FFTs in Graphics and Vision

Groups and Representations

Outline

Groups

Representations

Schur’s Lemma

Correlation

Groups

A group is a set of elements G with a binary operation (often denoted "·") such that for all $f,g,h \in G$, the following properties are satisfied:

Groups

A <u>group</u> is a set of elements G with a binary operation (often denoted "·") such that for all $f,g,h\in G$, the following properties are satisfied:

- <u>Closure</u>:

$$g\cdot h\in G$$

- <u>Associativity</u>:

$$f\cdot(g\cdot h)=(f\cdot g)\cdot h$$

- <u>Identity</u>: There exists an identity element $1\in G$ s.t.:

$$1\cdot g=g\cdot 1=g$$

- <u>Inverse</u>: Every element g has an inverse g^{-1} s.t.:

$$g\cdot g^{-1}=g^{-1}\cdot g=1$$

If it is also true that $f\cdot g=g\cdot f$ for all $f,g\in G$, the group is called <u>commutative</u>, or <u>abelian</u>.

Groups

<u>Examples</u>

Under what binary operations are the following groups, what is the identity element, and what is the inverse:

- Integers?
- Positive real-numbers?
- Pairs of real numbers in the half-open region $[0,2\pi)\times[0,2\pi)$?
- Vectors in a fixed vector space?
- Invertible linear transformations of a vector space?

Groups

Examples

Are these groups commutative:

- Integers under addition?
- Positive real-numbers under multiplication?
- Pairs of real numbers in the region $[0,2\pi)$x $[0,2\pi)$ under addition modulo $(2\pi,2\pi)$?
- Vectors under addition?
- Linear transformations under composition?
- Orthogonal transformations under composition?

Representations

Often, we think of a group as a set of elements that act on some space:

E.g.:

- Invertible linear transformations act on vector spaces
- 2D rotations act on the 2D arrays
- 3D rotations act on 3D arrays

Representations

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A representation is a way of formalizing this…

Representations

A <u>representation</u> of a group G on a vector space V is a map ρ that sends every element in G to an invertible linear transformation on V, satisfying:

$$\rho(g \cdot h) = \rho(g) \cdot \rho(h)$$

for all $g, h \in G$.

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<u>Analogy</u>:

Linear maps are functions between vector spaces that preserve the vector space structure:

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For simplicity, we will often write:

$$\rho(g) = \rho_g$$

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Unitary Representations

If the vector space V has a Hermitian inner product, and the representation preserves the inner product:

$$\langle v, w \rangle = \langle \rho_g v, \rho_g w \rangle \qquad \forall g \in G, \text{and } v, w \in V$$

the representation is called <u>unitary</u>.

Unitary Representations

<u>Examples</u>

For the group G, and Hermitian vector space V, is the map ρ a representation?

Is it unitary?

Unitary Representations

<u>Examples</u>

- G is the group of invertible nxn matrices
- V is the space of (complex) n-dimensional arrays
- ρ is the map:

$$\rho_M v = Mv$$

Representation?

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Unitary Representations

Examples

- G is the group of invertible *n*x*n* matrices
- V is the space of *n*-dimensional arrays
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Representation?

Unitary?

Unitary Representations

<u>Examples</u>

- G is the group of unitary transformations on V
- V is a complex Hermitian inner product space
- ρ is the map:

$$\rho_U v = Uv$$

Representation?

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Unitary Representations

Examples

- G is the group of 2D/3D rotations
- V is the space of functions on a circle/sphere
- ρ is the map:

$$\left[\rho_R f\right](p) = f\left(R^{-1}p\right)$$

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Representation?

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Unitary Representations

Examples

- *G* Is the group of pairs of real numbers in the region [0,2π)x [0,2π)
- *V* is the space of continuous, periodic, complex-value functions in the plane
- ρ is the map:

$$\left[\rho_{(a,b)} f\right](x,y) = f\left(x-a, y-b\right)$$

Representation?

Unitary?

Big Picture

Our goal is to try to better understand how a group acts on a vector space:

- How translational shifts act on periodic functions,
- How rotations act on functions on a sphere/circle
- Etc.

To do this we would like to simplify the “action” of the group into bite-size chunks.

Big Picture

Our goal is to try to better understand how a group acts on a vector space:

- How translational shifts act on periodic functions,
- How rotations act on functions on a sphere/circle
- Etc.

To do this we would like to simplify the “action” of the group into bite-size chunks.

Unless otherwise stated we will always be assuming that our representations are unitary

Sub-Representation

Given a representation, ρ, of a group, G, on a vector space, V, if there exists a subspace $W \subset V$, such that the representation fixes W:

$$\rho_g w \in W \qquad \forall g \in G \text{ and } w \in W$$

then we say that W is a sub-representation of V.

Sub-Representation

Claim:

If *W* is a sub-representation of *V*, then the perpendicular space $W^\perp$ will also be a sub-representation of *V*.

$W^\perp$ is defined by the property that every vector in $W^\perp$ is perpendicular to every vector in *W*:

$$\langle w, w' \rangle = 0 \qquad \forall w \in W \text{ and } w' \in W^\perp$$

Sub-Representation

Claim: $W^{\perp}$ will also be a sub-representation of V.

Proof: (By contradiction)

We would like to show that the representation ρ sends $W^{\perp}$ back into itself…

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<u>Proof</u>: (By contradiction)

We would like to show that the representation ρ sends $W^{\perp}$ back into itself… Assume not.

There exists $w' \in W^{\perp}$, $w \in W$, and $g \in G$ such that:

$$\left\langle w, \rho_g w' \right\rangle \neq 0$$

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Since ρ is unitary, this implies that:

$$\left\langle \rho_{g^{-1}} w, \rho_{g^{-1}} \rho_g w' \right\rangle \neq 0$$

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$$\Updownarrow$$

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There exists $w' \in W^{\perp}$, $w \in W$, and $g \in G$ such that:

$$\langle w, \rho_g w' \rangle \neq 0$$

Since ρ is unitary, this implies that:

But this would contradict the assumption that the representation ρ maps W back into itself!

$$\langle \rho_{g^{-1}} w, w' \rangle \neq 0$$

Sub-Representation

Example:

1. Consider the group of 2D rotations, acting on vectors in 3D by rotating around the *y*-axis. What are two sub-representations?

