-im\alpha}=1$ for all α, which would imply that $m=0$, or
- $\hat{g}(l,m)=0$

This implies that in terms of the spherical harmonics, we can express the function g as:

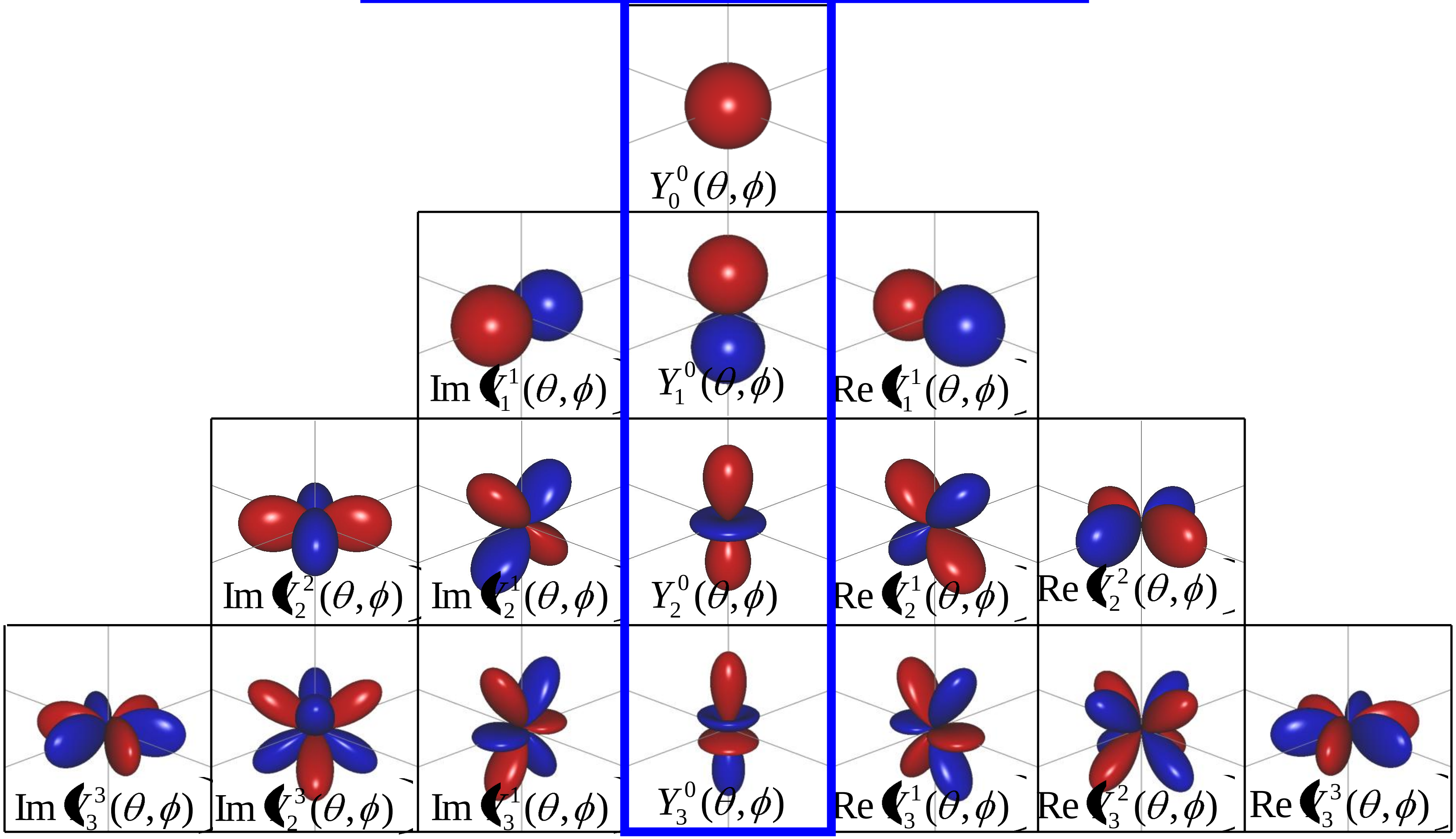
$$g(\theta,\phi) = \sum_{l} \hat{g}(l,0) Y_l^0(\theta,\phi)$$

Spherical Convolution



Spherical Convolution

$$g(\theta,\phi)=\sum_l \hat{g}(l,0)Y_l^0(\theta,\phi)$$



Spherical Convolution

Convolution:

Thus, the expression for the functions in terms of their spherical harmonic decomposition becomes:

$$f(\theta,\phi)=\sum_{l}\sum_{m=-l}^{l}\hat{f}(l,m)Y_l^m(\theta,\phi)$$

$$g(\theta,\phi)=\sum_{l}\hat{g}(l,0)Y_l^0(\theta,\phi)$$

Spherical Convolution

Convolution:

Thus, the expression for the functions in terms of their spherical harmonic decomposition becomes:

$$f(\theta,\phi) = \sum_{l} \sum_{m=-l}^{l} \hat{f}(l,m) Y_l^m(\theta,\phi)$$

$$g(\theta,\phi) = \sum_{l} \hat{g}(l,0) Y_l^0(\theta,\phi)$$

and we get an expression for the convolution:

$$(f * g)(\theta,\phi) = \left\langle \sum_{l} \sum_{m=-l}^{l} \hat{f}(l,m) Y_l^m, \rho_{R(\theta,\phi)}\left(\sum_{l} \hat{g}(l,0) Y_l^0 \right) \right\rangle$$

