Harmonic Coordinates for Character Articulation

PIXAR

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• We have a complex source mesh inside of a simpler cage mesh

• We want vertex deformations applied to the cage to be applied appropriately to the source mesh
Mean Value Coordinates

Ju, Schaefer, Warren, 2005

\[ p' = \sum_{i} g_i(p)C_i' \]

- \( p' \): New object vertices
- \( g_i(p) \): Mean value coordinate Weighting functions
- \( C_i' \): Deformed cage vertices
To compute $g_i(p)$ for each interior point $p$:
- Consider each point $x$ on the boundary
- Multiply $f(x)$ by the reciprocal distance from $x$ to $p$
- Average over all $x$

From the Ju paper:

$$g(p) = \frac{\int_x w(x, v) f(x) dS_p}{\int_x w(x, v) dS_p}$$

where

$$w(x, p) = \frac{1}{|x - p|}$$

and

$S_p$ is the unit sphere centered at $p$
To compute $g_i(p)$ for each interior point $p$:
- Consider each point $x$ on the boundary
- Multiply $f(x)$ by the reciprocal distance from $x$ to $p$
- Average over all $x$

Good things:
- Topological flexibility in designing the cage (any closed tri-mesh)
- Deformations are smooth
- Functions are linear, so no popping

Bad things:
- Does not respect the visibility of $x$ from $p$
- If a cage vertex has a negative weight associated with it, then the object vertex and cage vertex will move in opposite directions
Mean Value Coordinate Field

Desired Coordinate Field
• Instead, let’s average over all Brownian paths leaving $p$
  – This will consider the visibility of $x$ from $p$
  – Essential for any concave mesh

• Interestingly enough…
  – This is the same as solving Laplace’s equation

\[ \Delta h_i(p) = 0 \quad p \in \text{Interior}(C) \]

Solve for every cage vertex $p$

Let us first approach things in two dimensions
Boundary conditions:

- Let \( \partial p \) denote a point on the boundary \( \partial C \) of \( C \)
- Then:

\[
h_i(\partial p) = \phi_i(\partial p), \quad \text{for all} \quad \partial p \in \partial C
\]

where

\[
\phi_i(\partial p) \quad \text{is the piecewise linear function such that} \quad \phi_i(C_j) = \delta_{i,j}
\]
Properties:

- Interpolation
  \[ h_i(C_j) = \delta_{i,j} \]

- Smoothness
  The functions \( h_i(p) \) are smooth in the interior of the cage

- Non-negativity
  \[ h_i(p) \geq 0 \quad \text{for all} \quad p \in C \]

- Interior locality
  Interior locality holds if we have the non-negativity property and no interior extrema

- Linear reproduction
  Given an arbitrary \( f(p) \), the coordinate functions can be used to define:
  \[
  H[f](p) = \sum_i h_i(p)f(C_i)
  \]
  This is the ‘no popping’ condition
Properties:
  
  - Affine invariance
    \[
    \sum_{i} h_i(p) = 1 \quad \text{for all}\quad p \in C
    \]
  
  - Generalization of barycentric coordinates
    \[ h_i(p) \] is the barycentric coordinate of \( p \) with respect to \( C_i \)
Interpolation:

\[ h_i(C_j) = \phi_i(C_j) = \delta_{i,j} \]

Smoothness:

Away from the boundary, harmonic coordinates are solutions, so they are smooth in the cage interior.

On the boundary, they are only as smooth as the boundary conditions.

Non-negativity:

Harmonic functions achieve extreme at their boundaries.

Boundary values are restricted to \([0,1]\).

So interior values are restricted to \([0,1]\).

Interior locality:

Harmonic functions possess no interior extrema.
Linear reproduction:

This holds for everywhere on the boundary of \( C \), by definition:

\[ H[f](\partial p) = \sum_i h_i (\partial p) f(C) = \sum_i \phi_i (\partial p) f(C) \]

Since \( f(p) \) is linear, second derivatives vanish, ie:

\[ \nabla^2 f(p) = 0 \]

and \( f(p) \) satisfies Laplace’s equation on the interior of \( C \)

Since \( H[f](p) \) is a linear combination of harmonic functions, it also satisfies Laplace’s equation

Use proof by induction to generalize to any \( n \)-dimension
## Results

<table>
<thead>
<tr>
<th>Cage Vertices</th>
<th>325</th>
<th>112</th>
<th>39</th>
<th>27</th>
</tr>
</thead>
<tbody>
<tr>
<td>Object Vertices</td>
<td>9775</td>
<td>8019</td>
<td>269</td>
<td>136</td>
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<tr>
<td>Grid resolution</td>
<td>5</td>
<td>5</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>Solve time</td>
<td>57.4</td>
<td>17.6</td>
<td>5.85</td>
<td>0.83</td>
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<tr>
<td>Pose time</td>
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<td>0.026</td>
<td>0.0001</td>
<td>0.0007</td>
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<tr>
<td>Solution size (MB)</td>
<td>9.2</td>
<td>3.7</td>
<td>0.32</td>
<td>0.048</td>
</tr>
</tbody>
</table>

Error: < 0.005

Total footprint: < 90MB
Future Work:

- Compute the harmonic coordinates for each cage vertex independently and in parallel

- Better solvers (currently using MultiGrid)

- Octrees

- Localize re-solves