FFTs in Graphics and Vision

Spherical Harmonics
and
Legendre Polynomials
Outline

Math Stuff

Gram-Schmidt Orthogonalization
Height Functions
Completing Homogenous Polynomials

Review

Defining the Harmonics
Gram–Schmidt Orthogonalization

Given an inner product space $V$, and given a basis $\{v_1, \ldots, v_n\}$ we can define an orthonormal basis $\{w_1, \ldots, w_n\}$ for $V$:

$$\left\langle w_i, w_j \right\rangle = \delta_{ij}$$
Gram–Schmidt Orthogonalization

Given an inner product space $V$, and given a basis $\{v_1, \ldots, v_n\}$ we can define an orthonormal basis $\{w_1, \ldots, w_n\}$ for $V$:

$$\langle w_i, w_j \rangle = \delta_{ij}$$

Algorithm:

Start by making $v_1$ a unit vector:

$$w_1 = \frac{v_1}{\|v_1\|}$$
Gram–Schmidt Orthogonalization

Given an inner product space $V$, and given a basis \{\textbf{v}_1, \ldots, \textbf{v}_n\} we can define an orthonormal basis \{\textbf{w}_1, \ldots, \textbf{w}_n\} for $V$:

$$\langle \textbf{w}_i, \textbf{w}_j \rangle = \delta_{ij}$$

Algorithm:

To get the 2\textsuperscript{nd} basis element, subtract off from \textbf{v}_2 the $\textbf{w}_1$ component and then normalize:

$$\textbf{w}_2 = \frac{\textbf{v}_2 - \langle \textbf{v}_2, \textbf{w}_1 \rangle \textbf{w}_1}{\| \textbf{v}_2 - \langle \textbf{v}_2, \textbf{w}_1 \rangle \textbf{w}_1 \|}$$
Gram–Schmidt Orthogonalization

Given an inner product space \( V \), and given a basis \( \{v_1, \ldots, v_n\} \) we can define an orthonormal basis \( \{w_1, \ldots, w_n\} \) for \( V \):

\[
\langle w_i, w_j \rangle = \delta_{ij}
\]

Algorithm:

To get the \( i \)-th basis element, subtract off all the earlier components from \( v_i \) and then normalize:

\[
w_i = \frac{v_i - \langle v_i, w_{i-1} \rangle w_{i-1} - \cdots - \langle v_i, w_1 \rangle w_1}{\|v_i - \langle v_i, w_{i-1} \rangle w_{i-1} - \cdots - \langle v_i, w_1 \rangle w_1\|}
\]
Gram–Schmidt Orthogonalization

Example:

Consider the space of polynomial functions of degree $N$ on the interval $[-1,1]$, with the standard inner-product:

$$\langle f(x), g(x) \rangle = \int_{-1}^{1} f(x)g(x)dx$$

We would like to obtain an orthogonal basis:

$$\{p_0(x), \ldots, p_N(x)\}$$
Example:

Consider the space of polynomial functions of degree \( N \) on the interval \([-1,1]\), with the standard inner-product:

\[
\langle f(x), g(x) \rangle = \int_{-1}^{1} f(x) g(x) \, dx
\]

We would like to obtain an orthogonal basis:

\[
\{ p_0(x), \cdots, p_N(x) \}
\]

An easy basis to start with is the monomials:

\[
\{1, x, x^2, \ldots, x^N\}
\]
Example:

Starting with the constant term, we get:

\[ p_0(x) = \frac{1}{\|1\|} \]
Example:

Starting with the constant term, we get:

\[ p_0(x) = \frac{1}{\|1\|} = \frac{1}{\sqrt{\int_{-1}^{1} dx}} \]
Example:

Starting with the constant term, we get:

\[ p_0(x) = \frac{1}{\|1\|} = \frac{1}{\sqrt{\int_{-1}^{1} dx}} = \frac{1}{\sqrt{2}} \]
Example:

Moving on to the linear term, we get:

\[ p_1(x) = \frac{x - \langle x, p_0(x) \rangle p_0(x)}{\|x - \langle x, p_0(x) \rangle p_0(x)\|} \]
Gram–Schmidt Orthogonalization

Example:

Moving on to the linear term, we get:

\[ p_1(x) = \frac{x - \langle x, p_0(x) \rangle p_0(x)}{\| x - \langle x, p_0(x) \rangle p_0(x) \|} \]

\[ = \frac{x - \left( \int_{-1}^{1} x \frac{1}{\sqrt{2}} \, dx \right) \frac{1}{\sqrt{2}}}{\| x - \langle x, p_0(x) \rangle p_0(x) \|} \]
Gram–Schmidt Orthogonalization