Spherical Convolution

<u>Convolution</u>:

By leveraging the conjugate-linearity of the inner product and using the fact that the transformation ρ_R is linear, we get:

$$(f * g)(\theta,\phi) = \left\langle \sum_{l} \sum_{m=-l}^{l} \hat{f}(l,m) Y_l^m , \rho_{R(\theta,\phi)} \left(\sum_{l} \hat{g}(l,0) Y_l^0 \right) \right\rangle$$

$$(f * g)(\theta,\phi) = \sum_{l,l'} \sum_{m=-l}^{l} \hat{f}(l,m) \overline{\hat{g}(l',0)} \left\langle Y_l^m , \rho_{R(\theta,\phi)} Y_{l'}^0 \right\rangle$$

Spherical Convolution

Convolution:

Additionally, we know that:

- A rotation of an *l*-th frequency function will still be an *l*-th frequency function
- The space of *l*-th frequency functions is orthogonal to the space of *l'*-th frequency functions

Thus, for all $l \neq l'$, we must have:

$$\left\langle Y_l^m, \rho_R Y_{l'}^{m'} \right\rangle = 0$$

Spherical Convolution

Convolution:

This lets us simplify the expression for the convolution:

$$(f * g)(\theta,\phi) = \sum_{l,l'} \sum_{m=-l}^{l} \hat{f}(l,m)\overline{\hat{g}(l',0)}\left\langle Y_l^m, \rho_{R(\theta,\phi)} Y_{l'}^0 \right\rangle$$

$$\Downarrow$$

$$(f * g)(\theta,\phi) = \sum_{l} \sum_{m=-l}^{l} \hat{f}(l,m)\overline{\hat{g}(l,0)}\left\langle Y_l^m, \rho_{R(\theta,\phi)} Y_l^0 \right\rangle$$

Spherical Convolution

Convolution:

$$(f * g)(\theta,\phi) = \sum_{l} \sum_{m=-l}^{l} \hat{f}(l,m)\overline{\hat{g}(l,0)}\left\langle Y_l^m, \rho_{R(\theta,\phi)} Y_l^0 \right\rangle$$

To compute the convolution, we need to be able to evaluate the inner product:

$$\left\langle Y_l^m, \rho_{R(\theta,\phi)} Y_l^0 \right\rangle$$

Spherical Convolution

Convolution:

What is the meaning of the function:

$$\left\langle Y_l^m, \rho_{R(\theta,\phi)} Y_l^0 \right\rangle$$

Spherical Convolution

Convolution:

What is the meaning of the function:

$$\left\langle Y_l^m, \rho_{R(\theta,\phi)} Y_l^0 \right\rangle$$

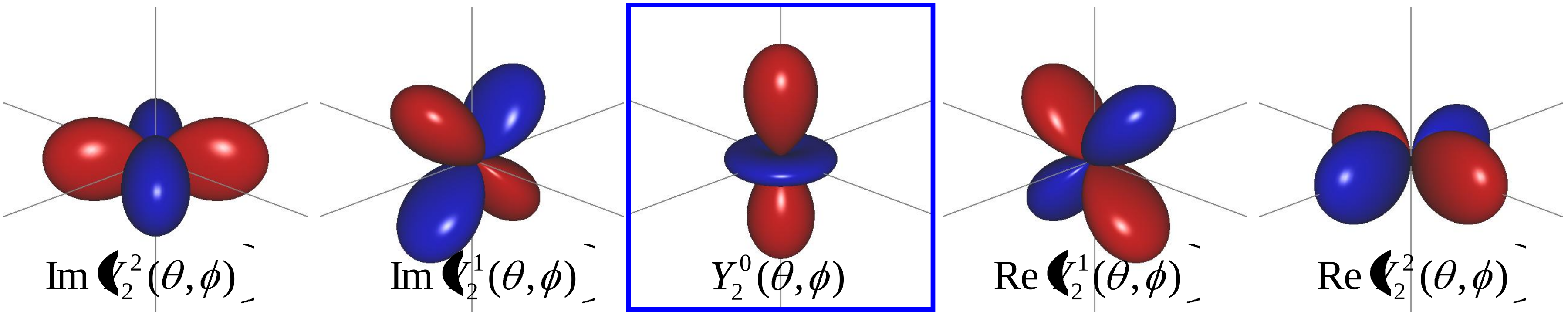
This is a function on the sphere whose value at the point $p=\Phi(\theta,\phi)$ is the inner product of Y_l^m with the rotation of Y_l^0, where the rotation takes the North pole to p.

Spherical Convolution

Convolution:

We would like to show that this function acts very simply on the spherical harmonics:

$$\left\langle Y_l^m, \rho_{R(\theta,\phi)} Y_l^0 \right\rangle = \lambda_l Y_l^m(\theta,\phi)$$



Spherical Convolution

Convolution:

Let's consider the operator C_l that maps spherical functions to spherical functions, defined by:

$$\left(C_l(f)\right)(\theta,\phi) = \left\langle f, \rho_{R(\theta,\phi)} Y_l^0 \right\rangle$$

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Thus, there exists an orthonormal basis with respect to which C_l is diagonal.

