FFTs in Graphics and Vision

The Spherical Laplacian
Outline

• Stokes’ Theorem
• Tangent Spaces
• Gradients
• The Spherical Laplacian
• Applications
Stokes’ Theorem

Stokes’ Theorem equates the integral of the divergence of a vector field over a region to the surface integral of the vector field over the boundary:

\[ \int \nabla \cdot F \, dV = \int_{\partial V} F \cdot dA \]

where \( F \cdot dA \) is defined by:

\[ F \cdot dA = \langle F, \hat{n} \rangle \, |dA| \]
Stokes’ Theorem

Stokes’ Theorem equates the integral of the divergence of a vector field over a region to the surface integral of the vector field over the boundary:

\[ \int_{\mathcal{V}} \nabla \cdot \mathbf{F} \, d\mathbf{V} = \oint_{\partial \mathcal{V}} \mathbf{F} \cdot d\mathbf{A} \]
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Tangent Spaces

Given a curve $C(t)=(x(t), y(t))$, the tangent line to the curve at a point $p_0=C(t_0)$ is the line passing through $p_0$ with direction $C'(t_0)=(x'(t_0), y'(t_0))$. 
Tangent Spaces

Given a curve $C(t)=(x(t),y(t))$, the tangent line to the curve at a point $p_0=C(t_0)$ is the line passing through $p_0$ with direction $C'(t_0)=(x'(t_0),y'(t_0))$.

This is the line that most closely approximates the curve $C(t)$ at the point $p_0$. 
Tangent Spaces

Often, what we want is a unit vector specifying the tangent direction.

In this case, we need to normalize:

\[ T_C(t) = \frac{C'(t)}{|C'(t)|} \]
Tangent Spaces

Given a surface \( S(u,v) \) the tangent plane to the curve at a point \( p_0 = S(u_0,v_0) \) is the plane passing through \( p_0 \), parallel to the plane spanned by:

\[
\left. \frac{\partial S(u,v)}{\partial u} \right|_{(u_0,v_0)} \quad \text{and} \quad \left. \frac{\partial S(u,v)}{\partial v} \right|_{(u_0,v_0)}
\]

\( \Phi(\theta, \phi) = \cos \theta \sin \phi, \cos \phi, \sin \theta \sin \phi \)
Given a surface $S(u,v)$ the tangent plane to the curve at a point $p_0 = S(u_0,v_0)$ is the plane passing through $p_0$, parallel to the plane spanned by:

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\frac{\partial S(u,v)}{\partial u}_{(u_0,v_0)} \quad \text{and} \quad \frac{\partial S(u,v)}{\partial v}_{(u_0,v_0)}
$$

This is the plane that most closely approximates $S(u,v)$ at the point $p_0$.

$$
\Phi(\theta, \phi) = \cos \theta \sin \phi, \cos \phi, \sin \theta \sin \phi
$$
Tangent Spaces

In the case of the sphere, points are parameterized by the equation:

$$\Phi(\theta, \phi) = \cos \theta \sin \phi, \cos \phi, \sin \theta \sin \phi$$

and the two tangent directions are:

$$\frac{\partial \Phi}{\partial \theta} = \sin \theta \sin \phi, 0, \cos \theta \sin \phi$$

$$\frac{\partial \Phi}{\partial \phi} = \cos \theta \cos \phi, -\sin \phi, \sin \theta \cos \phi$$
Tangent Spaces

If we look at the dot-product of the two vectors:

\[
\frac{\partial \Phi}{\partial \theta} = \begin{bmatrix} \sin \theta \sin \phi, 0, \cos \theta \sin \phi \end{bmatrix}
\]

\[
\frac{\partial \Phi}{\partial \phi} = \begin{bmatrix} \cos \theta \cos \phi, -\sin \phi, \sin \theta \cos \phi \end{bmatrix}
\]

we get:

\[
\left\langle \frac{\partial \Phi}{\partial \theta}, \frac{\partial \Phi}{\partial \theta} \right\rangle = \sin^2 \theta \sin^2 \phi + \cos^2 \theta \sin^2 \phi
\]
Tangent Spaces

If we look at the dot-product of the two vectors:

\[
\frac{\partial \Phi}{\partial \theta} = \langle \sin \theta \sin \phi, 0, \cos \theta \sin \phi \rangle
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\[
\frac{\partial \Phi}{\partial \phi} = \langle \cos \theta \cos \phi, -\sin \phi, \sin \theta \cos \phi \rangle
\]

we get:

\[
\left\langle \frac{\partial \Phi}{\partial \theta}, \frac{\partial \Phi}{\partial \theta} \right\rangle = \langle \sin^2 \theta + \cos^2 \theta \rangle \sin^2 \phi
\]
Tangent Spaces