Example:

Moving on to the linear term, we get:

\[
p_1(x) = \frac{x - \langle x, p_0(x) \rangle p_0(x)}{\| x - \langle x, p_0(x) \rangle p_0(x) \|}
\]

\[
= \frac{x - \left( \int_{-1}^{1} x \frac{1}{\sqrt{2}} \, dx \right) \frac{1}{\sqrt{2}}}{\| x - \langle x, p_0(x) \rangle p_0(x) \|}
\]

\[
= \frac{x}{\sqrt{\int_{-1}^{1} x^2 \, dx}} = \sqrt{\frac{3}{2}} x
\]
Gram–Schmidt Orthogonalization

Example:

And the quadratic term:

\[
p_2(x) = \frac{x^2 - \langle x^2, p_1(x) \rangle p_1(x) - \langle x^2, p_0(x) \rangle p_0(x)}{\| x^2 - \langle x^2, p_1(x) \rangle p_1(x) - \langle x^2, p_0(x) \rangle p_0(x) \|}
\]
Gram–Schmidt Orthogonalization

Example:

And the quadratic term:

\[ p_2(x) = \frac{x^2 - \langle x^2, p_1(x) \rangle p_1(x) - \langle x^2, p_0(x) \rangle p_0(x)}{\left\| x^2 - \langle x^2, p_1(x) \rangle p_1(x) - \langle x^2, p_0(x) \rangle p_0(x) \right\|} \]

These polynomials are called the Legendre Polynomials.
Legendre Polynomials

Claim:

The degrees of the monomials comprising the Legendre polynomials have the same parity as $k$:

$$p_k(x) = a_k x^k + a_{k-2} x^{k-2} + ...$$
Legendre Polynomials

Claim:
The degrees of the monomials comprising the Legendre polynomials have the same parity as $k$:

$$p_k(x) = a_k x^k + a_{k-2} x^{k-2} + ...$$

Proof by Induction:
Legendre Polynomials

Claim:

The degrees of the monomials comprising the Legendre polynomials have the same parity as $k$:

$$p_k(x) = a_k x^k + a_{k-2} x^{k-2} + ...$$

Proof by Induction ($k=0$):

$$p_0(x) = \frac{1}{\sqrt{2}}$$
Legendre Polynomials

Claim:

The degrees of the monomials comprising the Legendre polynomials have the same parity as $k$:

$$p_k(x) = a_k x^k + a_{k-2} x^{k-2} + ...$$

Proof by Induction (assume true for $k=n$):

$$p_{n+1}(x) = \frac{x^{n+1} - \langle x^{n+1}, p_n(x) \rangle p_n(x) - \cdots - \langle x^{n+1}, p_0(x) \rangle p_0(x)}{\left\| x^{n+1} - \langle x^{n+1}, p_n(x) \rangle p_n(x) - \cdots - \langle x^{n+1}, p_0(x) \rangle p_0(x) \right\|}$$
Legendre Polynomials

Claim:

The degrees of the monomials comprising the Legendre polynomials have the same parity as $k$:

$$p_k(x) = a_k x^k + a_{k-2} x^{k-2} + ...$$

Proof by Induction (assume true for $k=n$):

$$p_{n+1}(x) = x^{n+1} - \left< x^{n+1}, p_n(x) \right> p_n(x) - \cdots - \left< x^{n+1}, p_0(x) \right> p_0(x)$$

Recall that:

$$\left< x^{n+1}, p_m(x) \right> = \int_{-1}^{1} x^{n+1} p_m(x) \, dx$$
Legendre Polynomials

Claim:

The degrees of the monomials comprising the Legendre polynomials have the same parity as $k$:

$$p_k(x) = a_k x^k + a_{k-2} x^{k-2} + ...$$

Proof by Induction:

$$\left\langle x^{n+1}, p_m(x) \right\rangle = \int_{-1}^{1} x^{n+1} p_m(x) \, dx$$

Since $m \leq n$ we can assume that the monomials comprising $p_m(x)$ are all even if $m$ is even and all odd if $m$ is odd.
Legendre Polynomials

Claim:

The degrees of the monomials comprising the Legendre polynomials have the same parity as $k$:

$$p_k(x) = a_k x^k + a_{k-2} x^{k-2} + ...$$

Proof by Induction:

So if $n$ and $m$ are both either odd or both even, the polynomial $x^{n+1} p_m(x)$ is comprised of strictly odd-powered monomials:

$$\left\langle x^{n+1}, p_m(x) \right\rangle = \int_{-1}^{1} x^{n+1} p_m(x) dx = 0$$
Legendre Polynomials