Spherical Convolution

Convolution:

Let's consider the operator C_l that maps spherical functions to spherical functions, defined by:

$$\left(C_l(f)\right)(\theta,\phi)=\left\langle f,\rho_{R(\theta,\phi)}Y_l^0\right\rangle$$

This operator also has the property that it commutes with rotations:

- Rotating a spherical function and then convolving with Y_l^0 is the same as first convolving with Y_l^0 and then rotating.

Spherical Convolution

Convolution:

So, as with the Laplacian, we have a case in which we are given a symmetric operator which commutes with rotations.

Spherical Convolution

Convolution:

Thus, the subspace of l'-th frequency functions is a space of functions that are eigenvectors of C_l, all with the same eigenvalue.

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Spherical Convolution

Convolution:

Thus, the subspace of *l'*-th frequency functions is a space of functions that are eigenvectors of C_l, all with the same eigenvalue.

$$C_l(Y_{l'}^m) = \lambda_{l,l'} Y_{l'}^m$$

Since we already know that for *l*≠*l'*, we must have:

$$\left(C_l(Y_l^m)\right)(\theta,\phi) = \left\langle Y_l^m, \rho_{R(\theta,\phi)} Y_{l'}^{m'} \right\rangle = 0$$

This must imply that for *l*≠*l'* we must have:

$$\lambda_{l,l'} = 0$$

Spherical Convolution

Convolution:

Putting this all together, we get the desired expression:

$$\left\langle Y_l^m, \rho_{R(\theta,\phi)} Y_l^0 \right\rangle = \lambda_l Y_l^m(\theta,\phi)$$

Spherical Convolution

Convolution:

Putting this all together, we get the desired expression:

$$\left\langle Y_l^m, \rho_{R(\theta,\phi)} Y_l^0 \right\rangle = \lambda_l Y_l^m(\theta,\phi)$$

Thus, the general equation for the convolution becomes:

$$(f * g)(\theta,\phi) = \sum_l \sum_{m=-l}^{l} \hat{f}(l,m)\overline{\hat{g}(l,0)}\left\langle Y_l^m, \rho_{R(\theta,\phi)} Y_l^0 \right\rangle$$

$$(f * g)(\theta,\phi) = \sum_l \sum_{m=-l}^{l} \hat{f}(l,m)\overline{\hat{g}(l,0)}\lambda_l Y_l^m(\theta,\phi)$$

Spherical Convolution

Convolution:

$$\left(f * g\right)(\theta,\phi) = \sum_{l} \sum_{m=-l}^{l} \hat{f}(l,m)\lambda_l \overline{\hat{g}(l,0)} Y_l^m(\theta,\phi)$$

Thus, the spherical harmonic coefficients of the convolution of *f* with *g* can be obtained by multiplying the (*l*,*m*)-th coefficients of *f* by $\lambda_l \overline{\hat{g}(l,0)}$.

Spherical Convolution

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Thus, the spherical harmonic coefficients of the convolution of *f* with *g* can be obtained by multiplying the (*l*,*m*)-th coefficients of *f* by $\lambda_l \overline{\hat{g}(l,0)}$.

As in the case of functions on a circle, this means that convolution in the spatial domain amounts to multiplication in the frequency domain.

Spherical Convolution

Convolution:

In order to be able to use the convolution theorem for spherical functions, we need to know what the eigenvalues λ_l are.

Spherical Convolution

Convolution:

In order to be able to use the convolution theorem for spherical functions, we need to know what the eigenvalues λ_l are.

It turns out that these are:

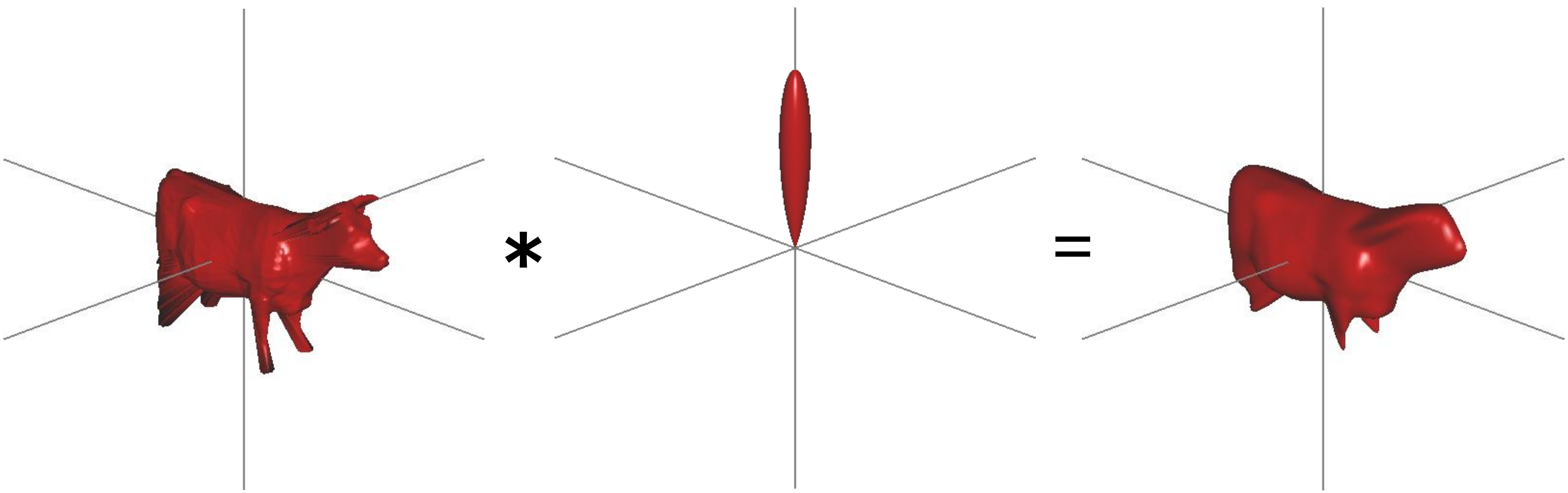
$$\lambda_l = \sqrt{\frac{4\pi}{2l+1}}$$