If we look at the dot-product of the two vectors:

\[ \frac{\partial \Phi}{\partial \theta} = \begin{pmatrix} \sin \theta \sin \phi, 0, \cos \theta \sin \phi \end{pmatrix} \]

\[ \frac{\partial \Phi}{\partial \phi} = \begin{pmatrix} \cos \theta \cos \phi, -\sin \phi, \sin \theta \cos \phi \end{pmatrix} \]

we get:

\[ \left\langle \frac{\partial \Phi}{\partial \theta}, \frac{\partial \Phi}{\partial \theta} \right\rangle = \sin^2 \phi \]
Tangent Spaces

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\frac{\partial \Phi}{\partial \phi} = \langle \cos \theta \cos \phi, -\sin \phi, \sin \theta \cos \phi \rangle
\]

we get:

\[
\left\langle \frac{\partial \Phi}{\partial \theta} , \frac{\partial \Phi}{\partial \theta} \right\rangle = \sin^2 \phi
\]

\[
\left\langle \frac{\partial \Phi}{\partial \phi} , \frac{\partial \Phi}{\partial \phi} \right\rangle = \cos^2 \theta \cos^2 \phi + \sin^2 \phi + \sin^2 \theta \cos^2 \phi
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Tangent Spaces

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\]

\[
\left\langle \frac{\partial \Phi}{\partial \phi}, \frac{\partial \Phi}{\partial \phi} \right\rangle = \cos^2 \theta + \sin^2 \theta \cos^2 \phi + \sin^2 \phi
\]
Tangent Spaces

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\]

we get:

\[
\begin{aligned}
\left\langle \frac{\partial \Phi}{\partial \theta}, \frac{\partial \Phi}{\partial \theta} \right\rangle &= \sin^2 \phi \\
\left\langle \frac{\partial \Phi}{\partial \phi}, \frac{\partial \Phi}{\partial \phi} \right\rangle &= \cos^2 \phi + \sin^2 \phi
\end{aligned}
\]
Tangent Spaces

If we look at the dot-product of the two vectors:

\[
\begin{align*}
\frac{\partial \Phi}{\partial \theta} &= \begin{pmatrix} \sin \theta \sin \phi, 0, \cos \theta \sin \phi \end{pmatrix} \\
\frac{\partial \Phi}{\partial \phi} &= \begin{pmatrix} \cos \theta \cos \phi, -\sin \phi, \sin \theta \cos \phi \end{pmatrix}
\end{align*}
\]

we get:

\[
\begin{align*}
\left\langle \frac{\partial \Phi}{\partial \theta}, \frac{\partial \Phi}{\partial \theta} \right\rangle &= \sin^2 \phi \\
\left\langle \frac{\partial \Phi}{\partial \phi}, \frac{\partial \Phi}{\partial \phi} \right\rangle &= 1
\end{align*}
\]
Tangent Spaces

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\frac{\partial \Phi}{\partial \theta} &= \langle \sin \theta \sin \phi, 0, \cos \theta \sin \phi \rangle \\
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\left\langle \frac{\partial \Phi}{\partial \theta}, \frac{\partial \Phi}{\partial \theta} \right\rangle &= \sin^2 \phi \\
\left\langle \frac{\partial \Phi}{\partial \phi}, \frac{\partial \Phi}{\partial \phi} \right\rangle &= 1 \\
\left\langle \frac{\partial \Phi}{\partial \theta}, \frac{\partial \Phi}{\partial \phi} \right\rangle &= -\sin \theta \cos \theta \sin \phi \cos \phi + \sin \theta \cos \theta \sin \phi \cos \phi
\end{align*}
\]
Tangent Spaces

If we look at the dot-product of the two vectors:

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\frac{\partial \Phi}{\partial \theta} &= \langle \sin \theta \sin \phi, 0, \cos \theta \sin \phi \rangle \\
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\left\langle \frac{\partial \Phi}{\partial \theta}, \frac{\partial \Phi}{\partial \phi} \right\rangle &= 0
\end{align*}
\]
Tangent Spaces

So, the vectors:

\[ \Phi_\theta(\theta, \phi) = \frac{1}{\sin \phi} \frac{\partial \Phi}{\partial \theta} \]

\[ \Phi_\phi(\theta, \phi) = \frac{\partial \Phi}{\partial \phi} \]

form an orthonormal basis for the tangent plane to the sphere at the point \( \Phi(\theta, \phi) \).
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Function Gradients