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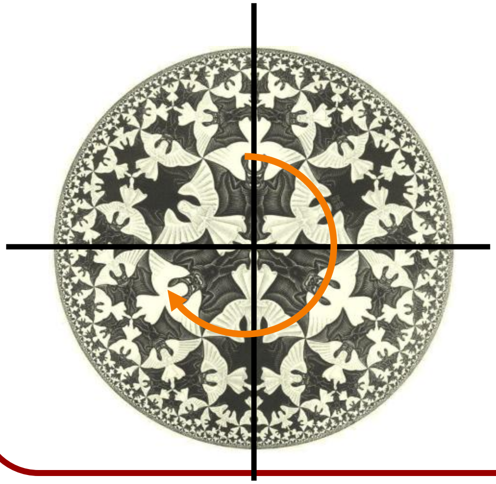
1. Consider the group of 2D rotations, acting on vectors in 3D by rotating around the *y*-axis. What are two sub-representations?
   a) The *y*-axis: The group acts on this sub-space trivially, mapping every vector to itself
   b) The *xz*-plane: The group acts as a 2D rotation on this 2D space.

Sub-Representation

Example:

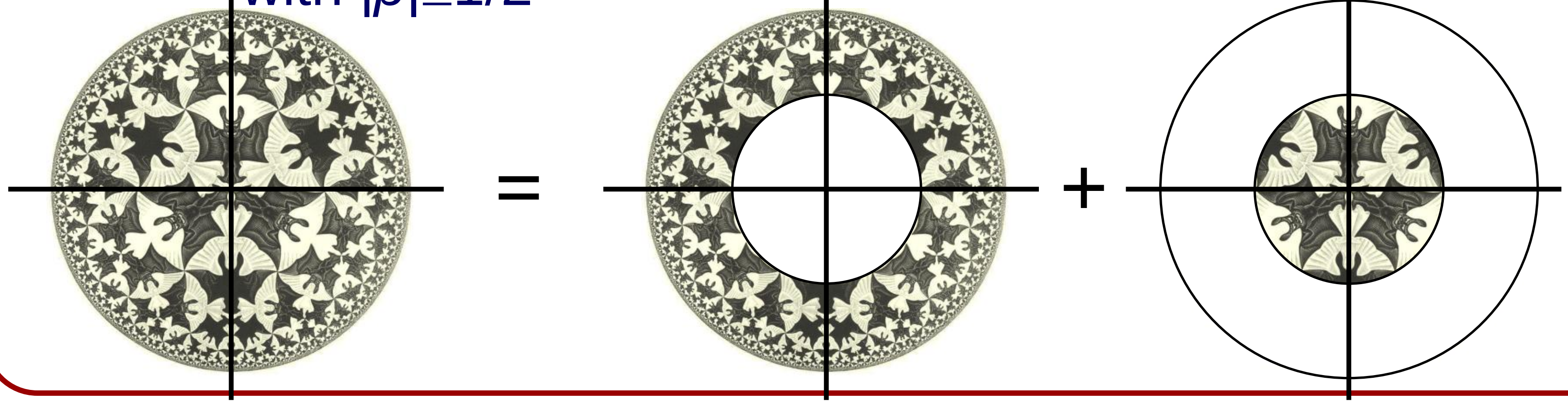
2. Consider the group of 2D rotations, acting on functions on the unit disk.
   What are two sub-representations?



Sub-Representation

<u>Example</u>:

2. Consider the group of 2D rotations, acting on functions on the unit disk.
   What are two sub-representations?
   a) The space of functions that are zero for all points p with $|p|<1/2$
   b) The space of functions that are zero for all points p with $|p|\geq 1/2$



Irreducible Representations

Given a representation, ρ, of a group, G, on a vector space, V, the representation is said to be irreducible if the only subspaces of V that are sub-representations are:

$$W = V \quad \text{and} \quad W = \emptyset$$

Structure Preservation

We had talked about linear transformations as maps between vector spaces, that preserve the underlying vector space structure:

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We had talked about a representation of a group as a map from a group into the group of invertible linear transformations that preserve the group structure:

$$\rho(g \cdot h) = \rho(g) \cdot \rho(h)$$

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We ha… roup as a m… ertible linear… up structure:

It doesn't matter if we perform the group/vector-space operations before or after we apply the map.

$$\rho(g \cdot h) = \rho(g) \cdot \rho(h)$$

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Given a representation ρ of a group G onto a vector space V, what does it mean for a map $\Phi:V\rightarrow V$ to preserve the representation structure?

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- Since Φ is a map between vector spaces, it should preserve the vector space structure:
  $\Rightarrow \Phi$ is a linear transformation.

Structure Preservation

Given a representation ρ of a group G onto a vector space V, what does it mean for a map $\Phi : V \to V$ to preserve the representation structure?

- Since Φ is a map between vector spaces, it should preserve the vector space structure:
  $\Rightarrow$ Φ is a linear transformation.
- Φ should also preserve the group action structure:

$$\Phi(\rho_g v) = \rho_g(\Phi(v))$$

Schur's Lemma

Given an irreducible representation ρ of a group G onto a vector space V, if Φ preserves the representation structure than Φ is just scalar multiplication:

$$\Phi = \lambda \mathrm{Id}$$

Schur's Lemma

Proof:

1. Since Φ is a linear transformation, it has a (complex) eigenvalue λ.

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2. Since Φ preserves the representation structure, $(\Phi-\lambda \mathrm{Id})$ must also preserve the representation structure:

$$(\Phi - \lambda \mathrm{Id})\left(\rho_g(v)\right) = \Phi\left(\rho_g(v)\right) - \lambda\left(\rho_g(v)\right)$$

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1. Since Φ is a linear transformation, it has a (complex) eigenvalue λ.

2. Since Φ preserves the representation structure, (Φ-λId) must also preserve the representation structure:

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\begin{aligned}
(\Phi - \lambda \mathrm{Id})\left(\rho_g(v)\right) &= \Phi\left(\rho_g(v)\right) - \lambda\left(\rho_g(v)\right) \\
&= \rho_g\left(\Phi(v)\right) - \rho_g(\lambda v)
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&= \rho_g\left(\Phi(v)\right)-\left(\rho_g(\lambda v)\right) \\
&= \rho_g\left((\Phi-\lambda\mathrm{Id})(v)\right)
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3. Since λ is an eigenvalue of Φ, $(\Phi-\lambda\text{Id})$ must have a kernel $W \subset V$.