Claim:

The degrees of the monomials comprising the Legendre polynomials have the same parity as \( k \):

\[
p_k(x) = a_k x^k + a_{k-2} x^{k-2} + \ldots
\]

Proof by Induction:

\[
p_{n+1}(x) = \frac{x^{n+1} - \langle x^{n+1}, p_n(x) \rangle p_n(x) - \cdots - \langle x^{n+1}, p_0(x) \rangle p_0(x)}{\left\| x^{n+1} - \langle x^{n+1}, p_n(x) \rangle p_n(x) - \cdots - \langle x^{n+1}, p_0(x) \rangle p_0(x) \right\|}
\]

\[
p_{n+1}(x) = \frac{x^{n+1} - \langle x^{n+1}, p_{n-1}(x) \rangle p_{n-1}(x) - \langle x^{n+1}, p_{n-3}(x) \rangle p_{n-3}(x) - \cdots}{\left\| x^{n+1} - \langle x^{n+1}, p_{n-1}(x) \rangle p_{n-1}(x) - \langle x^{n+1}, p_{n-3}(x) \rangle p_{n-3}(x) - \cdots \right\|}
\]
Legendre Polynomials

Claim:

The degrees of the monomials comprising the Legendre polynomials have the same parity as \( k \):

\[
p_k(x) = a_k x^k + a_{k-2} x^{k-2} + \ldots
\]

Proof by Induction:

\[
p_{n+1}(x) = \frac{x^{n+1} - \langle x^{n+1}, p_{n-1}(x) \rangle p_{n-1}(x) - \langle x^{n+1}, p_{n-3}(x) \rangle p_{n-3}(x) - \cdots}{\|x^{n+1} - \langle x^{n+1}, p_{n-1}(x) \rangle p_{n-1}(x) - \langle x^{n+1}, p_{n-3}(x) \rangle p_{n-3}(x) - \cdots\|}
\]

So \( p_{n+1}(x) \) is obtained by starting with the monomial \( x^{n+1} \) and subtracting off monomials with the same parity.
Example:

Consider the space of polynomials of degree $N$ on the interval $[-1,1]$, with an inner-product giving more weight to points near the origin:

$$\langle f(x), g(x) \rangle_m = \int_{-1}^{1} (1 - x^2)^m f(x) g(x) \, dx$$
Gram–Schmidt Orthogonalization

Example:

Consider the space of polynomials of degree $N$ on the interval $[-1,1]$, with an inner-product giving more weight to points near the origin:

$$\langle f(x), g(x) \rangle_m = \int_{-1}^{1} (1 - x^2)^m f(x) g(x) \, dx$$

We would like to obtain an orthogonal basis:

$$\{ p_0^m(x), \ldots, p_N^m(x) \}$$
Gram–Schmidt Orthogonalization

Example:

\[ \langle f(x), g(x) \rangle_m = \int_{-1}^{1} \left(1 - x^2\right)^m f(x) g(x) \, dx \]

We proceed exactly as before but now using the new inner-product.
Gram–Schmidt Orthogonalization

Example:

$$\langle f(x), g(x) \rangle_m = \int_{-1}^{1} (1-x^2)^m f(x) g(x) \, dx$$

We proceed exactly as before but now using the new inner-product.

Since the weighting function is even, if $f$ is an even function and $g$ is an odd function (or vice-versa), the inner product must be zero:

$$\langle f(x), g(x) \rangle_m = 0$$
Gram–Schmidt Orthogonalization

Example:

\[ \langle f(x), g(x) \rangle_m = \int_{-1}^{1} (1 - x^2)^m f(x) g(x) \, dx \]

We proceed exactly as before but now using the new inner-product.

Since the weighting function is even, if \( f \) is an even function and \( g \) is an odd function (or vice-versa), the inner product must be zero:

\[ \langle f(x), g(x) \rangle_m = 0 \]

Thus, as before, the degree of the monomials comprising \( p_i^m(x) \) must all have the same parity.
Height Functions

Given a function \( f(x) \) on the interval \([-1,1]\), we can turn the function into a function on the sphere:

\[
F(x, y, z) = \frac{f(y)}{\sqrt{2\pi}}
\]
Height Functions

Given a function $f(x)$ on the interval $[-1,1]$, we can turn the function into a function on the sphere:

$$F(x, y, z) = \frac{f(y)}{\sqrt{2\pi}}$$

Given two such functions $f(x)$ and $g(y)$, the mapping from 1D functions to spherical functions preserves inner products:

$$\int_{|p|=1} F(p) \cdot G(p) dp = \int_{-1}^{1} f(y) \cdot g(y) dy$$
Height Functions