The gradient of a function is a vector which tells us how the function changes as we move in different directions.
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Given a function $f$ and given a direction $v$:

$$f(p + v) \approx f(p) + \langle \nabla f(p), v \rangle$$
Function Gradients

To compute the gradient, we choose two orthogonal unit vectors $u$ and $v$, and we set:

$$\nabla f(p) = \frac{d}{dt} f(p + tu)u + \frac{d}{dt} f(p + tv)v$$
Curve Gradients

Given a curve $C(t)$, and given a function $f(t)$ the gradient of the function is a vector field on the curve telling us how the function changes as we move along the curve.
Curve Gradients

Example:

Let $C$ be the curve defined by:

$$C(t) = t^2$$

and let $f(t)$ be the function on the curve defined by:

$$f(t) = t$$

What is the gradient of $f(t)$?
Curve Gradients

Example:

The gradient is not the function $\nabla_C f = 1$!

This would imply that at any point on the curve moving a unit forward would change the value by a constant amount.
Curve Gradients

Example:

The gradient is **not** the function $\nabla_C f = 1$!

As we move from $t=1$ to $t=2$, the function changes by a value of 1. Similarly, as we move from $t=10$ to $t=11$, the function changes by a value of 1.
Curve Gradients

Example:

The gradient is **not** the function $\nabla_C f = 1$!

As we move from $t=1$ to $t=2$, the function changes by a value of 1. Similarly, as we move from $t=10$ to $t=11$, the function changes by a value of 1.

But in the first case, we have moved a distance of:

\[
d_1 \approx \| C(2) - C(1) \| = \sqrt{1^2 + 3^2}
\]
Curve Gradients

Example:

The gradient is \textbf{not} the function $\nabla_C f = 1$!

As we move from $t=1$ to $t=2$, the function changes by a value of 1. Similarly, as we move from $t=10$ to $t=11$, the function changes by a value of 1.

In the second case, we have moved a distance of:

$$d_2 \approx \left\| C(11) - C(10) \right\| = \sqrt{1^2 + 21^2}$$
Curve Gradients

Example:

We need to measure the ratio of the change in $f$ over the distance traveled:

$$\nabla_C f(t) \approx \frac{f(t + \varepsilon) - f(t)}{|C(t + \varepsilon) - C(t)|}$$

$$\nabla_C f(t) = \frac{f'(t)}{|C'(t)|}$$
Curve Gradients

Example:

We need to measure the ratio of the change in $f$ over the distance traveled:

$$\nabla_{C}f(t) = \frac{1}{\sqrt{1+2t}}$$
Spherical Gradients

Given a function on the sphere, \( f(\theta, \phi) \), we would like to compute the gradient:

\[
\nabla f (\theta, \phi)
\]
Spherical Gradients

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We need to pick two directions in parameter space whose images on the surface of the sphere are orthogonal.
Spherical Gradients

Given a function on the sphere, \( f(\theta,\phi) \), we would like to compute the gradient:

\[
\nabla f (\theta, \phi)
\]

We need to pick two directions in parameter space whose images on the surface of the sphere are orthogonal.

The directions \( \theta \) and \( \phi \) are two such directions:
Spherical Gradients

We could try taking the partial derivatives in the θ and φ directions:

\[ \nabla f(\theta, \phi) = \left( \frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial \phi} \right) \]
Spherical Gradients

We could try taking the partial derivatives in the $\theta$ and $\phi$ directions:

$$\nabla f (\theta, \phi) = \left( \frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial \phi} \right)$$

But this introduces bias!

Shifting by a constant $\Delta \theta$ will move us different distances depending on where we are on the sphere.
Spherical Gradients

How does the scale change as we change $\theta$ or $\phi$ by a value of $\varepsilon$?
Spherical Gradients

How does the scale change as we change $\theta$ or $\phi$ by a value of $\varepsilon$?

At the point $p=\Phi(\theta, \phi)$, changing the value of $\theta$ by $\varepsilon$, moves us a distance of $\varepsilon r$ along the circle about the $y$-axis, where $r$ is the radius of the circle:
Spherical Gradients

How does the scale change as we change $\theta$ or $\phi$ by a value of $\varepsilon$?

At the point $p = \Phi(\theta, \phi)$, changing the value of $\theta$ by $\varepsilon$, moves us a distance of $\varepsilon r$ along the circle about the $y$-axis, where $r$ is the radius of the circle.

On the sphere, the radius is defined by:

$$r(\phi) = \sin \phi$$
Spherical Gradients

How does the scale change as we change $\theta$ or $\phi$ by a value of $\varepsilon$?