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$$\rho_g(w)\in W$$

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5. This implies that the kernel of (Φ-λId) must be a sub-representation of *V*.

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6. Since ρ is an irreducible representation and since the kernel of (Φ-λId) is not empty, this must imply that W=V.

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Proof:

5. This implies that the kernel of $(\Phi - \lambda \mathrm{Id})$ must be a sub-representation of *V*.

6. Since ρ is an irreducible representation and since the kernel of $(\Phi - \lambda \mathrm{Id})$ is not empty, this must imply that $W = V$.

7. Since the kernel is the entire vector space, this implies that

$$(\Phi - \lambda \mathrm{Id}) = 0 \quad \Leftrightarrow \quad \Phi = \lambda \mathrm{Id}$$

Schur's Lemma (Corollary)

Corollary:

If a representation of a commutative group is irreducible, it must be one-dimensional.

Schur's Lemma (Corollary)

Proof:

1. Fix some element $h \in G$.

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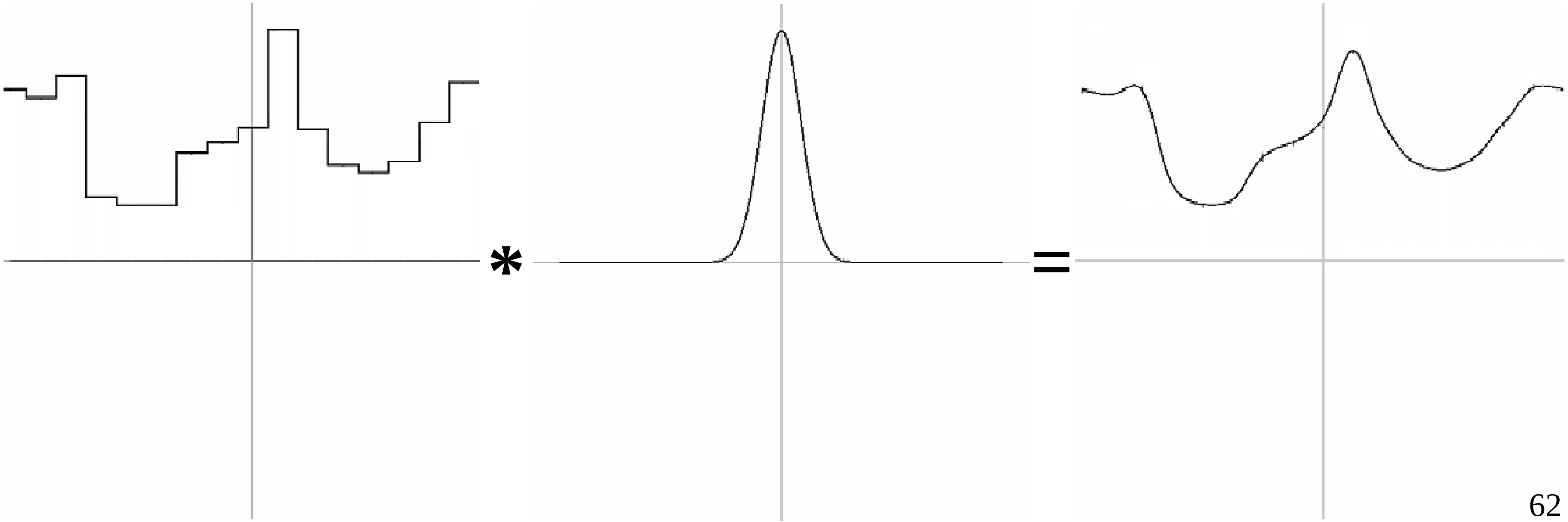
4. Since this is true for any $h \in G$, this implies that any subspace $W \subset V$ must be a sub-representation.
5. Since V is irreducible, this must imply that V is one-dimensional.

Why do we care?

In signal/image/voxel processing, we are often interested in applying a filter to some initial data.

E.g. Smoothing:



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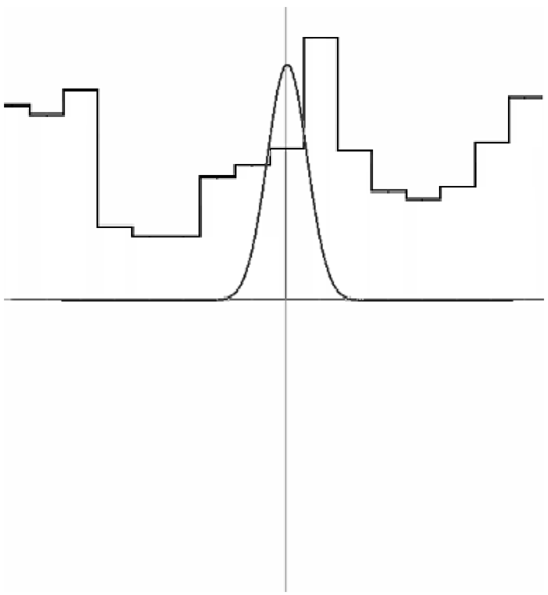