Spherical Convolution

<u>Convolution</u>:

Which gives us the equation:

$$\widehat{(f * g)}(l,m) = \sqrt{\frac{4\pi}{2l+1}}\,\hat{f}(l,m)\,\overline{\hat{g}(l,0)}$$

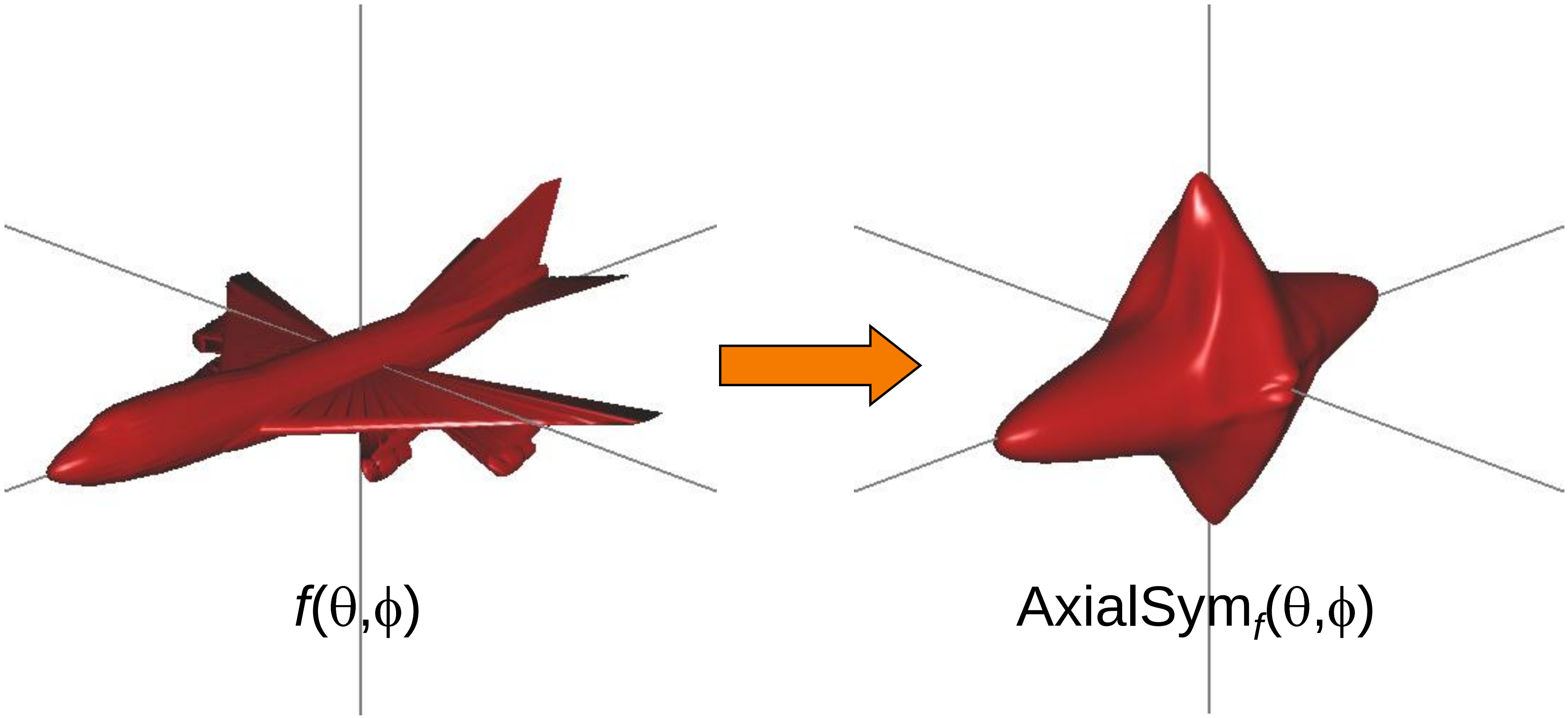


Outline

- Math Review
- Spherical Convolution
- Axial Symmetry Detection

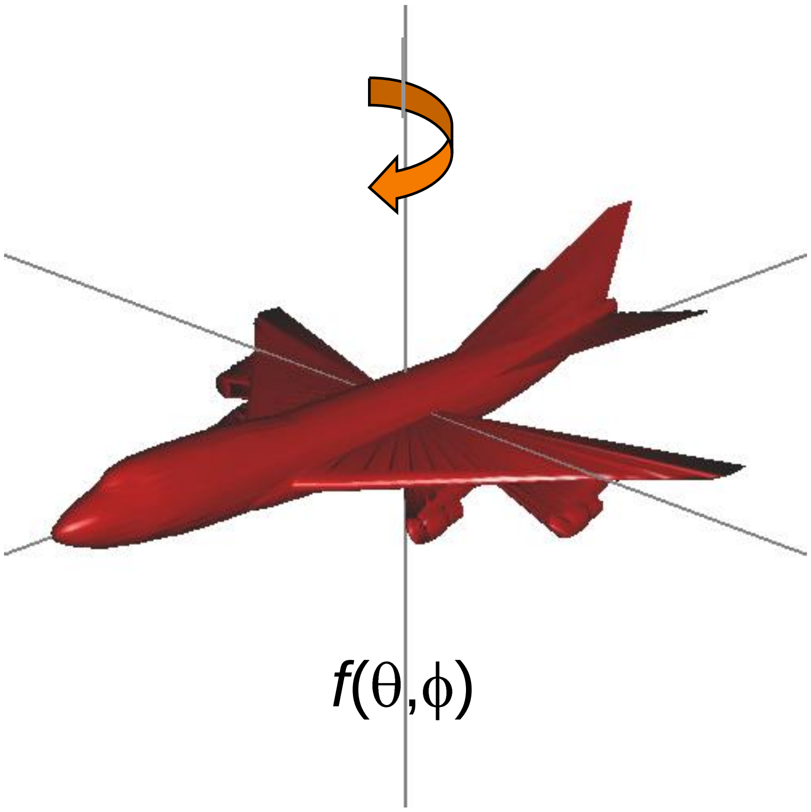
Axial Symmetry Detection

