FFTs in Graphics and Vision

More Math Review
Outline

Inner Product Spaces
- Real Inner Products
- Hermitian Inner Products
- Orthogonal Transforms
- Unitary Transforms
- Function Spaces
Inner Product Spaces

Given a real vector space $V$, an inner product is a function $\langle \cdot, \cdot \rangle$ that takes a pair of vectors and returns a real value.
Inner Product Spaces

An inner product is a map from $V \times V$ into the real numbers that is:

1. **Linear**: For all $u, v, w \in V$ and any real scalar $\lambda$
   
   $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$

   $\langle \lambda v, w \rangle = \lambda \langle v, w \rangle$

2. **Symmetric**: For all $u, v \in V$

   $\langle v, w \rangle = \langle w, v \rangle$

3. **Positive Definite**: For all $v \in V$:

   $\langle v, v \rangle \geq 0$

   $\langle v, v \rangle = 0 \iff v = 0$
Inner Product Spaces

An inner product defines a notion of distance on a vector space by setting:

\[ D(v, w) = \sqrt{\langle v - w, v - w \rangle} \equiv \|v - w\|^{1/2} \]
Inner Product Spaces

Examples:

1. On the space of $n$-dimensional arrays, the standard inner product is:
   \[
   \langle (a_1, \cdots, a_n), (b_1, \cdots, b_n) \rangle = a_1 b_1 + \cdots + a_n b_n
   = (a_1, \cdots, a_n)(b_1, \cdots, b_n)^t
   \]
Inner Product Spaces

Examples:

1. On the space of $n$-dimensional arrays, the standard inner product is:
   \[
   \langle (a_1, \cdots, a_n), (b_1, \cdots, b_n) \rangle = a_1 b_1 + \cdots + a_n b_n
   \]
   
   \[
   = (a_1, \cdots, a_n)(b_1, \cdots, b_n)^t
   \]

2. On the space of continuous, real-valued functions, defined on a circle, the standard inner product is:
   \[
   \langle f, g \rangle = \int_0^{2\pi} f(\theta) g(\theta) \, d\theta
   \]
Inner Product Spaces

Examples:

3. Suppose we have the space of $n$-dimensional arrays, and suppose we have a matrix:

$$M = \begin{pmatrix}
M_{11} & \cdots & M_{1n} \\
\vdots & \ddots & \vdots \\
M_{n1} & \cdots & M_{nn}
\end{pmatrix}$$

Does the map:

$$\langle v, w \rangle_M = v^t M w$$

define an inner product?
Inner Product Spaces

Examples:

3. Does the map:

\[ \langle v, w \rangle_M = v^t Mw \]

define an inner product?

- Is it linear?
- Is it symmetric?
- Is it positive definite?
Inner Product Spaces

Examples:

\[ \langle v, w \rangle_M = v^t M w \]

• Is it linear? Yes!

\[ \langle u + v, w \rangle_M = (u + v)^t M w \]

\[ \langle \lambda v, w \rangle_M = (\lambda v)^t M w \]
Inner Product Spaces

Examples:

\[ \langle v, w \rangle_M = v^t M w \]

- Is it symmetric? Only if \( M \) is symmetric (\( M=M^t \))

\[ \langle w, v \rangle_M = w^t M v \]
Inner Product Spaces

Examples:

\[ \langle v, w \rangle_M = v^t M w \]

• Is it positive definite?

If, \( M \) is symmetric, there exists an orthogonal basis \( \{\nu_1, \ldots, \nu_n\} \) with respect to which it is diagonal:

\[
M = B^t \begin{pmatrix}
\lambda_1 & 0 & \cdots & 0 & 0 \\
0 & \lambda_2 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \lambda_{n-1} & 0 \\
0 & 0 & \cdots & 0 & \lambda_n \\
\end{pmatrix} B
\]
Inner Product Spaces

Examples:

\[ \langle v, w \rangle_M = v^t M w \]

- Is it positive definite?

Only if the eigenvalues are all positive

If we express \( v \) in terms of this basis:

\[ v = a_1 v_1 + \cdots + a_n v_n \]

then

\[ \langle v, v \rangle_M = \lambda_1 a_1^2 + \cdots + \lambda_n a_n^2 \]
Examples:

4. On the space of continuous, real-valued functions, defined on a circle, does the map:

\[
\langle f, g \rangle = \int_{0}^{2\pi} f(\theta)g(\theta) \omega(\theta) d\theta
\]

define an inner product? No!
Examples:

4. On the space of continuous, real-valued functions, defined on a circle, does the map:

\[ \langle f, g \rangle = \int_0^{2\pi} f(\theta) g(\theta) \omega(\theta) d\theta \]

define an inner product? No!