At the point $p=\Phi(\theta,\phi)$, changing the value of $\phi$ by $\varepsilon$, moves us a distance of $\varepsilon$ along a great circle, regardless of where on the sphere we are:
Spherical Gradients

Taking the scaling into account, we get:

\[
\nabla f(\theta, \phi) \approx \begin{pmatrix}
\frac{f(\theta + \epsilon, \phi) - f(\theta, \phi)}{\epsilon \sin \phi} & \frac{f(\theta, \phi + \epsilon) - f(\theta, \phi)}{\epsilon}
\end{pmatrix}
\]
Spherical Gradients

Taking the scaling into account, we get:

\[ \nabla f(\theta, \phi) \approx \left( \frac{f(\theta + \epsilon, \phi) - f(\theta, \phi)}{\epsilon \sin \phi}, \frac{f(\theta, \phi + \epsilon) - f(\theta, \phi)}{\epsilon} \right) \]

\[ \nabla f(\theta, \phi) = \left( \frac{1}{\sin \phi} \frac{\partial f}{\partial \theta} \right) \Phi_\theta + \left( \frac{\partial f}{\partial \phi} \right) \Phi_\phi \]
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The Spherical Laplacian

Recall:
The Laplacian operator is self-adjoint (symmetric)
⇒ There is an orthogonal basis of eigenvectors.
The Spherical Laplacian

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The Laplacian operator is self-adjoint (symmetric)
⇒ There is an orthogonal basis of eigenvectors.

The Laplacian operator commutes with rotations
⇒ If $F_\lambda$ are the eigenfunctions of the Laplacian with eigenvalue $\lambda$, rotations fix $F_\lambda$. 
The Spherical Laplacian

Recall:

The Laplacian operator is self-adjoint (symmetric)

⇒ There is an orthogonal basis of eigenvectors.

The Laplacian operator commutes with rotations

⇒ If $F_\lambda$ are the eigenfunctions of the Laplacian with eigenvalue $\lambda$, rotations fix $F_\lambda$.

⇒ The irreducible representations are subspaces of the $F_\lambda$. 
The Spherical Laplacian

All this implies that for a fixed degree $l$, the spherical harmonics of degree $l$:

$$ Y_l^k (\theta, \phi) = e^{ik\theta} P_l^k (\cos \phi) $$

($-k \leq k \leq l$) must be eigenvectors of the Laplacian with the same eigenvalue.
The Spherical Laplacian

All this implies that for a fixed degree $l$, the spherical harmonics of degree $l$:

$$Y_l^k (\theta, \phi) = e^{ik\theta} P_l^k (\cos \phi)$$

($-l \leq k \leq l$) must be eigenvectors of the Laplacian with the same eigenvalue.

1. What is the Laplacian?
2. What are the eigenvalues?
The Spherical Laplacian

How do we compute the Laplacian of a spherical function $f(\theta, \phi)$?
The Spherical Laplacian

How do we compute the Laplacian of a spherical function $f(\theta, \phi)$?

Recall:
The Laplacian of a function is the divergence of its gradient:

$$\nabla^2 f = \nabla \cdot (\nabla f)$$
The Spherical Laplacian

By Stokes’ Theorem, we can compute the integral of the Laplacian (the divergence of the gradient) over a region by computing the surface integral of the gradient field over the boundary:
The Spherical Laplacian

Consider the “square” on the sphere with endpoints \((\theta, \phi), (\theta + \varepsilon, \phi), (\theta + \varepsilon, \phi + \varepsilon)\) and \((\theta, \phi + \varepsilon)\):
The Spherical Laplacian

Consider the “square” on the sphere with end-points \((\theta, \phi), (\theta + \varepsilon, \phi), (\theta + \varepsilon, \phi + \varepsilon)\) and \((\theta, \phi + \varepsilon)\):

The integral of the Laplacian is approximately:

\[
\int_{R} \nabla^{2} f \, dR \approx \text{Area}(R) \nabla^{2} f (\theta, \phi)
\]
The Spherical Laplacian

Consider the “square” on the sphere with endpoints \((\theta, \phi), (\theta + \varepsilon, \phi), (\theta + \varepsilon, \phi + \varepsilon)\) and \((\theta, \phi + \varepsilon)\):

The integral of the Laplacian is approximately:

\[
\int_{R} \nabla^2 f \, dR \approx \text{Area}(R) \nabla^2 f (\theta, \phi)
\]

\[= \varepsilon^2 \sin \phi \nabla^2 f (\theta, \phi)\]
The Spherical Laplacian

Consider the “square” on the sphere with endpoints \((\theta, \phi), (\theta + \varepsilon, \phi), (\theta + \varepsilon, \phi + \varepsilon)\) and \((\theta, \phi + \varepsilon)\):