To see this, we can parameterize points on the sphere by spherical angle:

$$\Phi(\theta, \phi) = (\cos \theta \sin \phi, \cos \phi, \sin \theta \sin \phi)$$

Then the integral on the left-hand side becomes:

$$\int_{|p|=1} F(p) \cdot G(p) dp = \int_{0}^{\pi} \int_{0}^{2\pi} F(\Phi(\theta, \phi)) \cdot G(\Phi(\theta, \phi)) d\theta \sin(\phi) d\phi$$
Height Functions

To see this, we can parameterize points on the sphere by spherical angle:

$$
\Phi(\theta, \phi) = (\cos \theta \sin \phi, \cos \phi, \sin \theta \sin \phi)
$$

Then the integral on the left-hand side becomes:

$$
\int_{|p|=1} F(p) \cdot G(p) dp = \int_{0}^{\pi} \int_{0}^{2\pi} F(\Phi(\theta, \phi)) \cdot G(\Phi(\theta, \phi)) d\theta \sin(\phi) d\phi
$$

$$
= \frac{1}{2\pi} \int_{0}^{\pi} \int_{0}^{2\pi} f(\cos \phi) \cdot g(\cos \phi) d\theta \sin(\phi) d\phi
$$
Height Functions

To see this, we can parameterize points on the sphere by spherical angle:

\[ \Phi(\theta, \phi) = (\cos \theta \sin \phi, \cos \phi, \sin \theta \sin \phi) \]

Then the integral on the left-hand side becomes:

\[ \int_{|p|=1} F(p) \cdot G(p) dp = \int_0^\pi \int_0^{2\pi} F(\Phi(\theta, \phi)) \cdot G(\Phi(\theta, \phi)) d\theta \sin(\phi) d\phi \]

\[ = \frac{1}{2\pi} \int_0^{2\pi} \int_0^\pi f(\cos \phi) \cdot g(\cos \phi) d\theta \sin(\phi) d\phi \]

\[ = \int_0^\pi f(\cos \phi) \cdot g(\cos \phi) \sin(\phi) d\phi \]
Height Functions

On the other hand, setting $y = \cos \phi$, the left-hand side can be expressed as:

$$
\int_{-1}^{1} f(y) \cdot g(y) dy = \int_{0}^{\pi} f(\cos \phi) \cdot g(\cos \phi) \frac{dy}{d\phi} d\phi
$$
Height Functions

On the other hand, setting \( y = \cos \phi \), the left-hand side can be expressed as:

\[
\int_{-1}^{1} f(y) \cdot g(y) dy = \int_{0}^{\pi} f(\cos \phi) \cdot g(\cos \phi) \frac{dy}{d\phi} \, d\phi
\]

\[
= \int_{\pi}^{\pi} f(\cos \phi) \cdot g(\cos \phi)(-\sin(\phi)) \, d\phi
\]
Height Functions

On the other hand, setting \( y = \cos \phi \), the left-hand side can be expressed as:

\[
\int_{-1}^{1} f(y) \cdot g(y) dy = \int_{0}^{\pi} f(\cos \phi) \cdot g(\cos \phi) \frac{dy}{d\phi} d\phi
\]

\[
= \int_{0}^{\pi} f(\cos \phi) \cdot g(\cos \phi)(-\sin(\phi)) d\phi
\]

\[
= \int_{0}^{\pi} f(\cos \phi) \cdot g(\cos \phi) \sin(\phi) d\phi
\]
Height Functions

On the other hand, setting \( y = \cos \phi \), the left-hand side can be expressed as:

\[
\int_{-1}^{1} f(y) \cdot g(y) dy = \int_{0}^{\pi} f(\cos \phi) \cdot g(\cos \phi) \frac{dy}{d\phi} d\phi \\
= \int_{0}^{\pi} f(\cos \phi) \cdot g(\cos \phi)(-\sin(\phi)) d\phi \\
= \int_{0}^{\pi} f(\cos \phi) \cdot g(\cos \phi) \sin(\phi) d\phi \\
= \int_{|p|=1} F(p) \cdot G(p) dp
\]
Completing Homogenous Polynomials

Given a polynomial $p(x, y, z)$ of degree $d$, consisting of monomials of powers $d, d-2, d, \ldots$:

$$p(x, y, z) = \sum_{k=0}^{\left\lfloor d/2 \right\rfloor} \left( \sum_{l+m+n=d-2k} a_{l,m,n} x^l y^m z^n \right)$$
Completing Homogenous Polynomials