What if \( \omega(\theta) > 0 \)? Yes!
Hermitian Inner Product Spaces

Given a complex vector space $V$, a Hermitian inner product is a function $\langle \cdot , \cdot \rangle$ that takes a pair of vectors and returns a complex value.
A Hermitian inner product is a map from $V \times V$ into the complex numbers that is:

1. **Linear**: For all $u, v, w \in V$ and any real scalar $\lambda$
   
   \[
   \langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle \\
   \langle \lambda v, w \rangle = \lambda \langle v, w \rangle 
   \]

2. **Conjugate Symmetric**: For all $u, v \in V$
   
   \[
   \langle v, w \rangle = \overline{\langle w, v \rangle} 
   \]

3. **Positive Definite**: For all $v \in V$
   
   \[
   \langle v, v \rangle \geq 0 \\
   \langle v, v \rangle = 0 \iff v = 0
   \]
Inner Product Spaces

As in the real case, a Hermitian inner product defines a notion of distance on a complex vector space by setting:

\[ D(v, w) = \sqrt{\langle v - w, v - w \rangle} \equiv \| v - w \|\frac{1}{2} \]
Hermitian Inner Product Spaces

Examples:

1. On complex-valued, $n$-dimensional arrays, the standard Hermitian inner product is:

$$\langle (a_1, \cdots, a_n), (b_1, \cdots, b_n) \rangle = a_1 \overline{b_1} + \cdots + a_n \overline{b_n} = (a_1, \cdots, a_n)^t (b_1, \cdots, b_n)^t$$
Hermitian Inner Product Spaces

Examples:

1. On complex-valued, $n$-dimensional arrays, the standard Hermitian inner product is:

$$\langle (a_1, \cdots, a_n), (b_1, \cdots, b_n) \rangle = a_1 \overline{b}_1 + \cdots + a_n \overline{b}_n$$

$$= (a_1, \cdots, a_n)(b_1, \cdots, b_n)^t$$

2. On the space of continuous, complex-valued functions, defined on a circle, the standard Hermitian inner product is:

$$\langle f, g \rangle = \int_0^{2\pi} f(\theta) \overline{g}(\theta) \, d\theta$$
Structure Preservation

Recall:

If we have an $n$-dimensional vector space $V$ then a linear map $L$ is just a function from $V$ to $V$ that preserves the linear structure:

$$L(av_1 + bv_2) = a \cdot L(v_1) + b \cdot L(v_2)$$

for all $v, w \in V$ and all scalars $a$ and $b$. 
Structure Preservation

Recall:

If we have an $n$-dimensional vector space $V$ then a linear map $L$ is just a function from $V$ to $V$ that preserves the linear structure:

$$L(av_1 + bv_2) = a \cdot L(v_1) + b \cdot L(v_2)$$

for all $v, w \in V$ and all scalars $a$ and $b$.

If $L$ is invertible, then we can think of $L$ as a function that renames all the elements in $V$ while preserving the underlying vector space structure.
Structure Preservation

Orthogonal Transformations:

For a real vector space $V$ that has an inner product, we would also like to consider those functions that rename the elements of $V$ while preserving the underlying structure.
Structure Preservation

Orthogonal Transformations:

For a real vector space $V$ that has an inner product, we would also like to consider those functions that rename the elements of $V$ while preserving the underlying structure.

If $R$ is such a function, then:

- $R$ must be an invertible linear operator, in order to preserve the underlying vector space structure.
- $R$ must also preserve the underlying inner product.
Structure Preservation

Orthogonal Transformations:

For a real vector space $V$, an invertible linear operator $R$ is called **orthogonal** if it preserves the inner product:

$$\langle R(v), R(w) \rangle = \langle v, w \rangle$$

for all $v, w \in V$. 
Structure Preservation

Example:

On the space of real-valued, $n$-dimensional arrays, a matrix is orthogonal if:

$$\langle Rv, Rw \rangle = \langle v, w \rangle$$

$$(Rv)^t (Rw) = v^t w$$

$$v^t R^t Rw = v^t w$$

$$R^t = R^{-1}$$
Structure Preservation

Example:

On the space of real-valued, \( n \)-dimensional arrays, a matrix is orthogonal if:

\[ R^t = R^{-1} \]

**Note:** The determinant of an orthogonal matrix always has absolute value 1.
Structure Preservation

Example:

On the space of real-valued, $n$-dimensional arrays, a matrix is orthogonal if:

$$R^t = R^{-1}$$

Note: The determinant of an orthogonal matrix always has absolute value 1.