On the curve \(c_1\), the surface integral of the gradient is approximately:

\[
\int_{c_1} \nabla f \cdot dA \approx \text{Length } (c_1) \left\langle \nabla f, \Phi_{\phi} \right\rangle
\]
The Spherical Laplacian

Consider the “square” on the sphere with endpoints \((\theta, \phi), (\theta + \varepsilon, \phi), (\theta + \varepsilon, \phi + \varepsilon)\) and \((\theta, \phi + \varepsilon)\):

On the curve \(c_1\), the surface integral of the gradient is approximately:

\[
\int_{c_1} \nabla f \cdot dA \approx \text{Length } (c_1) \langle \nabla f, \Phi_\phi \rangle
\]

\[
= \varepsilon \sin(\phi + \varepsilon) \left\langle \left( \frac{1}{\sin \phi + \varepsilon} \frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial \phi} \right), (0, 1) \right\rangle
\]
The Spherical Laplacian

Consider the “square” on the sphere with end-points \((\theta, \phi), (\theta + \varepsilon, \phi), (\theta + \varepsilon, \phi + \varepsilon)\) and \((\theta, \phi + \varepsilon)\):

On the curve \(c_1\), the surface integral of the gradient is approximately:

\[
\int_{c_1} \nabla f \cdot dA \approx \text{Length} \left( c_1 \right) \langle \nabla f, \Phi_\phi \rangle
\]

\[
= \varepsilon \sin(\phi + \varepsilon) \left\langle \left( \frac{1}{\sin \phi + \varepsilon} \frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial \phi} \right), (0,1) \right\rangle
\]

\[
= \varepsilon \sin(\phi + \varepsilon) \frac{\partial f}{\partial \phi} (\theta, \phi + \varepsilon)
\]
The Spherical Laplacian

Consider the “square” on the sphere with endpoints \((\theta, \phi), (\theta + \varepsilon, \phi), (\theta + \varepsilon, \phi + \varepsilon)\) and \((\theta, \phi + \varepsilon)\):

Similarly, on the curve \(c_2\), the surface integral of the gradient is approximately:

\[
\int_{c_2} \nabla f \cdot dA \approx -\varepsilon \sin(\phi) \frac{\partial f}{\partial \phi}(\theta, \phi)
\]
The Spherical Laplacian

Consider the “square” on the sphere with endpoints \((\theta, \phi), (\theta + \varepsilon, \phi), (\theta + \varepsilon, \phi + \varepsilon)\) and \((\theta, \phi + \varepsilon)\):

On the curve \(c_3\), the surface integral of the gradient is approximately:

\[
\int_{c_3} \nabla f \cdot dA \approx \text{Length } (c_3) \left\langle \nabla f, \Phi_\theta \right\rangle
\]
The Spherical Laplacian

Consider the “square” on the sphere with end-points \((\theta, \phi), (\theta + \varepsilon, \phi), (\theta + \varepsilon, \phi + \varepsilon)\) and \((\theta, \phi + \varepsilon)\):

On the curve \(c_3\), the surface integral of the gradient is approximately:

\[
\int_{c_3} \nabla f \cdot dA \approx \text{Length} \ (c_3) \left\langle \nabla f, \Phi_{\theta} \right\rangle
\]

\[
= \varepsilon \left\langle \left( \frac{1}{\sin \phi} \frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial \phi} \right) \right\rangle, (1,0)
\]
The Spherical Laplacian

Consider the “square” on the sphere with endpoints \((\theta, \phi), (\theta + \varepsilon, \phi), (\theta + \varepsilon, \phi + \varepsilon)\) and \((\theta, \phi + \varepsilon)\):

On the curve \(c_3\), the surface integral of the gradient is approximately:

\[
\int_{c_3} \nabla f \cdot dA \approx \text{Length}(c_3) \langle \nabla f, \Phi_\theta \rangle
\]

\[
= \varepsilon \left\langle \left( \frac{1}{\sin \phi} \frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial \phi} \right), (1,0) \right\rangle
\]

\[
= \varepsilon \frac{1}{\sin \phi} \frac{\partial f}{\partial \theta} \Phi + \varepsilon, \phi
\]
The Spherical Laplacian

Consider the “square” on the sphere with endpoints \((\theta, \phi), (\theta + \varepsilon, \phi), (\theta + \varepsilon, \phi + \varepsilon)\) and \((\theta, \phi + \varepsilon)\):