Given a spherical function *f*, we would like to compute the measure of the axial symmetry of *f* with respect to every axis through the origin.



Axial Symmetry Detection

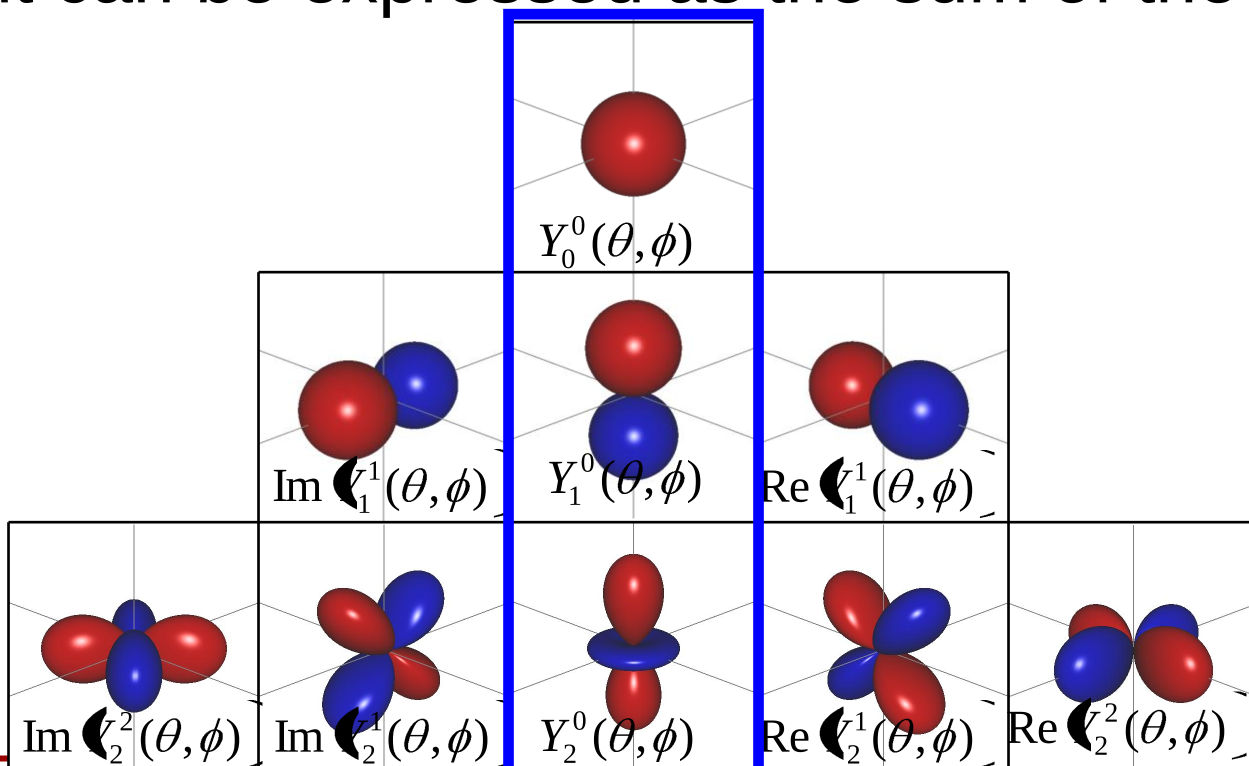
What is the measure of the axial symmetry of *f* about the *y*-axis?



Axial Symmetry Detection

What is the measure of the axial symmetry of *f* about the *y*-axis?