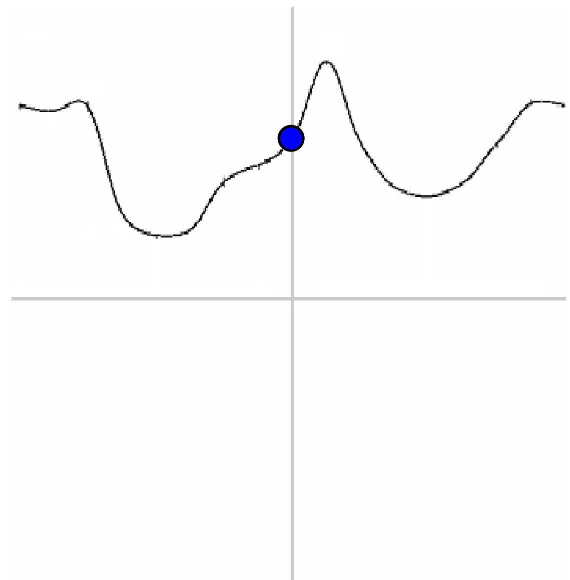

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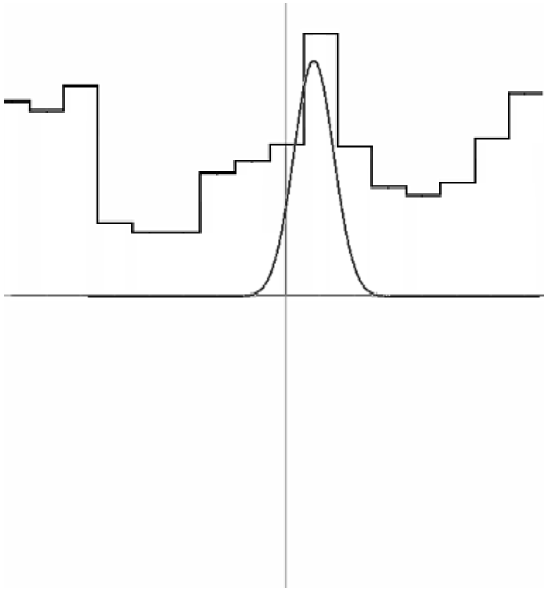

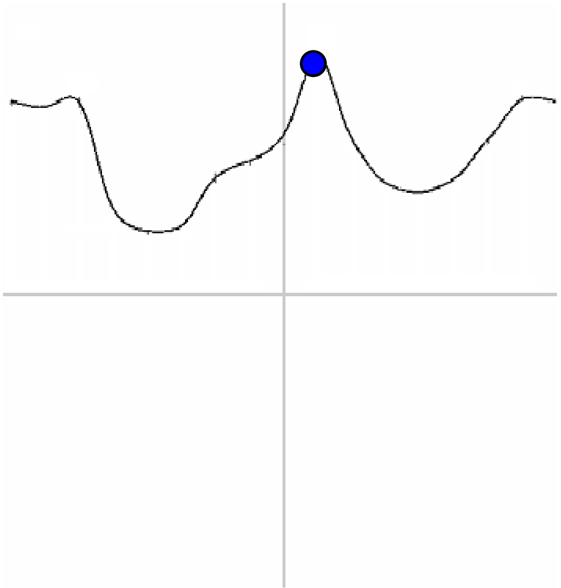

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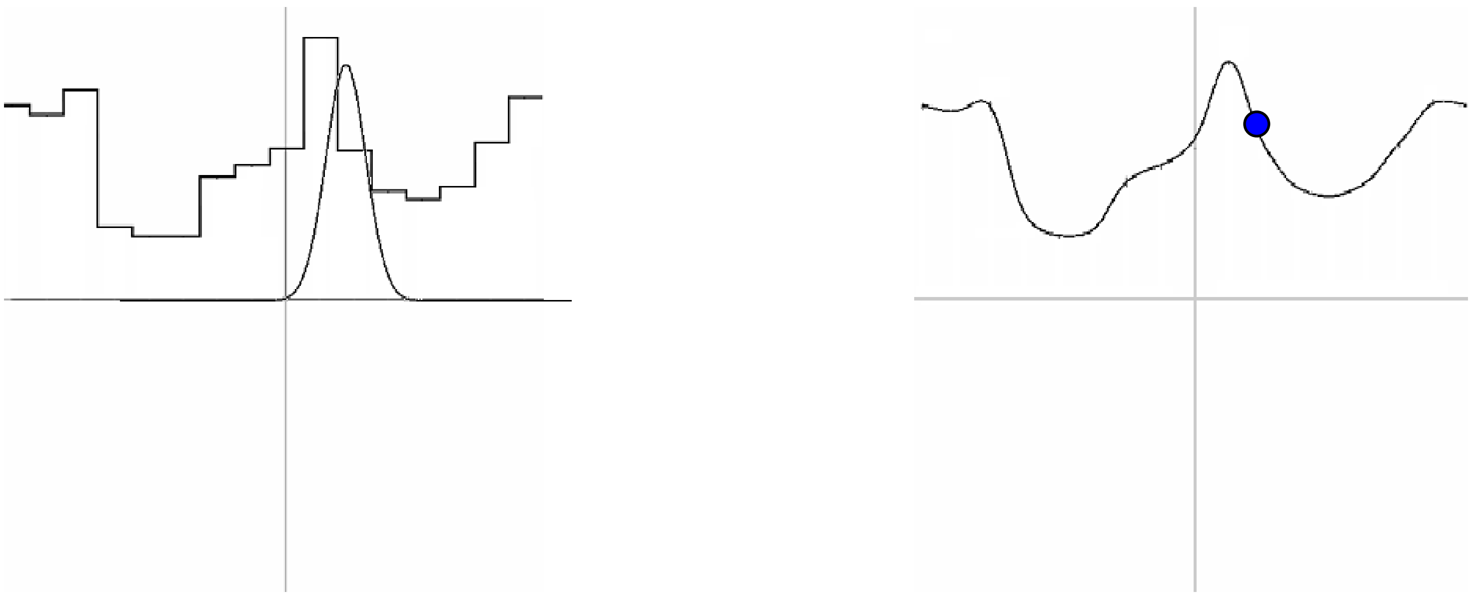

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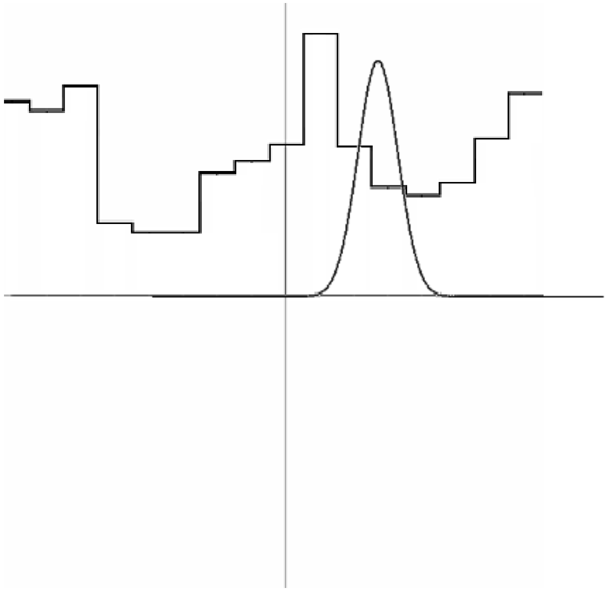