Similarly, on the curve \(c_4\), the surface integral of the gradient is approximately:

\[
\int_{c_4} \nabla f \cdot dA \approx -\varepsilon \frac{1}{\sin(\phi)} \frac{\partial f}{\partial \theta} \theta, \phi
\]
The Spherical Laplacian

Consider the “square” on the sphere with end-points \((\theta, \phi), (\theta + \varepsilon, \phi), (\theta + \varepsilon, \phi + \varepsilon)\) and \((\theta, \phi + \varepsilon)\):

Summing these together, we can approximate the boundary integral by:

\[
\int_{\partial R} \nabla f \cdot dA \approx \varepsilon \left( \frac{1}{\sin \phi} \left[ \frac{\partial f}{\partial \theta} \theta + \varepsilon, \phi - \frac{\partial f}{\partial \theta} \theta, \phi \right] \right) + \\
\varepsilon \left( \sin(\phi + \varepsilon) \frac{\partial f}{\partial \phi} (\theta, \phi + \varepsilon) - \sin(\phi) \frac{\partial f}{\partial \phi} (\theta, \phi) \right)
\]
The Spherical Laplacian

Consider the “square” on the sphere with endpoints \((\theta, \phi), (\theta + \varepsilon, \phi), (\theta + \varepsilon, \phi + \varepsilon)\) and \((\theta, \phi + \varepsilon)\):

Summing these together, we can approximate the boundary integral by:

\[
\int_{\partial R} \nabla f \cdot dA \approx \varepsilon \left( \frac{1}{\sin \phi} \varepsilon \frac{\partial}{\partial \theta} \left[ \frac{\partial f}{\partial \theta} \phi, \phi \right] \right) + \varepsilon \left( \varepsilon \frac{\partial}{\partial \phi} \left[ \sin(\phi) \frac{\partial f}{\partial \phi}(\theta, \phi) \right] \right)
\]
The Spherical Laplacian

Consider the “square” on the sphere with end-points \((\theta, \phi), (\theta + \varepsilon, \phi), (\theta + \varepsilon, \phi + \varepsilon)\) and \((\theta, \phi + \varepsilon)\):

Summing these together, we can approximate the boundary integral by:

\[
\int_{\partial R} \nabla f \cdot dA \approx \varepsilon^2 \left( \frac{1}{\sin \phi} \frac{\partial^2 f}{\partial \theta^2} \phi, \phi + \frac{\partial}{\partial \phi} \left[ \sin \phi \frac{\partial f}{\partial \phi}(\theta, \phi) \right] \right)
\]
The Spherical Laplacian

Using the fact that the boundary integral can be approximated by:

$$\int_{\partial R} \nabla f \cdot dA \approx \varepsilon^2 \left( \frac{1}{\sin \phi} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial}{\partial \phi} \right) \left[ \sin \phi \frac{\partial f}{\partial \phi} (\theta, \phi) \right]$$

and the surface integral can be approximated by:

$$\int_R \nabla^2 f \cdot dR \approx \varepsilon^2 \sin \phi \nabla^2 f (\theta, \phi)$$
The Spherical Laplacian

Using the fact that the boundary integral can be approximated by:

$$\int_{\partial R} \nabla f \cdot dA \approx \varepsilon^2 \left( \frac{1}{\sin \phi} \frac{\partial^2 f}{\partial \theta^2} \phi, \phi + \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial f}{\partial \phi} (\theta, \phi) \right) \right)$$

and the surface integral can be approximated by:

$$\int_{R} \nabla^2 f \cdot dR \approx \varepsilon^2 \sin \phi \nabla^2 f (\theta, \phi)$$

we can apply Stokes’ Theorem to get:

$$\nabla^2 f (\theta, \phi) = \frac{1}{\sin^2 \phi} \frac{\partial^2 f}{\partial \theta^2} \phi, \phi + \frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial f}{\partial \phi} (\theta, \phi) \right)$$
The Spherical Laplacian

\[ \nabla^2 f(\theta, \phi) = \frac{1}{\sin^2 \phi} \frac{\partial^2 f}{\partial \theta^2} \theta, \phi \hat{\theta} + \frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left[ \sin \phi \frac{\partial f}{\partial \phi} (\theta, \phi) \right] \]

In order to compute the eigenvalues of the Laplacian associated to the spherical harmonics, we need to compute:

\[ \nabla^2 Y_l^k (\theta, \phi) = \nabla^2 \left( \left( \Re^{ik\theta} P_l^k \right) \cos \phi \right) \]
The Spherical Laplacian

\[ \nabla^2 f(\theta, \phi) = \frac{1}{\sin^2 \phi} \frac{\partial^2 f(\theta, \phi)}{\partial \theta^2} + \frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left[ \sin \phi \frac{\partial f(\theta, \phi)}{\partial \phi} \right] \]