We know that *f* is axially symmetric about the *y*-axis if it can be expressed as the sum of the Y_l^0:



Axial Symmetry Detection

What is the measure of the axial symmetry of *f* about the *y*-axis?

We know that *f* is axially symmetric about the *y*-axis if it can be expressed as the sum of the Y_l^0.

We also know that for $m \neq 0$:

$$\left\langle Y_l^0, Y_l^m \right\rangle = 0$$

Axial Symmetry Detection

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We also know that for $m \neq 0$:

$$\left\langle Y_l^0, Y_l^m \right\rangle = 0$$

So the projection onto the space of functions that are axially symmetric about the *y*-axis is obtained by zeroing out the appropriate coefficients:

$$\pi_y\left(\sum_l \sum_{m=-l}^{l} \hat{f}(l,m) Y_l^m\right) = \sum_l \hat{f}(l,0) Y_l^0$$

Axial Symmetry Detection

What is the measure of the axial symmetry of *f* about the *y*-axis?

Thus, the measure of the axial symmetry of *f* about the *y*-axis is defined as:

$$\mathrm{YAxialSym}^2(f) = \left\| \sum_l \hat{f}(l,0) Y_l^0 \right\|^2$$

Axial Symmetry Detection

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Axial Symmetry Detection

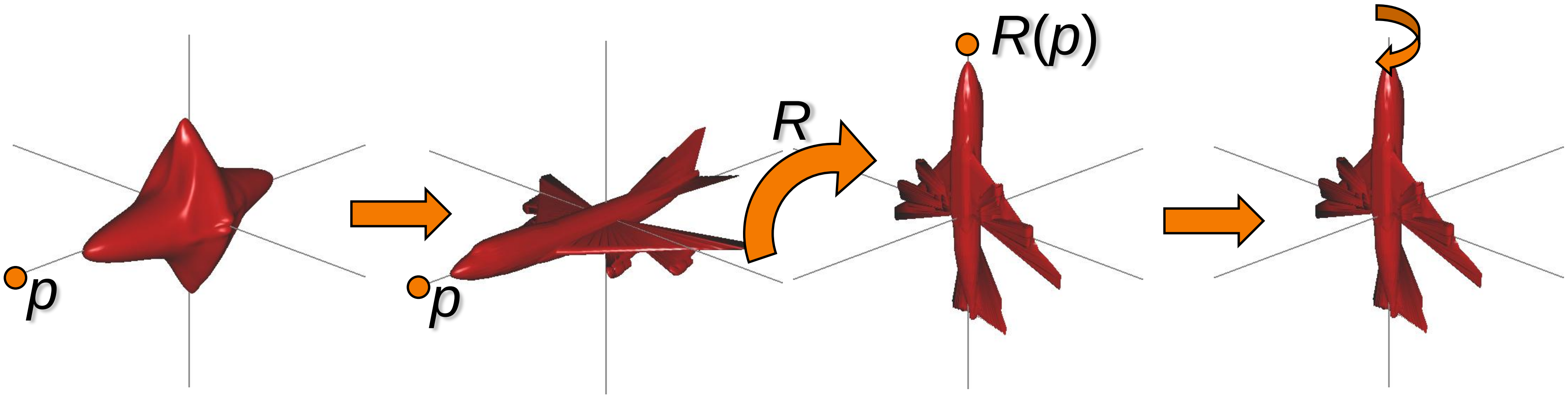
More generally, we would like to be able to compute the measure of the axial symmetry of f with respect to any axis.

Axial Symmetry Detection

More generally, we would like to be able to compute the measure of the axial symmetry of *f* with respect to any axis.

To compute the symmetry measure about the line through $p=\Phi(\theta,\phi)$ we:

- Rotate so that *p* goes to the North pole, and
- Compute the symmetry measure about the *y*-axis.



Axial Symmetry Detection

More generally, we would like to be able to compute the measure of the axial symmetry of *f* with

Since the rotation $R(\theta,\phi)$ maps the North pole to *p*, the rotation we are interested in is the inverse, $R^{-1}(\theta,\phi)$.

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- Rotate so that *p* goes to the North pole, and
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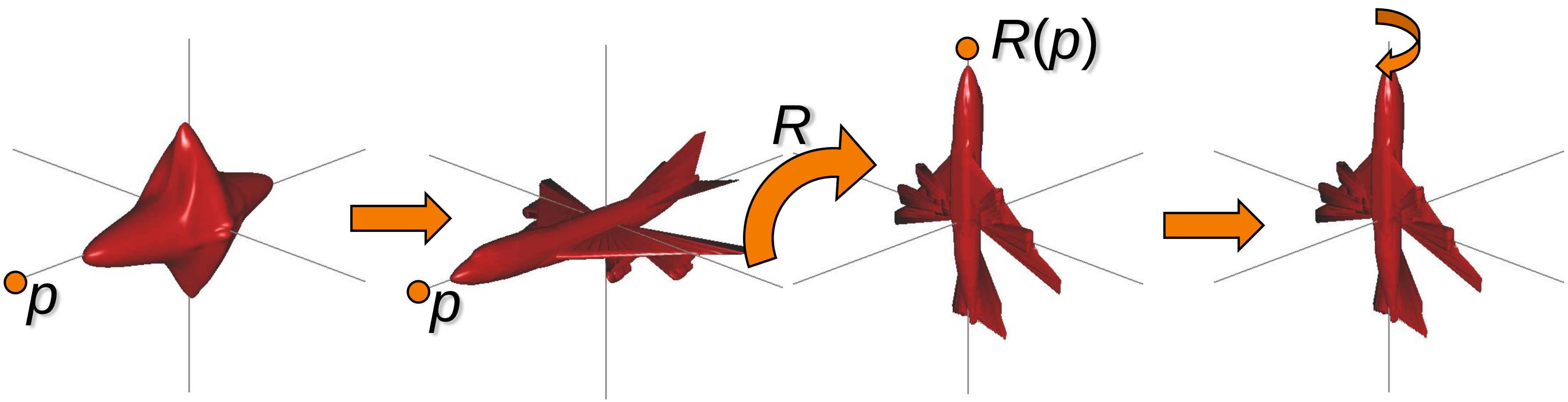