Given a polynomial $p(x,y,z)$ of degree $d$, consisting of monomials of powers $d,d-2,d,…$:

$$p(x, y, z) = \sum_{k=0}^{\left\lfloor d/2 \right\rfloor} \left( \sum_{l+m+n=d-2k} a_{lmn} x^l y^m z^n \right)$$

This is not a homogenous polynomial.
Completing Homogenous Polynomials

Given a polynomial $p(x,y,z)$ of degree $d$, consisting of monomials of powers $d,d-2,d,\ldots$:

$$p(x, y, z) = \sum_{k=0}^{\lfloor d/2 \rfloor} \left( \sum_{l+m+n=d-2k} a_{lmn} x^l y^m z^n \right)$$

This is not a homogenous polynomial.

However, if we restrict it to the sphere, we can think of it as homogenous:

$$p(x, y, z) \to \sum_{k=0}^{\lfloor d/2 \rfloor} \left( x^2 + y^2 + z^2 \right)^k \left( \sum_{l+m+n=d-2k} a_{lmn} x^l y^m z^n \right)$$
Completing Homogenous Polynomials

Example:

\[ p(x, y, z) = x^2 y + y + z \]

Is not a homogenous polynomial.
Completing Homogenous Polynomials

Example:

\[ p(x, y, z) = x^2 y + y + z \]

Is not a homogenous polynomial.

But on the unit-sphere, the polynomial:

\[ q(x, y, z) = x^2 y + (y + z)(x^2 + y^2 + z^2) \]

has identical values and is homogenous of degree 3.
Outline

Math Stuff

Review
  Spherical Harmonics

Defining the Harmonics
Spherical Harmonics

For each non-negative integer $l$, there are $2l+1$ spherical harmonics of degree $l$ satisfying:

1. Each spherical harmonic of degree $l$ can be expressed as the restriction of a homogenous polynomial of degree $l$ to the unit-sphere.

2. The different spherical harmonics are orthogonal to each other.
Spherical Harmonics

We had seen that by considering just the rotations about the $y$-axis, we could factor the spherical harmonics as:

$$Y_l^m(\theta, \phi) = e^{im\theta} P_l^m(\phi)$$

$$= (\cos \theta + i \sin \theta)^m P_l^m(\phi)$$

where $|m| \leq l$. 
Outline

Math Stuff

Review

Defining the Harmonics
Defining the Harmonics \((m \geq 0)\)

To define the spherical harmonics, we would like to express the function:

\[
Y_{lm}(\theta, \phi) = (\cos \theta + i \sin \theta)^m P_l^m(\phi)
\]

as the restriction of a homogenous polynomial of degree \(l\) to the unit sphere.
Defining the Harmonics \((m \geq 0)\)

Using the parameterization of the unit-sphere

\[ \Phi(\theta, \phi) = (\cos \theta \sin \phi, \cos \phi, \sin \theta \sin \phi) \]

we get:

\[ Y_l^m(\theta, \phi) = \left( \frac{x}{\sin \phi} + i \frac{z}{\sin \phi} \right)^m P_l^m(\phi) \]
Defining the Harmonics \((m \geq 0)\)

Using the parameterization of the unit-sphere

\[\Phi(\theta, \phi) = (\cos \theta \sin \phi, \cos \phi, \sin \theta \sin \phi)\]

we get:

\[Y_l^m(\theta, \phi) = \left( \frac{x}{\sin \phi} + i \frac{z}{\sin \phi} \right)^m P_l^m(\phi)\]

\[= (x + iz)^m \frac{P_l^m(\phi)}{\sin^m \phi}\]
Defining the Harmonics \((m \geq 0)\)

Using the parameterization of the unit-sphere

\[
\Phi(\theta, \phi) = (\cos \theta \sin \phi, \cos \phi, \sin \theta \sin \phi)
\]

we get:

\[
Y_l^m(\theta, \phi) = \left( \frac{x}{\sin \phi} + i \frac{z}{\sin \phi} \right)^m P_l^m(\phi)
\]

\[
= (x + iz)^m \frac{P_l^m(\phi)}{\sin^m \phi}
\]

\[
= (x + iz)^m \frac{P_l^m(\cos^{-1} y)}{(\sqrt{1 - y^2})^m}
\]
Defining the Harmonics ($m \geq 0$)

\[
Y_l^m(\theta, \phi) = (x + iz)^m \frac{P_l^m(\cos^{-1} y)}{(\sqrt{1 - y^2})^m}
\]

This is a homogenous polynomial of degree $m$. 
Defining the Harmonics \((m \geq 0)\)

\[
Y_l^m(\theta, \phi) = \frac{P_l^m(\cos^{-1} y)}{(\sqrt{1 - y^2})^m}(x + iz)^m
\]

This is a homogenous polynomial of degree \(m\).