If the determinant of an orthogonal matrix is equal to 1, the matrix is called a rotation.
Orthogonal Matrices and Eigenvalues

If $R$ is an orthogonal transformation and $R$ has an eigenvalue $\lambda$, then $|\lambda| = 1$. 
Orthogonal Matrices and Eigenvalues

If $R$ is an orthogonal transformation and $R$ has an eigenvalue $\lambda$, then $|\lambda|=1$.

To see this, let $v$ be the eigenvector corresponding to the eigenvalue $\lambda$. Then since $R$ is orthogonal, we have:

$$\langle v, v \rangle = \langle Rv, Rv \rangle$$
Structure Preservation

Unitary Transformations:

For a complex vector space $V$, an invertible linear operator $R$ is called unitary if it preserves the hermitian inner product:

$$\langle R(v), R(w) \rangle = \langle v, w \rangle$$

for all $v, w \in V$. 
Structure Preservation

Example:

On the space of complex-valued, \( n \)-dimensional arrays, a matrix is unitary if:

\[
\langle Rv, Rw \rangle = \langle v, w \rangle
\]

\[
(Rv)^t (Rw) = v^t \overline{w}
\]

\[
v^t R^t \overline{Rw} = v^t \overline{w}
\]

\[
\overline{R}^t = R^{-1}
\]
Structure Preservation

Example:

On the space of complex-valued, $n$-dimensional arrays, a matrix is unitary if:

$$\overline{R^t} = R^{-1}$$

Note: The determinant of a unitary matrix always has norm 1.
Unitary Matrices and Eigenvalues

If $R$ is a unitary transformation and $R$ has an eigenvalue $\lambda$, then $|\lambda|=1$. 
Unitary Matrices and Eigenvalues

If $R$ is a unitary transformation and $R$ has an eigenvalue $\lambda$, then $|\lambda|=1$.

To see this, let $v$ be the eigenvector corresponding to the eigenvalue $\lambda$. Then since $R$ is unitary, we have:

$$\langle v, v \rangle = \langle Rv, Rv \rangle$$
In this course, the vector spaces we will be looking at most often are the vector spaces of functions defined on some domain:

- Continuous functions on the unit circle ($S^1$)
- Continuous functions on the unit disk ($D^2$)
- Continuous, periodic functions on the plane ($\mathbb{R}^2$)
- Continuous functions on the unit sphere ($S^2$)
- Continuous functions on the unit ball ($B^3$)
Function Spaces

Continuous functions on the unit circle ($S^1$):
This is the set of points $(x,y)$ such that $x^2+y^2=1$.

If we have functions $f(x,y)$, and $g(x,y)$ the inner product is:

$$\langle f, g \rangle = \int_{p \in S^1} f(p) \overline{g}(p) \, dp$$
Function Spaces

Continuous functions on the unit circle ($S^1$):
This is the set of points $(x, y)$ such that $x^2 + y^2 = 1$.

If we have functions $f(x, y)$, and $g(x, y)$ the inner product is:
$$\langle f, g \rangle = \int_{p \in S^1} f(p)g(p) \, dp$$

Or, we can represent points on the circle in terms of angle $\theta \in [0, 2\pi]$:
$$\theta \rightarrow \cos \theta, \sin \theta$$

For functions $f(\theta)$ and $g(\theta)$ the inner product is:
$$\langle f, g \rangle = \int_{0}^{2\pi} f(\theta)g(\theta) \, d\theta$$
Function Spaces

Continuous functions on the unit disk ($D^2$):
This is the set of points $(x,y)$ such that $x^2+y^2 \leq 1$.

If we have functions $f(x,y)$, and $g(x,y)$ the inner product is:

$$\langle f, g \rangle = \int_{p \in D^2} f(p) \bar{g}(p) \, dp$$
Function Spaces

Continuous functions on the unit disk ($D^2$):

This is the set of points $(x,y)$ such that $x^2+y^2\leq 1$.