Axial Symmetry Detection

Using the fact that the spherical harmonics form an orthonormal basis, we know that the (*l*,*m*)-th harmonic coefficient of *f* is defined by:

$$\hat{f}(l,m) = \left\langle f, Y_l^m \right\rangle$$

Axial Symmetry Detection

Using the fact that the spherical harmonics form an orthonormal basis, we know that the (*l*,*m*)-th harmonic coefficient of *f* is defined by:

$$\hat{f}(l,m) = \left\langle f, Y_l^m \right\rangle$$

Thus, to compute the measure of axial symmetry about the axis through *p* we need to compute:

$$\text{AxialSym}_f^2(\theta,\phi) = \sum_l \left\| \left\langle \rho_{R^{-1}(\theta,\phi)} f, Y_l^0 \right\rangle \right\|^2$$

Axial Symmetry Detection

Using the fact that ρ is a unitary representation we can re-write this equation as:

$$\text{AxialSym}_f^2(\theta,\phi) = \sum_l \left\| \left\langle \rho_{R^{-1}(\theta,\phi)} f, Y_l^0 \right\rangle \right\|^2$$

$$\downarrow$$

$$\text{AxialSym}_f^2(\theta,\phi) = \sum_l \left\| \left\langle f, \rho_{R(\theta,\phi)} Y_l^0 \right\rangle \right\|^2$$

Axial Symmetry Detection

Thus, if we express *f* in terms of its spherical harmonic decomposition, we get:

$$\text{AxialSym}_f^2(\theta,\phi) = \sum_l \left\| \left\langle f, \rho_{R(\theta,\phi)} Y_l^0 \right\rangle \right\|^2$$

$$\text{AxialSym}_f^2(\theta,\phi) = \sum_l \left\| \sum_{m=-l}^{l} \hat{f}(l,m) \left\langle Y_l^m, \rho_{R(\theta,\phi)} Y_l^0 \right\rangle \right\|^2$$

Axial Symmetry Detection

But now we can apply the identity:

$$\left\langle Y_l^m, \rho_{R(\theta,\phi)} Y_l^0 \right\rangle = \sqrt{\frac{4\pi}{2l+1}} Y_l^m(\theta,\phi)$$

to get an expression for the symmetry measure:

$$\text{AxialSym}_f^2(\theta,\phi) = \sum_l \left\| \sum_{m=-l}^{l} \hat{f}(l,m) \left\langle Y_l^m, \rho_{R(\theta,\phi)} Y_l^0 \right\rangle \right\|^2$$

$$\downarrow$$

$$\text{AxialSym}_f^2(\theta,\phi) = \sum_l \frac{4\pi}{2l+1} \left\| \sum_{m=-l}^{l} \hat{f}(l,m) Y_l^m(\theta,\phi) \right\|^2$$

Axial Symmetry Detection

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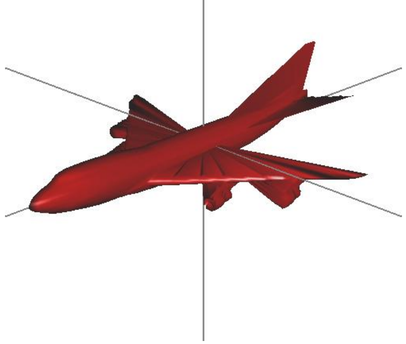
Thus, the measure of axial symmetry can be computed by taking the (weighted) sum of the square norms of the different frequency components of *f*.

Axial Symmetry Detection

$$\text{AxialSym}_f^2(\theta,\phi)=\sum_l \frac{4\pi}{2l+1}\left\|\sum_{m=-l}^{l}\hat{f}(l,m)Y_l^m(\theta,\phi)\right\|^2$$