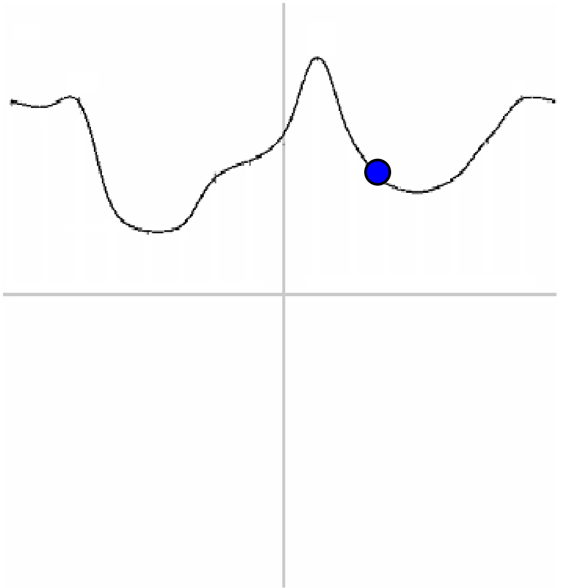

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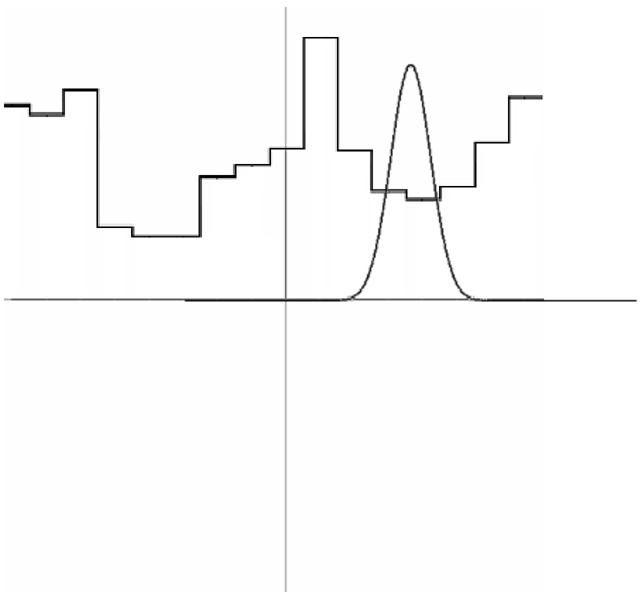

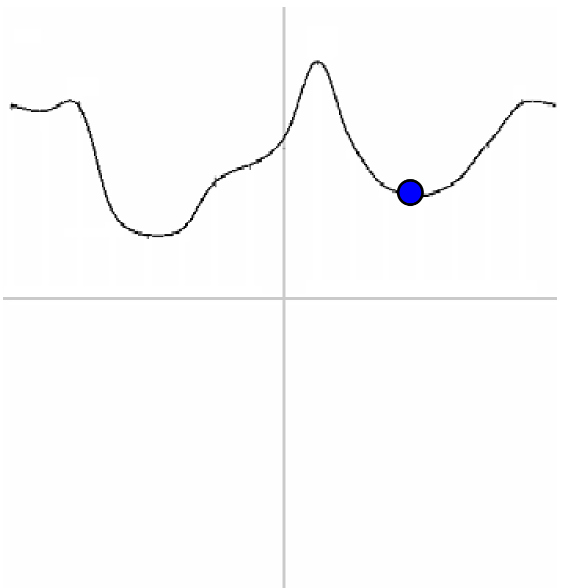

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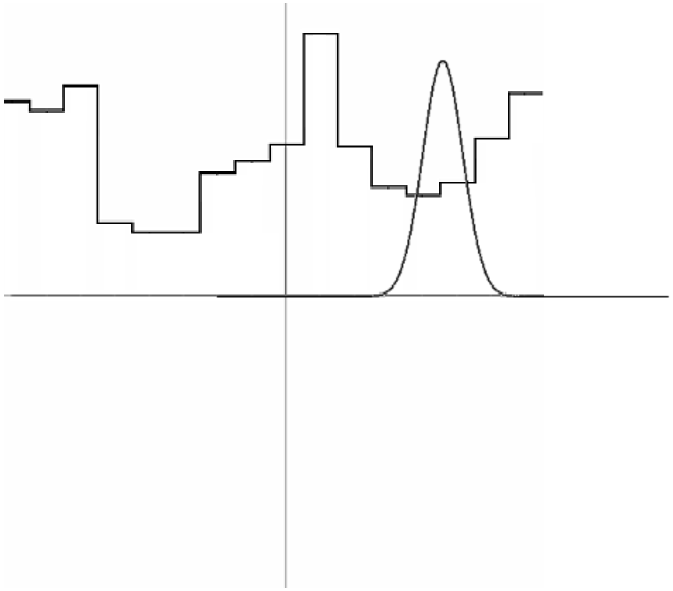