If we have functions $f(x,y)$, and $g(x,y)$ the inner product is:

$$\langle f, g \rangle = \int_{p\in S^1} f(p) \overline{g}(p) \, dp$$

Or, we can represent points on the circle in terms of radius $r\in[0,1]$ and angle $\theta\in[0,2\pi]$:

$$(r, \theta) \rightarrow (\cos \theta, r \sin \theta)$$

For functions $f(r,\theta)$ and $g(r,\theta)$ the inner product is:

$$\langle f, g \rangle = \int_{0}^{2\pi} \int_{0}^{1} f(r,\theta) \overline{g}(r,\theta) \, r \, dr \, d\theta$$
Function Spaces

Continuous, periodic functions on the plane ($\mathbb{R}^2$):
This is the set of functions $f(x,y)$ satisfying the property that:

$$f(x, y) = f(x + 2\pi, y) = f(x, y + 2\pi)$$

If we have functions $f(x,y)$, and $g(x,y)$ the inner product is:

$$\langle f, g \rangle = \int_0^{2\pi} \int_0^{2\pi} f(x, y) g(x, y) \, dy \, dx$$
Function Spaces

Continuous functions on the unit sphere ($S^2$):

This is the set of points $(x,y,z)$ such that $x^2+y^2+z^2=1$.

If we have functions $f(x,y,z)$, and $g(x,y,z)$ the inner product is:

$$\langle f, g \rangle = \int_{p \in S^2} f(p) \overline{g}(p) \, dp$$
Function Spaces

Continuous functions on the unit sphere ($S^2$):
This is the set of points $(x,y,z)$ such that $x^2+y^2+z^2=1$.

If we have functions $f(x,y,z)$, and $g(x,y,z)$ the inner product is:

$$\langle f, g \rangle = \int_{p \in S^2} f(p) \overline{g}(p) \, dp$$

Or, we can represent points on the sphere in terms of spherical angle $\theta \in [0,\pi]$ and $\phi \in [0,2\pi]$:

$$(\theta, \phi) \rightarrow \hat{\sin} \theta \cos \phi, \cos \theta, \sin \theta \sin \phi \hat{\sin}$$

For functions $f(\theta,\phi)$ and $g(\theta,\phi)$ the inner product is:

$$\langle f, g \rangle = \int_0^{2\pi} \int_0^\pi f(\theta, \phi) \overline{g}(\theta, \phi) \sin \theta \, d\theta \, d\phi$$
Function Spaces

Continuous functions on the unit ball ($B^3$):

This is the set of points $(x,y,z)$ such that $x^2+y^2+z^2 \leq 1$.

If we have functions $f(x,y,z)$, and $g(x,y,z)$ the inner product is:

$$\langle f, g \rangle = \int_{p \in B^3} f(p)\bar{g}(p) \, dp$$
Function Spaces

Continuous functions on the unit ball ($B^3$):
This is the set of points $(x,y,z)$ such that $x^2+y^2+z^2 \leq 1$.

If we have functions $f(x,y,z)$, and $g(x,y,z)$ the inner product is:

$$\langle f, g \rangle = \int_{p \in B^3} f(p) \overline{g}(p) \, dp$$

Or, we can represent points in the ball in terms of radius $r \in [0,1]$ and spherical angle $\theta \in [0,\pi]$, $\varphi \in [0,2\pi]$:

$$(r, \theta, \varphi) \rightarrow (\sin \theta \cos \varphi, r \cos \theta, r \sin \theta \sin \varphi)$$

For functions $f(\theta, \varphi)$ and $g(\theta, \varphi)$ the inner product is:

$$\langle f, g \rangle = \int_{0}^{2\pi} \int_{0}^{\pi} \int_{0}^{1} f(r, \theta, \varphi) \overline{g}(r, \theta, \varphi) r^2 \sin \theta \, dr \, d\theta \, d\varphi$$
Function Spaces

Examples

If we consider the space of continuous, complex-valued functions on the unit circle:

- Is the map:
  \[ f(p) \rightarrow f(p) + 1 \]

  a linear transformation? **No!**
Function Spaces

Examples

If we consider the space of continuous, complex-valued functions on the unit circle:

- For any scalar value $\lambda$, is:
  \[ f(p) \rightarrow \lambda f(p) \]

  a linear transformation? Yes!
Function Spaces

Examples

If we consider the space of continuous, complex-valued functions on the unit circle:

- For any scalar value $\lambda$, is:
  \[ f(p) \rightarrow \lambda f(p) \]
  a linear transformation? **Yes!**

- Is it unitary? **No!**
Function Spaces

Examples

If we consider the space of continuous, complex-valued functions on the unit circle:

- For any scalar value $\lambda$, is:
  \[ f(p) \rightarrow \lambda f(p) \]
  
  a linear transformation? Yes!