Initial Function



Axial Symmetry Detection

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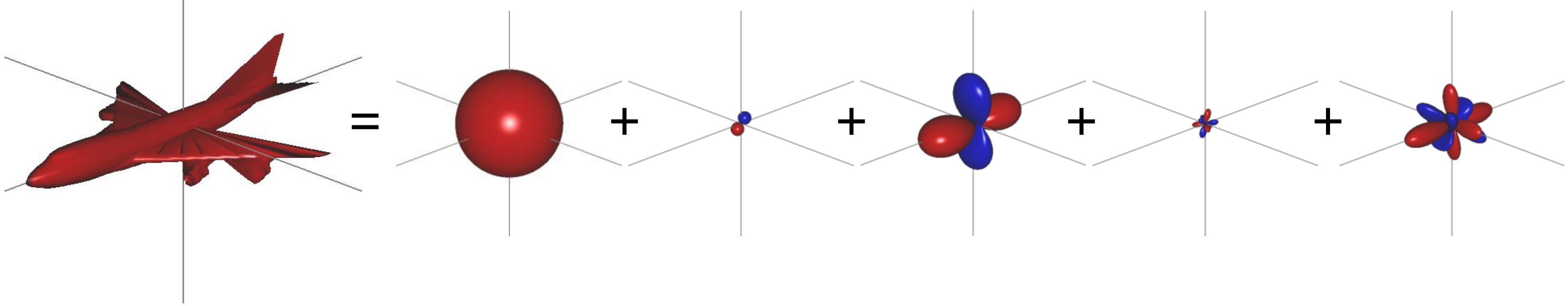
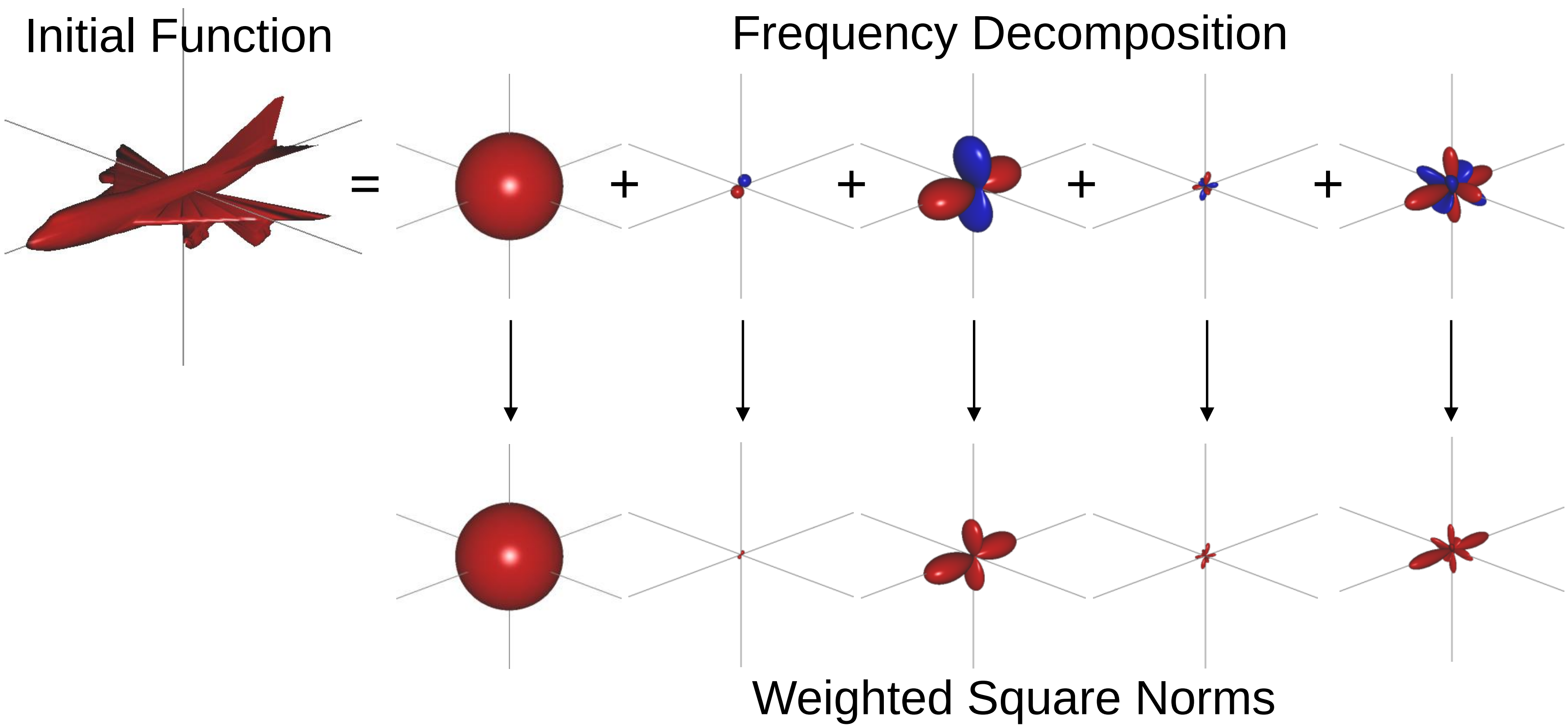
Initial Function

Frequency Decomposition



Axial Symmetry Detection

$$\text{AxialSym}_f^2(\theta,\phi) = \sum_l \frac{4\pi}{2l+1} \left\| \sum_{m=-l}^{l} \hat{f}(l,m) Y_l^m(\theta,\phi) \right\|^2$$



Axial Symmetry Detection

$$\text{AxialSym}_f^{\,2}(\theta,\phi)=\sum_l \frac{4\pi}{2l+1}\left\|\sum_{m=-l}^{l}\hat{f}(l,m)Y_l^m(\theta,\phi)\right\|^2$$