Taking the derivative with respect to \( \theta \) is easy:

\[ \frac{1}{\sin^2 \phi} \frac{\partial^2 Y_l^k(\theta, \phi)}{\partial \theta^2} = \frac{1}{\sin^2 \phi} \frac{\partial^2}{\partial \theta^2} (i^{ik\theta} P_l^k \cos \phi) \]
The Spherical Laplacian

\[ \nabla^2 f(\theta, \phi) = \frac{1}{\sin^2 \phi} \frac{\partial^2 f}{\partial \theta^2} \Theta, \phi \vec{\phi} + \frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left[ \sin \phi \frac{\partial f}{\partial \phi}(\theta, \phi) \right] \]

Taking the derivative with respect to \( \theta \) is easy:

\[
\frac{1}{\sin^2 \phi} \frac{\partial^2 Y_l^k(\theta, \phi)}{\partial \theta^2} = \frac{1}{\sin^2 \phi} \frac{\partial^2}{\partial \theta^2} e^{ik\theta} P_l^k(\cos \phi) = \frac{-k^2}{\sin^2 \phi} e^{ik\theta} P_l^k(\cos \phi) \]
The Spherical Laplacian

\[ \nabla^2 f(\theta, \phi) = \frac{1}{\sin^2 \phi} \frac{\partial^2 f}{\partial \theta^2} \phi, \phi + \frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left[ \sin \phi \frac{\partial f}{\partial \phi} (\theta, \phi) \right] \]

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\[ \frac{1}{\sin^2 \phi} \frac{\partial^2 Y_l^k (\theta, \phi)}{\partial \theta^2} = \frac{1}{\sin^2 \phi} \frac{\partial^2}{\partial \theta^2} \left( e^{ik\theta} P_l^k \cos \phi \right) \]

\[ = \frac{-k^2}{\sin^2 \phi} e^{ik\theta} P_l^k \cos \phi \]

\[ = \frac{-k^2}{\sin^2 \phi} Y_l^k (\theta, \phi) \]
The Spherical Laplacian

\[
\nabla^2 f (\theta, \phi) = \frac{1}{\sin^2 \phi} \frac{\partial^2 f}{\partial \theta^2} + \frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left[ \sin \phi \frac{\partial f}{\partial \phi} (\theta, \phi) \right]
\]

Taking the derivative with respect to \( \phi \) is more complicated:

\[
\frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial Y_l^k (\theta, \phi)}{\partial \phi} \right) = \frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial}{\partial \phi} \left[ i^k \theta P_l^k \cos \phi \right] \right)
\]
The Spherical Laplacian

\[
\nabla^2 f(\theta, \phi) = \frac{1}{\sin^2 \phi} \frac{\partial^2 f}{\partial \theta^2} \cdot \hat{\theta}, \hat{\phi} + \frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left[ \sin \phi \frac{\partial f}{\partial \phi} (\theta, \phi) \right]
\]

Taking the derivative with respect to \( \phi \) is more complicated:

\[
\frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial Y_l^k (\theta, \phi)}{\partial \phi} \right) = \frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial}{\partial \phi} \left[ e^{i k \theta} P_l^k \cos \phi \right] \right)
\]

\[
= \frac{e^{i k \theta}}{\sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial P_l^k \cos \phi}{\partial \phi} \right)
\]

as it requires taking the derivatives of the associated Legendre polynomials.
Recall:

The associated Legendre polynomials are defined by the (somewhat hairy) formula:

\[ P^k_l(x) = \frac{(-1)^k}{2^l l!} \left( x^2 - 1 \right)^{\frac{l}{2}} \frac{d^{l+k}}{dx^{l+k}} \left( x^2 - 1 \right) \]
One can show, (but we won’t) that the associated Legendre polynomials satisfy the following two identities:

\[
\frac{dP^k_l \cos \phi}{d\phi} \cos \phi = l \cos(\phi) P^k_l \cos \phi - (l + k) P^k_{l-1} \cos \phi
\]

\[
0 = (l - k) P^k_l \cos \phi - \cos(2l - 1) P^k_{l-1} \cos \phi + (l + k - 1) P^k_{l-2} \cos \phi
\]
The Spherical Laplacian

Plugging these identities into the equation for the Laplacian, we get:

\[
\frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial Y_l^k (\theta, \phi)}{\partial \phi} \right) = \frac{k^2 Y_l^m (\theta, \phi)}{\sin^2 \phi} - l(l + 1)Y_l^m (\theta, \phi)
\]
The Spherical Laplacian