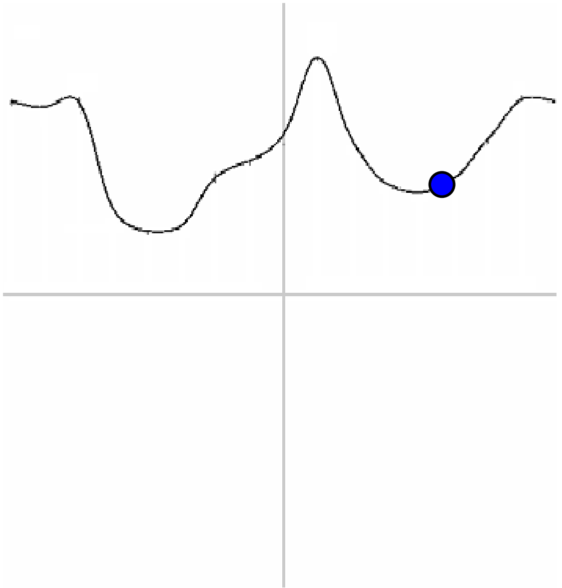

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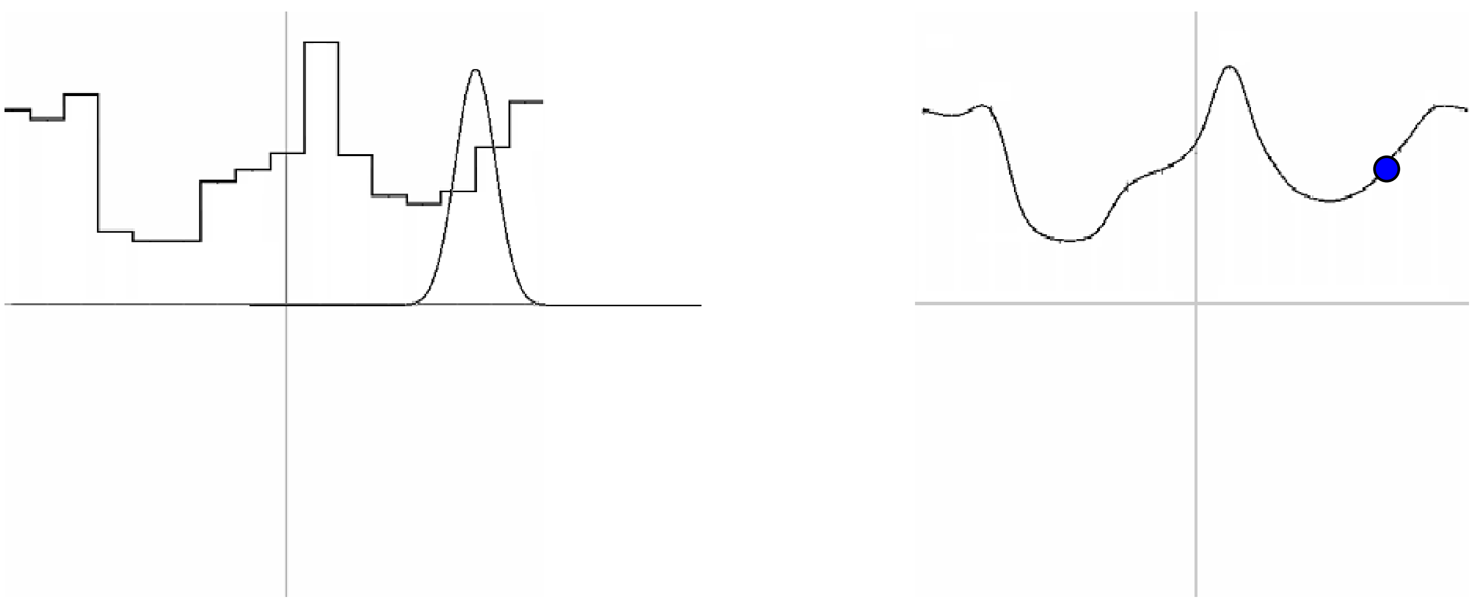
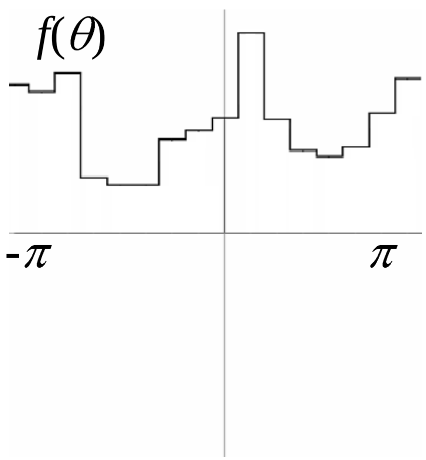
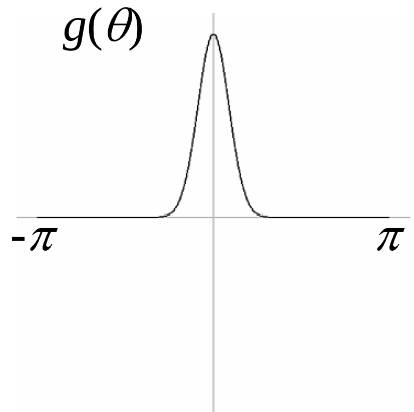
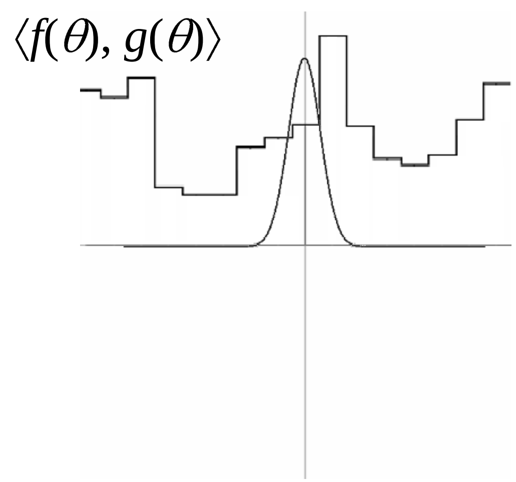


Smoothing

What we are really doing is computing a moving inner product:







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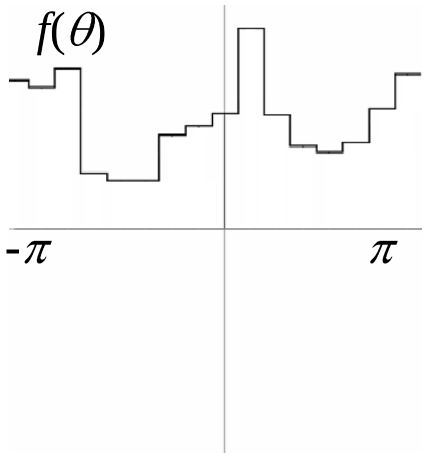
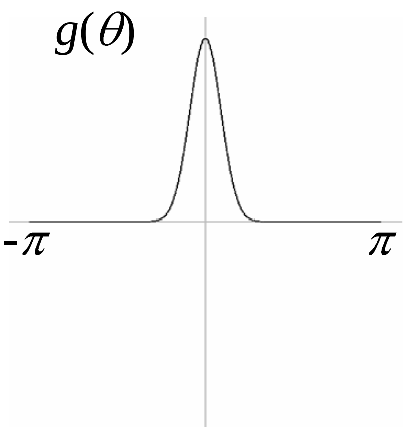
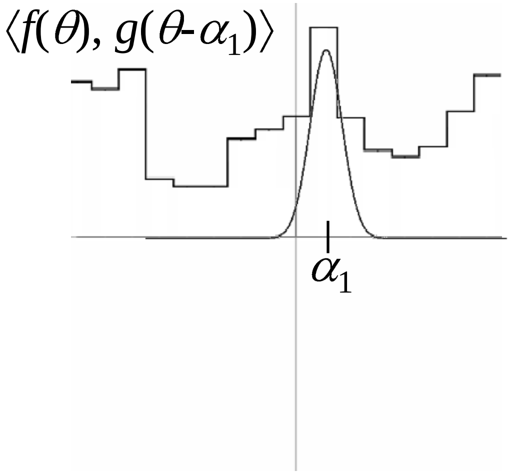
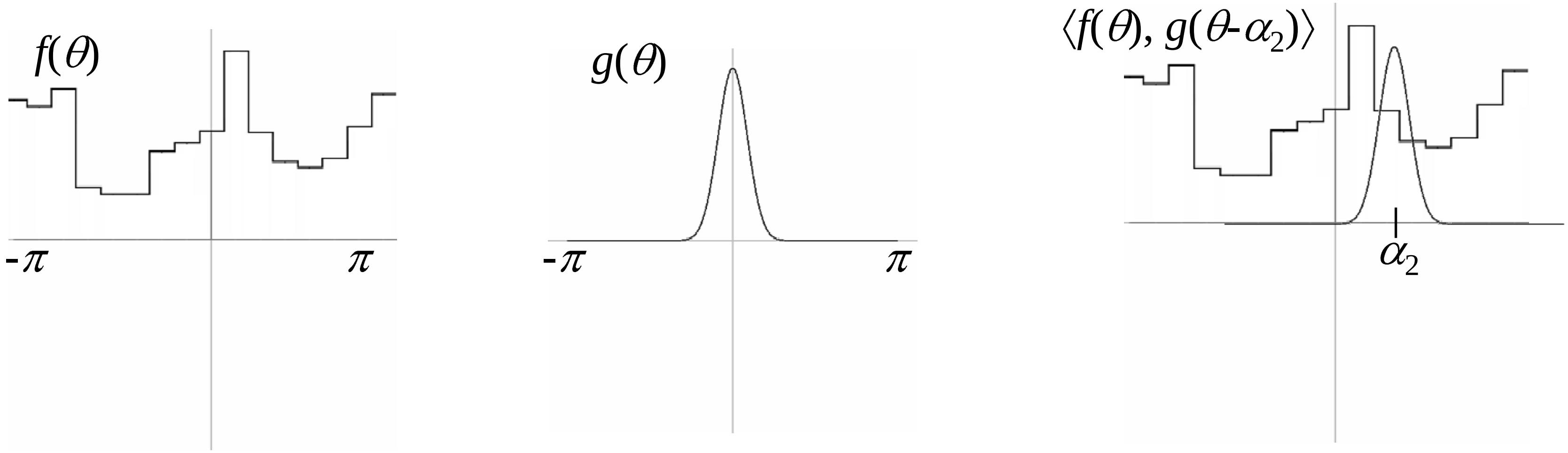






Smoothing

What we are really doing is computing a moving inner product:



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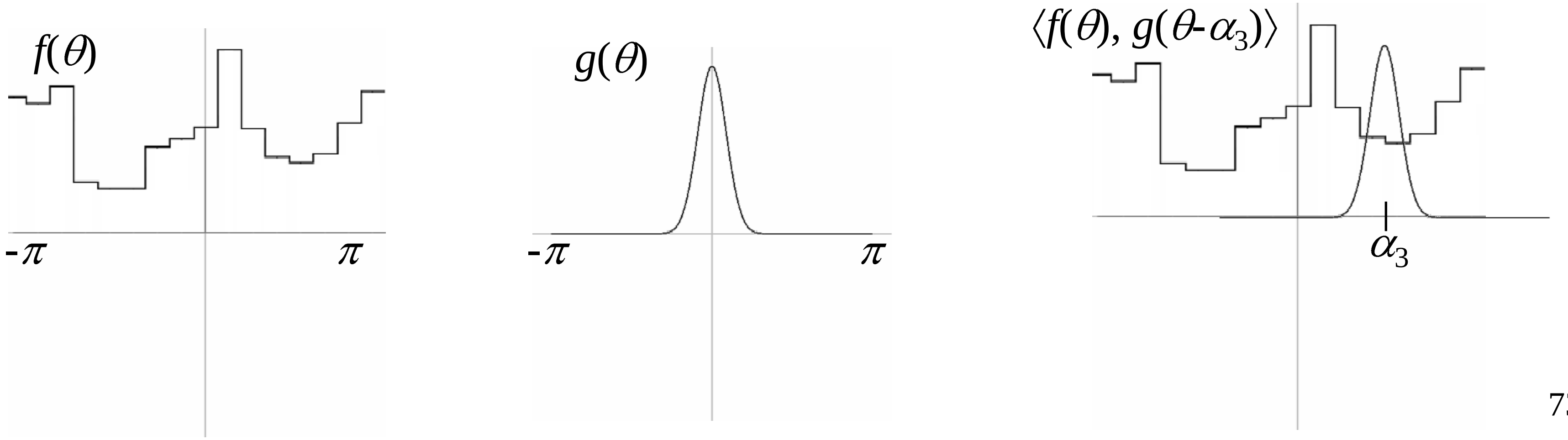
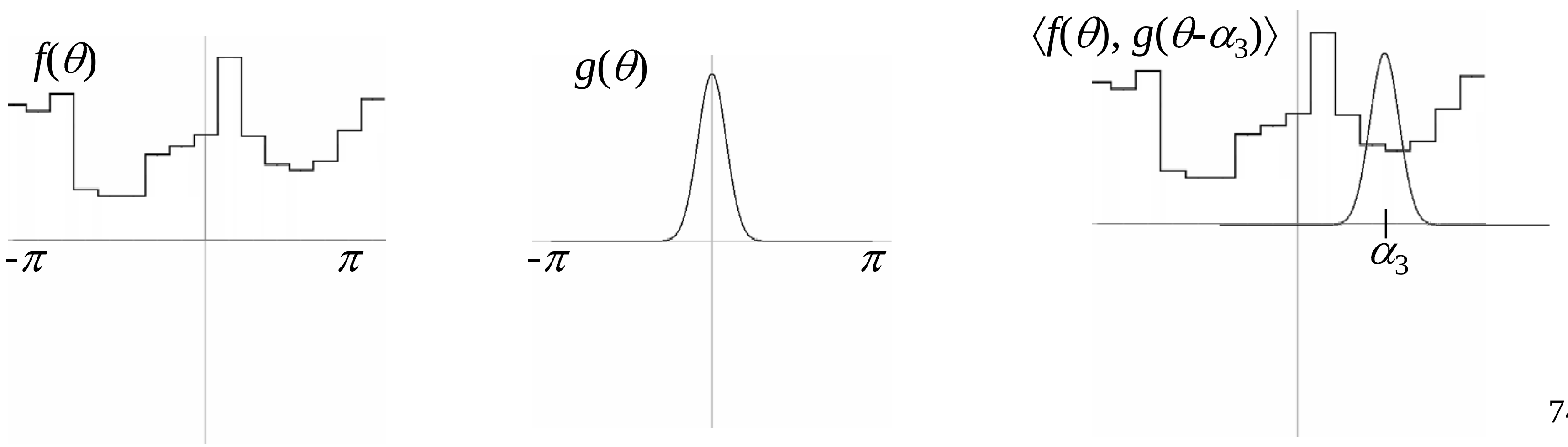