So we want:

1. This to complete to a homogenous polynomial of degree \(l-m\).
Defining the Harmonics \((m \geq 0)\)

\[
Y_l^m(\theta, \phi) = \frac{(x + iz)^m}{\left(\sqrt{1 - y^2}\right)^m} P_l^m(\cos^{-1} y)
\]

This is a homogenous polynomial of degree \(m\).

So we want:

1. This to complete to a **homogenous polynomial of degree** \(l-m\).

2. The different \(Y_l^m\) to be **orthogonal**.
Defining the Harmonics ($m \geq 0$)

$$Y_l^m(\theta, \phi) = (x + iz)^m \frac{P_l^m(\cos^{-1}y)}{\left(\sqrt{1 - y^2}\right)^m}$$

Homogeneity:

To satisfy the homogeneity constraint, we need:

$$\frac{P_l^m(\cos^{-1}y)}{\left(\sqrt{1 - y^2}\right)^m} = a_{l-m}y^{l-m} + a_{l-m-2}y^{l-m-2} + ...$$
Defining the Harmonics ($m \geq 0$)

$$Y_l^m(\theta, \phi) = (x + i z)^m \frac{P_l^m(\cos^{-1} y)}{(\sqrt{1 - y^2})^m}$$

**Homogeneity:**

To satisfy the homogeneity constraint, we need:

$$\frac{P_l^m(\cos^{-1} y)}{(\sqrt{1 - y^2})^m} = a_{l-m} y^{l-m} + a_{l-m-2} y^{l-m-2} + \ldots$$

Or equivalently:

$$P_l^m(\cos^{-1} y) = q_l^m(y) \left(\sqrt{1 - y^2}\right)^m$$

for a polynomial:

$$q_l^m(y) = a_{l-m} y^{l-m} + a_{l-m-2} y^{l-m-2} + \ldots$$
Defining the Harmonics \((m \geq 0)\)

\[
Y_l^m(\theta, \phi) = (x + iz)^m \frac{P_l^m(\cos^{-1} y)}{\left(\sqrt{1 - y^2}\right)^m}
\]

Orthogonality:

To satisfy the orthogonality constraint, we need:

\[
\langle Y_l^m(\theta, \phi), Y_{l'}^{m'}(\theta, \phi) \rangle = 0 \quad \forall l \neq l' \text{ or } m \neq m'
\]
Defining the Harmonics \((m \geq 0)\)

\[
Y_l^m(\theta, \phi) = (x + iz)^m P_l^m(\cos^{-1} y) \left(\sqrt{1 - y^2}\right)^m
\]

Orthogonality \((m \neq m')\):

Since we have separation of variables:

\[
Y_l^m(\theta, \phi) = e^{im\theta} P_l^m(\phi)
\]

we know that:

\[
\langle Y_l^m(\theta, \phi), Y_{l'}^{m'}(\theta, \phi) \rangle = \int_0^{2\pi} \int_0^{\pi} e^{im\theta} P_l^m(\phi)e^{im'\theta} P_{l'}^{m'}(\phi)d\theta \sin \phi d\phi
\]
Defining the Harmonics \((m \geq 0)\)

\[
Y_l^m(\theta, \phi) = (x + i z)^m \frac{P_l^m(\cos^{-1} y)}{\left(\sqrt{1 - y^2}\right)^m}
\]

Orthogonality \((m \neq m')\):

Since we have separation of variables:

\[
Y_l^m(\theta, \phi) = e^{im\theta} P_l^m(\phi)
\]

we know that:

\[
\langle Y_l^m(\theta, \phi), Y_{l'}^{m'}(\theta, \phi) \rangle = \int_0^{2\pi} \int_0^{2\pi} e^{im\theta} P_l^m(\phi) e^{im'\theta} P_{l'}^{m'}(\phi) d\theta \sin \phi d\phi
\]

\[
= \left(\int_0^\pi P_l^m(\phi) P_{l'}^{m'}(\phi) \sin \phi d\phi\right) \left(\int_0^{2\pi} e^{i(m-m')\theta} d\theta\right)
\]
Defining the Harmonics (m ≥0)

\[ Y_l^m(\theta, \phi) = (x + iz)^m \frac{P_l^m(\cos^{-1} y)}{(\sqrt{1 - y^2})^m} \]