Plugging these identities into the equation for the Laplacian, we get:

\[
\frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial Y_l^k(\theta, \phi)}{\partial \phi} \right) = \frac{-k^2 Y_l^m(\theta, \phi)}{\sin^2 \phi} - l(l+1)Y_l^m(\theta, \phi)
\]

In sum, this gives:

\[
\nabla^2 Y_l^k(\theta, \phi) = -l(l+1)Y_l^k(\theta, \phi)
\]
Outline

• Stokes’ Theorem
• Tangent Spaces
• Gradients
• The Spherical Laplacian
• Applications
Smoothing

In the case of a functions on a plane, we had Newton’s Law of Cooling:

“The rate of heat loss of a body is proportional to the difference in temperatures between the body and its surroundings.”
Smoothing

In the case of a functions on a plane, we had Newton’s Law of Cooling:

“The rate of heat loss of a body is proportional to the difference in temperatures between the body and its surroundings.”

This can be expresses as a PDE:

$$\frac{\partial F}{\partial t} = \lambda \nabla^2 F$$
Smoothing

Using the fact that the spherical harmonics are eigenvectors of the Laplacian, we get solutions of the form:

\[ F_l^k (\theta, \phi, t) = e^{-\lambda (l+1)t} Y_l^k (\theta, \phi) \]
Smoothing

Using the fact that the spherical harmonics are eigenvectors of the Laplacian, we get solutions of the form:

$$F^k_l (\theta, \phi, t) = e^{-\lambda(l+1)t} Y^k_l (\theta, \phi)$$

Since the linear sum of solutions is also a solution, solutions to the PDE will have the form:

$$F(\theta, \phi, t) = \sum_{l=0}^{\infty} \sum_{k=-l}^{l} a^k_l e^{-\lambda(l+1)t} Y^k_l (\theta, \phi)$$

and we have freedom in choosing the linear coefficients.
Smoothing

To satisfy the initial condition:

\[ F(\theta, \phi, 0) = f(\theta, \phi) \]

we need to compute the spherical harmonic decomposition of \( f \):

\[
f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{k=-l}^{l} \hat{f}(l, k) Y_l^k(\theta, \phi)
\]
Smoothing

To satisfy the initial condition:

\[ F(\theta, \phi, 0) = f(\theta, \phi) \]

we need to compute the spherical harmonic decomposition of \( f \):

\[
f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{k=-l}^{l} \hat{f}(l, k) Y_{l}^{k}(\theta, \phi)\]

and then we set the solution to be:

\[
F(\theta, \phi, t) = \sum_{l=0}^{\infty} \sum_{k=-l}^{l} \hat{f}(l, k) e^{-\lambda(l+1)t} Y_{l}^{k}(\theta, \phi)
\]
Smoothing

\[ F(\theta, \phi, t) = \sum_{l=0}^{\infty} \sum_{k=-l}^{l} \hat{f}(l, k) e^{-\lambda(l+1)t} Y_l^k (\theta, \phi) \]
The Spherical Wave Equation

We can repeat the same type of argument for the wave equation, where the acceleration is proportional to the difference in values:

\[
\frac{\partial^2 F}{\partial t^2} = \lambda \nabla^2 F
\]
The Spherical Wave Equation

Again, using the fact that the spherical harmonics $Y_l^k$ are eigenvectors of the Laplacian with eigenvalues $l(l+1)$, we get solutions of the form:

$$F_l^{k+} (\theta, \phi, t) = e^{i \sqrt{2l(l+1)} t} \cdot Y_l^m (\theta, \phi)$$

$$F_l^{k-} (\theta, \phi, t) = e^{-i \sqrt{2l(l+1)} t} \cdot Y_l^m (\theta, \phi)$$
The Spherical Wave Equation

Thus, given the initial conditions:

\[ F(\theta, \phi, 0) = f(\theta, \phi) \]
\[ \frac{\partial}{\partial t} F(\theta, \phi, 0) = 0 \]

we get the solution:

\[ F(\theta, \phi, t) = \sum_{l=0}^{\infty} \sum_{k=-l}^{l} \hat{f}(k, l) \cos \left( \sqrt{\lambda l(l+1)t} \right) Y_l^k(\theta, \phi) \]
The Spherical Wave Equation

\[ F(\theta, \phi, t) = \sum_{l=0}^{\infty} \sum_{k=-l}^{l} \hat{f}(k, l) \cos(\sqrt{\lambda l(l+1)t} \hat{Y}_l^k(\theta, \phi)) \]

Waving Cow

Waving Gaussians