Smoothing

We can write out the operation of smoothing a signal *f* by a filter *g* as:

$$(f * g)(\alpha) = \langle f, \rho_\alpha(g) \rangle$$

where ρ_α is the linear transformation that translates a periodic function by α.



Smoothing

We can think of this as a representation:

- V is the space of periodic functions on the line
- G is the group of real numbers in $[-\pi,\pi)$
- ρ_α is the representation translating a function by α.

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This is a representation of a commutative group…

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Setting $f_i(\theta)$ to be a unit-vector in V_i, we know that the group acts on $f_i(\theta)$ by scalar multiplication:

$$\rho_\alpha\left(f_i(\theta)\right) = \lambda_i(\alpha)\cdot f_i(\theta)$$

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$$\rho_\alpha\left(f_i(\theta)\right) = \lambda_i(\alpha)\cdot f_i(\theta)$$

If we write out the functions $f(\theta)$ and $g(\theta)$ as:

$$f(\theta) = a_1 f_1(\theta) + a_2 f_2(\theta) + \cdots + a_n f_n(\theta)$$
$$g(\theta) = b_1 f_1(\theta) + b_2 f_2(\theta) + \cdots + b_n f_n(\theta)$$

Smoothing

Then the moving dot-product can be written as:

$$(f * g)(\alpha) = \langle f, \rho_\alpha(g) \rangle$$

Smoothing

$$\left(f * g\right)(\alpha) = \left\langle f, \rho_\alpha(g) \right\rangle$$

Expanding *f* and *g* in terms of the basis $\{f_1,\ldots,f_n\}$:

$$\left(f * g\right)(\alpha) = \left\langle \sum_{i=1}^{n} a_i f_i, \rho_\alpha\left(\sum_{j=1}^{n} b_j f_j \right) \right\rangle$$

Smoothing

$$(f * g)(\alpha) = \left\langle \sum_{i=1}^{n} a_i f_i, \rho_\alpha \left(\sum_{j=1}^{n} b_j f_j \right) \right\rangle$$

Using the fact that ρ_α is a linear transformation:

$$(f * g)(\alpha) = \left\langle \sum_{i=1}^{n} a_i f_i, \sum_{j=1}^{n} b_j \rho_\alpha (f_j) \right\rangle$$

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Using the fact that the inner product is linear in the first term:

$$(f * g)(\alpha) = \sum_{i=1}^{n} a_i \left\langle f_i, \sum_{j=1}^{n} b_j \rho_\alpha(f_j) \right\rangle$$

Smoothing

$$\left(f * g\right)(\alpha) = \sum_{i=1}^{n} a_i \left\langle f_i, \sum_{j=1}^{n} b_j \rho_\alpha(f_j) \right\rangle$$

Using the fact that the inner product is conjugate-linear in the second term:

$$\left(f * g\right)(\alpha) = \sum_{i,j=1}^{n} a_i \overline{b_j} \left\langle f_i, \rho_\alpha(f_j) \right\rangle$$

Smoothing

$$\left(f*g\right)(\alpha)=\sum_{i,j=1}^{n}a_i\bar{b}_j\left\langle f_i,\rho_\alpha(f_j)\right\rangle$$

Using the fact that on V_i, the representation ρ_α is just scalar multiplication:

$$\left(f*g\right)(\alpha)=\sum_{i,j=1}^{n}a_i\bar{b}_j\left\langle f_i,\lambda_j(\alpha)f_j\right\rangle$$

Smoothing

$$(f * g)(\alpha) = \sum_{i,j=1}^{n} a_i \overline{b_j} \left\langle f_i, \lambda_j(\alpha) f_j \right\rangle$$

Again, using the fact that the inner product is conjugate-linear in the second term:

$$(f * g)(\alpha) = \sum_{i,j=1}^{n} a_i \overline{b_j} \overline{\lambda_j}(\alpha) \left\langle f_i, f_j \right\rangle$$

Smoothing

$$\left(f * g\right)(\alpha) = \sum_{i,j=1}^{n} a_i \overline{b_j} \overline{\lambda_j}(\alpha) \left\langle f_i, f_j \right\rangle$$

And finally, using the fact that the f_i are orthogonal unit-vectors:

$$\left(f * g\right)(\alpha) = \sum_{i=1}^{n} a_i \overline{b_i} \overline{\lambda_i}(\alpha)$$

Smoothing

$$\left(f * g\right)(\alpha) = \sum_{i=1}^{n} a_i \overline{b_i} \overline{\lambda_i}(\alpha)$$

This implies that we can compute the moving dot-product by multiplying the coefficients of *f* and *g*.

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This implies that we can compute the moving dot-product by multiplying the coefficients of *f* and *g*.

Convolution in the spatial domain is multiplication in the frequency domain!