- Is it unitary? No!

- How about if $|\lambda|=1$? Yes!
Function Spaces

Examples

If we consider the space of continuous, complex-valued functions on the unit circle:

- **Is the differentiation operator:**
  \[ f(p) \rightarrow f'(p) \]

  a linear transformation? **No!**
Function Spaces

Examples

If we consider the space of continuous, complex-valued functions on the unit circle:

- Is the differentiation operator:
  \[ f(p) \rightarrow f'(p) \]
  a linear transformation? **No!**

- What if we only consider the functions that are infinitely differentiable? **Yes!**
Function Spaces

Examples

If we consider the space of continuous, complex-valued functions on the unit circle:

- Is the differentiation operator:
  \[ f(p) \rightarrow f'(p) \]
  a linear transformation? **No!**

- What if we only consider the functions that are infinitely differentiable? **Yes!**

- Is it unitary? **No!**
Function Spaces

Examples

If we consider the space of continuous, complex-valued functions on the unit circle:

- For any 2D rotation $R$ is the transformation:

$$f(p) \rightarrow f(R^{-1}p)$$

a linear transformation? Yes!
Function Spaces

Examples

If we consider the space of continuous, complex-valued functions on the unit circle:

- For any 2D rotation $R$ is the transformation:
  $$f(p) \rightarrow f(R^{-1}p)$$

  a linear transformation? Yes!

- Is it unitary? Yes!
Function Spaces

Examples

If we consider the space of continuous, periodic, complex-valued functions on the plane:

- For any 2D point \((x_0, y_0)\), is the transformation:
  \[
  f(x, y) \rightarrow f(x - x_0, y - y_0)
  \]

  a linear transformation? Yes!
Function Spaces

Examples

If we consider the space of continuous, periodic, complex-valued functions on the plane:

- For any 2D point \((x_0, y_0)\), is the transformation:
  \[ f(x, y) \rightarrow f(x - x_0, y - y_0) \]

  a linear transformation? Yes!

- Is it unitary? Yes!
Function Spaces

Examples

If we consider the space of continuous, infinitely-differentiable, periodic, complex-valued functions on the plane:

- Is differentiation with respect to $x$: $\frac{\partial}{\partial x} f(x, y)$ a linear transformation? **Yes!**
Function Spaces

Examples

If we consider the space of continuous, infinitely-differentiable, periodic, complex-valued functions on the plane:

- Is differentiation with respect to $x$:
  $$f(x, y) \rightarrow \frac{\partial}{\partial x} f(x, y)$$
  a linear transformation? Yes!

- Is it unitary? No!
Function Spaces

Examples

If we consider the space of continuous, complex-valued functions on the sphere:

- For any rotation $R$, is the transformation:
  \[ f(p) \rightarrow f(R^{-1}p) \]

a linear transformation? Yes!
Function Spaces

Examples

If we consider the space of continuous, complex-valued functions on the sphere:

- For any rotation $R$, is the transformation:
  \[ f(p) \rightarrow f(R^{-1}p) \]
  a linear transformation? **Yes!**

- Is it unitary?
Function Spaces

Change of Variables:

Given a real/complex-valued function $f$ defined on some domain $D$, and given some differentiable, invertible, map:

$$\Phi : D \to \Phi(D)$$

We have:

$$\int_{x \in D} f(\Phi(x)) |\partial \Phi| \, dx = \int_{y \in \Phi(x)} f(y) \, dy$$

where $|\partial \Phi|$ denotes the Jacobian of $\Phi$ (i.e. the determinant of the derivative matrix)
Function Spaces

Examples

If we consider the space of continuous, complex-valued functions on the sphere:

- For any rotation $R$, is the transformation:
  \[ f(p) \rightarrow f(R^{-1}p) \]
  a linear transformation? Yes!

- Is it unitary? Yes!