Orthogonality (m≠m'):

\[ \left\langle Y_l^m(\theta, \phi), Y_l^{m'}(\theta, \phi) \right\rangle = \left( \int_0^\pi P_l^m(\phi) P_l^{m'}(\phi) \sin \phi d\phi \right) \left( \int_0^{2\pi} e^{i(m-m')\theta} d\theta \right) \]

But this is zero whenever m≠m':

\[ \int_0^{2\pi} e^{i(m-m')\theta} d\theta = \left. \frac{1}{i(m-m')} e^{i(m-m')\theta} \right|_0^{2\pi} = 0 \]
Defining the Harmonics ($m \geq 0$)

$$Y_l^m(\theta, \phi) = (x + iz)^m \frac{P_l^m(\cos^{-1} y)}{\left(\sqrt{1-y^2}\right)^m}$$

Orthogonality ($m=m'$ and $l \neq l'$):

We have to choose the function:

$$P_l^m(\cos^{-1} y) = q_l^m(y)\left(\sqrt{1-y^2}\right)^m$$

so that:

$$\left\langle Y_l^m(\theta, \phi), Y_{l'}^{m'}(\theta, \phi) \right\rangle = \int_0^\pi \int_0^{2\pi} e^{im\theta} P_l^m(\phi) e^{im'\theta} P_{l'}^{m'}(\phi) d\theta \sin \phi d\phi = 0$$
Defining the Harmonics \((m \geq 0)\)

\[ Y_l^m(\theta, \phi) = (x + iz)^m P_l^m(\cos^{-1} y) \left(\sqrt{1 - y^2}\right)^m \]

Orthogonality \((m=m' \text{ and } l\neq l')\):

We have to choose the function:

\[ P_l^m(\cos^{-1} y) = q_l^m(y)\left(\sqrt{1 - y^2}\right)^m \]

so that:

\[
\langle Y_l^m(\theta, \phi), Y_{l'}^m(\theta, \phi) \rangle = \int_0^\pi \int_0^{2\pi} e^{im\theta} P_l^m(\phi) e^{im'\theta} P_{l'}^{m'}(\phi) d\theta \sin \phi d\phi = 0
\]

Since \(m=m'\), this reduces to:

\[
\int_0^\pi P_l^m(\phi) P_{l'}^{m}(\phi) \sin \phi d\phi = 0
\]
Defining the Harmonics ($m \geq 0$)

\[
Y_l^m(\theta, \phi) = (x + iz)^m \frac{P_l^m(\cos^{-1} y)}{\left(\sqrt{1 - y^2}\right)^m}
\]

Orthogonality ($m=m'$ and $l \neq l'$):

We have to choose the function:

\[
P_l^m(\cos^{-1} y) = q_l^m(y)\left(\sqrt{1 - y^2}\right)^m
\]

Using the change of variable formulation we get:

\[
\int_0^\pi P_l^m(\phi)P_{l'}^m(\phi) \sin \phi d\phi = \int_{-1}^{1} P_l^m(\cos^{-1} y)P_{l'}^m(\cos^{-1} y)dy
\]
Defining the Harmonics \( (m \geq 0) \)

\[
Y_{l}^{m}(\theta, \phi) = (x + iz)^{m} \frac{P_{l}^{m}(\cos^{-1} y)}{(\sqrt{1 - y^2})^{m}}
\]

Orthogonality \( (m=m' \text{ and } l \neq l') \):

We have to choose the function:

\[
P_{l}^{m}(\cos^{-1} y) = q_{l}^{m}(y)(\sqrt{1 - y^2})^{m}
\]

Using the change of variable formulation we get:

\[
\int_{0}^{\pi} P_{l}^{m}(\phi) P_{l'}^{m}(\phi) \sin \phi d\phi = \int_{-1}^{1} P_{l}^{m}(\cos^{-1} y) P_{l'}^{m}(\cos^{-1} y) dy
\]

\[
= \int_{-1}^{1} q_{l}^{m}(y) q_{l'}^{m}(y)(1 - y^2)^{m} dy
\]
Defining the Harmonics \((m \geq 0)\)

\[
Y_l^m(\theta, \phi) = (x + iz)^m \frac{P_l^m(\cos^{-1} y)}{\left(\sqrt{1 - y^2}\right)^m}
\]

\[
P_l^m(\cos^{-1} y) = p_l^m(y)\left(\sqrt{1 - y^2}\right)^m
\]

Thus, we require:

1. The polynomials \(q_l^m(y)\) to complete to homogenous polynomials of degree \(l-m\):

\[
q_l^m(y) = a_{l-m}y^{l-m} + a_{l-m-2}y^{l-m-2} + \ldots
\]
Defining the Harmonics ($m \geq 0$)

$$Y_l^m(\theta, \phi) = (x + iz)^m \frac{P_l^m(\cos^{-1} y)}{(\sqrt{1 - y^2})^m}$$

$$P_l^m(\cos^{-1} y) = p_l^m(y)(\sqrt{1 - y^2})^m$$

Thus, we require:

1. The polynomials $q_l^m(y)$ to complete to homogenous polynomials of degree $l-m$:
   $$q_l^m(y) = a_{l-m} y^{l-m} + a_{l-m-2} y^{l-m-2} + ...$$

2. And they satisfy the orthogonality condition:
   $$\int_{-1}^{1} q_l^m(y) q_{l'}^m(y) (1 - y^2)^m \, dy = 0$$
Defining the Harmonics \((m \geq 0)\)

\[
Y_l^m(\theta, \phi) = (x + iz)^m \frac{P_l^m(\cos^{-1} y)}{(\sqrt{1 - y^2})^m}
\]

\[
P_l^m(\cos^{-1} y) = p_l^m(y)(\sqrt{1 - y^2})^m
\]

This is precisely what we get when we compute the G.S. orthogonalization of \(\{1, y, y^2, \ldots\}\) relative to the inner-product:

\[
\langle f(y), g(y) \rangle_m = \int_{-1}^{1} f(y)g(y)(1 - y^2)^m \, dy
\]

and set:

\[
q_l^m(y) = p_{l-m}^m(y)
\]
Defining the Harmonics \((m \geq 0)\)

\[
Y^m_l(\theta, \phi) = e^{im\theta} P^m_l(\phi)
\]

\[
P^m_l(\cos^{-1} y) = p^m_l(y)\left(\sqrt{1 - y^2}\right)^m
\]

In sum, we get an expression for the spherical harmonics as:

\[
Y^m_l(\theta, \phi) = e^{im\theta} p^m_{l-m}(\cos \phi)\left(\sqrt{1 - \cos^2 \phi}\right)^m
\]
Defining the Harmonics \((m \geq 0)\)

\[
Y_l^m(\theta, \phi) = e^{im\theta} P_l^m(\phi)
\]

\[
P_l^m(\cos^{-1} y) = p_l^m(y) \left(\sqrt{1 - y^2}\right)^m
\]

In sum, we get an expression for the spherical harmonics as:

\[
Y_l^m(\theta, \phi) = e^{im\theta} p_{l-m}^m(\cos \phi) \left(\sqrt{1 - \cos^2 \phi}\right)^m
\]

\[
= e^{im\theta} p_{l-m}^m(\cos \phi) \sin^m \phi
\]

where \(p_{l-m}^m(y)\) is a polynomial of degree \(l-m\).
The Spherical Harmonics

$$Y_l^m(\theta, \phi) = e^{im\theta} p_{l-m}^m(\cos \phi) \sin^m \phi$$

Examples ($l=0$):

$$Y_0^0(\theta, \phi) = \frac{1}{\sqrt{4\pi}}$$
The Spherical Harmonics

\[ Y_l^m(\theta, \phi) = e^{im\theta} p_{l-m}^m(\cos \phi) \sin^m \phi \]

Examples (l=1):

\[ Y_1^{-1}(\theta, \phi) = \sqrt{\frac{3}{8\pi}} \sin(\phi)e^{-i\theta} \]

\[ Y_1^0(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos(\phi) \]

\[ Y_1^1(\theta, \phi) = -\sqrt{\frac{3}{8\pi}} \sin(\phi)e^{i\theta} \]
The Spherical Harmonics

\[ Y_l^m(\theta, \phi) = e^{im\theta} p_{l-m}^m(\cos \phi) \sin^m \phi \]

Examples (l=2):

\[ Y_{2}^{-2}(\theta, \phi) = \sqrt{\frac{15}{32\pi}} \sin^2(\phi) e^{-2i\theta} \]

\[ Y_{2}^{-1}(\theta, \phi) = \sqrt{\frac{15}{8\pi}} \sin(\phi) \cos(\phi) e^{-i\theta} \]

\[ Y_{2}^{0}(\theta, \phi) = \sqrt{\frac{5}{16\pi}} (3\cos^2(\phi) - 1) \]

\[ Y_{2}^{1}(\theta, \phi) = -\sqrt{\frac{15}{8\pi}} \sin(\phi) \cos(\phi) e^{i\theta} \]

\[ Y_{2}^{2}(\theta, \phi) = \sqrt{\frac{15}{32\pi}} \sin^2(\phi) e^{2i\theta} \]